MA 660-3A, Dr Chernov — Assignment #9

. — Due Mon, Mar 22

1. Prove properties 1, 6, and 8 of condition numbers listed in Proposition 12.8.

2. (JPE, September 1997). Show that, given a matrix $A \in \mathbb{R}^{n\times n}$, one can choose vectors b and Δb so that if

$$Ax = b$$

$$A(x + \Delta x) = b + \Delta b$$

then

$$\frac{||\Delta x||_2}{||x||_2} = \kappa_2(A)\frac{||\Delta b||_2}{||b||_2}$$

Explain the significance of this result for the 'optimal' role of condition numbers in the sensitivity analysis of linear systems.
(Hint: use the SVD theorem to show that it is enough to consider the case where A is a diagonal matrix.)

3. (JPE, September 2002). Consider a linear system $Ax = b$. Let x^* be the exact solution, and let x_c be some computed approximate solution. Let $e = x^* - x_c$ be the error and $r = b - Ax_c$ the residual for x_c. Show that

$$\frac{1}{\kappa(A)}\,\frac{\|r\|}{\|b\|} \le \frac{\|e\|}{\|x^*\|} \le \kappa(A)\,\frac{\|r\|}{\|b\|}$$

Interpret the above inequality for $\kappa(A)$ close to 1 and for $\kappa(A)$ large.

4. (JPE, May 1997). Compute the condition numbers κ_1, κ_2 and κ_∞ for the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1.01 & 2 \end{pmatrix}$$

For extra credit: show that for typical 2×2 matrices A we have $\kappa_1(A) = \kappa_\infty(A)$. What exactly does 'typical' mean here?