

1. (JPE May, 1994). Let $X^{-1}AX = D$, where D is a diagonal matrix.
- (i) Show that the columns of X are right eigenvectors and the conjugate rows of X^{-1} are left eigenvectors of A .
 - (ii) Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A . Show that there are right eigenvectors x_1, \dots, x_n and left eigenvectors y_1, \dots, y_n such that

$$A = \sum_{i=1}^n \lambda_i x_i y_i^*$$

2. Let $A \in \mathbb{C}^{n \times n}$. Show that
- (i) λ is an eigenvalue of A iff $\bar{\lambda}$ is an eigenvalue of A^* .
 - (ii) if A is normal, then for each eigenvalue the left and right eigenspaces coincide;
 - (iii) if A is normal, then for any simple eigenvalue λ of A we have $K(\lambda) = 1$.
3. Let $A \in \mathbb{C}^{n \times n}$ and $B = Q^* A Q$, where Q is a unitary matrix. Show that if the left and right eigenspaces of A are equal, then B enjoys the same property. After that show that A is normal. Conclude that if A has all simple eigenvalues with $K(\lambda) = 1$, then A is normal.
4. If λ is an eigenvalue of geometric multiplicity ≥ 2 for a matrix A , show that for each right eigenvector x there is a left eigenvector y such that $y^* x = 0$.

[Bonus] Let $A \in \mathbb{C}^{n \times n}$ be Hermitean with eigenvalues $\lambda_1 \leq \dots \leq \lambda_n$. Let $\mu_1 \leq \dots \leq \mu_{n-1}$ be all the eigenvalues of the $(n-1)$ -st principal minor A_{n-1} of A . Use the Minimax theorem to prove the *interlacing property*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \dots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n$$

If you plan to take JPE this spring, see copies of some recent JPE tests at

http://www.math.uab.edu/programs/jpe_exams.html

Some older JPE tests are in the math office (ask Sue).