Thermodynamic Formalism, MA 793

Preface

Thermodynamic formalism is the theory of Gibbs Measures. Technically, those are defined on abstract spaces of symbolic sequences called subshifts of finite type (the elements of those spaces are infinite sequences of some symbols). In that context, it is hard to see the real meaning of Gibbs measures and their relevance for dynamical systems and classical physics. It is also difficult to learn Gibbs measures starting with that abstract context.

We adopt a different approach. First we describe two classes of dynamical systems: *Anosov diffeomorphisms* and *Axiom A diffeomorphisms*. Then we construct *Markov partitions*, which will naturally lead to a symbolic representation (coding) by *subshifts of finite type*. Gibbs measures will then correspond to invariant measures for Anosov and Axiom A diffeomorphisms.

Notation

- M denotes a smooth (of class C^∞) compact connected manifold with a Riemannian metric.
- d denotes the dimension of M, i.e. $\dim M = d$.
- $\mathcal{T}_x M$ denotes the tangent space to M at $x \in M$, and $\mathcal{T}M = \cup_x \mathcal{T}_x M$ the tangent bundle of M.
- $\|v\|$ denotes the norm of $v \in \mathcal{T}M$.
- $\angle(u, v)$ denotes the angle between $u, v \in \mathcal{T}_x M$.
- $T : M \to M$ denotes a diffeomorphisms of class C^p, $p \geq 1$.
- $D_x T : \mathcal{T}_x M \to \mathcal{T}_{T(x)} M$ denotes the derivative of T at $x \in M$.

1 Anosov Diffeomorphisms

1.1 Definition (Anosov Diffeomorphism)

Suppose that each tangent space $\mathcal{T}_x M$ with $x \in M$ is a direct sum

$$\mathcal{T}_x M = E_x^u \oplus E_x^s$$

of subspaces so that

(a) $D_x T(E_x^u) = E_{Tx}^u$ and $D_x T(E_x^s) = E_{Tx}^s$

(b) there exist constants $C > 0$ and $\lambda \in (0, 1)$ such that

$$\|D_x T^n(v)\| \leq C\lambda^n \|v\| \quad \text{when} \quad v \in E_x^s, \quad n \geq 0$$

$$\|D_x T^{-n}(v)\| \leq C\lambda^n \|v\| \quad \text{when} \quad v \in E_x^u, \quad n \geq 0$$

(c) The spaces E_x^u and E_x^s vary continuously with x, and the angle between E_x^u and E_x^s is uniformly bounded away from zero.

Note: the dimensions $d^u = \dim E_x^u$ and $d^s = \dim E_x^s$ are constant on M.

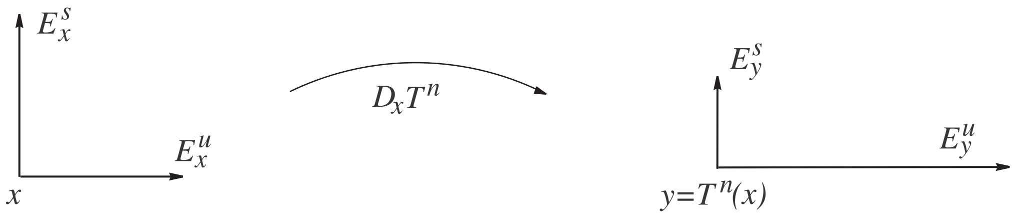

Figure 1: The action of $D_x T$ on the subspaces E_x^u and E_x^s.

1.2 Exercise

(a) Show that (a) and (b) of 1.1 imply $\|D_x T^{-n}(v)\| \geq C\lambda^{-n}\|v\|$ for all $v \in E_x^s$ and $n \geq 0$ and $\|D_x T^n(v)\| \geq C\lambda^{-n}\|v\|$ for all $v \in E_x^u$ and $n \geq 0$.

(b) Prove that $T^{-1} : M \to M$ is an Anosov diffeomorphism.

(c) Show that if $v \in \mathcal{T}_x M$ is such that $v \notin E_x^u$ and $v \notin E_x^s$, then there is a $C_v > 0$ such that $\|D_x T^n(v)\| \geq C_v \lambda^{-|n|}$ for all $n \in \mathbb{Z}$, i.e. the images of v grow exponentially both in the future and in the past.

(d) Prove that $T : M \to M$ has at least one invariant measure μ.

(e) Let μ be a T-invariant measure on M. Prove that at μ-almost every point $x \in M$ Lyapunov exponents exist (recall the Oseledec Theorem). Then show that there are exactly d^u positive Lyapunov exponents (all $\geq \log \lambda^{-1}$) and exactly d^s negative Lyapunov exponents (all $\leq \log \lambda$).

1.3 Remarks

The subspace E_x^u and its vectors are said to be *unstable*, the subspace E_x^s and its vectors are said to be *stable*. The assumption (b) in Definition 1.1 implies a uniform contraction of stable vectors and uniform expansion of unstable vectors (under the future iterations). The set of assumptions (a)–(c) specifies what is frequently called *uniform hyperbolicity*.

As Exercise 1.2 (e) shows, the assumptions (a)–(c) imply that there are no zero Lyapunov exponents for the map T (this property is often referred to as *hyperbolicity*). Hence, uniform hyperbolicity implies hyperbolicity.

The converse is not true, i.e. the hyperbolicity does not imply uniform hyperbolicity. Indeed, the hyperbolicity is only an asymptotic property meaning that the images of stable (unstable) vectors contract (respectively, expand) *eventually*, in the limit, as time goes to infinity. This does not prevent temporary opposite trends – when stable vectors expand and unstable vectors contract during arbitrarily long time periods.

1.4 Adapted Metric

Technically, the definition 1.1 does not prevent temporary expansion of stable vectors and contraction of unstable vectors (but the time periods of such reverses are strictly bounded by n such that $C\lambda^n < 1$). Interestingly, for every Anosov diffeomorphism there exist a Riemannian metric in which Definition 1.1 (b) holds with $C = 1$. This metric is called *adapted metric* or *Lyapunov metric*.

The adapted metric makes many calculations simpler. For our convenience, we will always use an adapted metric.

1.5 Proposition

The assumption (c) in Definition 1.1 follows from (a) and (b).

Proof. Suppose $x_n \to x$. By taking a subsequence, if necessary, we can assume that $k = \dim E_{x_n}^s$ is independent of n. Let $e_1^{(n)}, \dots, e_k^{(n)}$ be an orthonormal basis in $E_{x_n}^s$. By taking a subsequence, if necessary, we can enforce the convergence $e_i^{(n)} \to e_i$ as $n \to \infty$, for each $i = 1, \dots, k$. Let $E_x^1 = \text{span}\{e_1, \dots, e_k\}$. By continuity, $\|D_x T^n(v)\| \leq \lambda^n \|v\|$ and $\|D_x T^{-n}(v)\| \geq \lambda^{-n}\|v\|$ for all $v \in E_x^s$ and $n \geq 0$. Hence, $E_x^1 \subset E_x^s$. A similar argument shows that $E^u_{x_i}$ converge to a subspace $E^2_x \subset E^u_x$. Since $\dim E^1_x + \dim E^2_x = d$, then we have $E^1_x = E^s_x$ and $E^2_x = E^u_x$. $\Box$

1.6 Proposition

Let $\tilde{T} : M \to M$ be a small C^1 perturbation of an Anosov diffeomorphism $T : M \to M$. That is, there is a small $\varepsilon > 0$ such that $\mathrm{dist}(T(x), \tilde{T}(x)) < \varepsilon$ and $\|D_xT - D_x\tilde{T}\| < \varepsilon$ for all $x \in M$. Then $\tilde{T}$ is also an Anosov diffeomorphism.

Proof. The main part is to construct the subspaces $\tilde{E}^u_x$ and $\tilde{E}^s_x$ for the perturbed diffeomorphism $\tilde{T}$. We fix a small $\alpha > 0$ and take a cone C^u_x around E^u_x with opening α, i.e. $C^u_x = \{v \in \mathcal{T}_xM : \angle(v, E^u_x) < \alpha\}$. It is easy to check, by a perturbation argument, that $D_x\tilde{T}(C^u_x) \subset C^u_{\tilde{T}x}$ and $\|D_x\tilde{T}(v)\| \geq \lambda_1^{-1}\|v\|$ for all $v \in C^u_x$ and some constant $\lambda_1 \in (\lambda, 1)$ that depends on α. Then we define $\tilde{E}^u_x = \cap_{n\geq 0} D_{\tilde{T}^{-n}x}\tilde{T}^n(C^u_{\tilde{T}^{-n}x})$. Similarly, $\tilde{E}^s_x = \cap_{n\geq 0} D_{\tilde{T}^n x}\tilde{T}^{-n}(C^s_{\tilde{T}^n x})$.

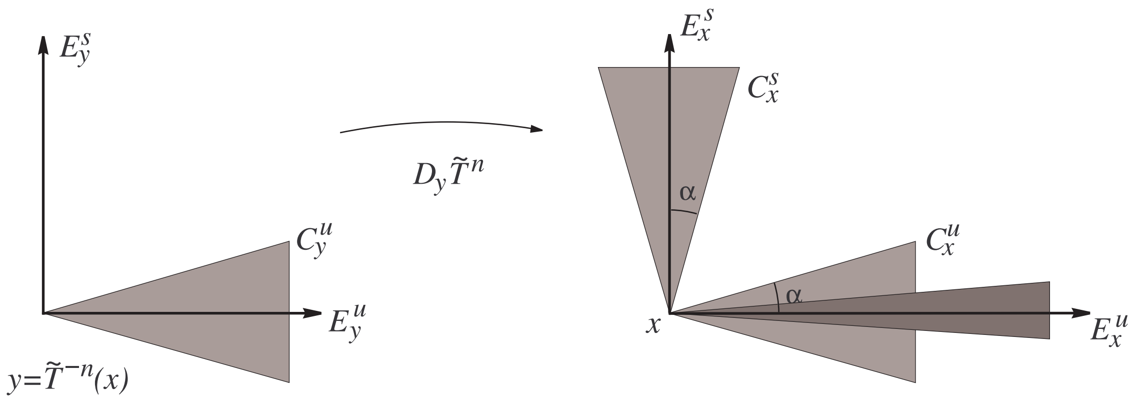

Figure 2: Stable and unstable cones.

1.7 Definition (local stable and unstable manifolds)

Let $x \in M$ and $\varepsilon > 0$. A *local unstable manifold* of x is

$$W^u_\varepsilon(x) = \{y \in M : \ \mathrm{dist}(T^{-n}x, T^{-n}y) < \varepsilon \quad \forall n \geq 0\}$$

Similarly, a *local stable manifold* of x is

$$W^s_\varepsilon(x) = \{y \in M : \ \mathrm{dist}(T^n x, T^n y) < \varepsilon \quad \forall n \geq 0\}$$

1.8 Remark

There is a natural duality between stable and unstable objects in this theory. These objects get interchanged if one replaces $T : M \to M$ with $T^{-1} : M \to M$. For brevity, we will sometimes mention just one of them and suppress the other.

1.9 Theorem

Let $T : M \to M$ be a C^p Anosov diffeomorphism. There is a small $\varepsilon > 0$ such that for any point $x \in M$

(a) $W^u_\varepsilon(x)$ and $W^s_\varepsilon(x)$ are embedded C^p disks in M with $\mathcal{T}_x W^{u,s}_\varepsilon(x) = E^{u,s}_x$.

(b) We have
$$\mathrm{dist}(T^{-n}x, T^{-n}y) \le \lambda^n \,\mathrm{dist}(x,y) \quad \forall y \in W^u_\varepsilon(x), \quad n \ge 0$$
and
$$\mathrm{dist}(T^{n}x, T^{n}y) \le \lambda^n \,\mathrm{dist}(x,y) \quad \forall y \in W^s_\varepsilon(x), \quad n \ge 0$$

(c) We have
$$\mathcal{T}_y W^{u,s}_\varepsilon(x) = E^{u,s}_y \quad \forall y \in W^{u,s}_\varepsilon(x)$$

(d) The disks $W^{u,s}(x)$ vary continuously with x.

Proof. A complete proof of this theorem is quite involved, and its details are not much relevant to our further studies. We only sketch a proof here. The idea of the proof resembles that of Proposition 1.6. Define a cone C^u_x with a small opening $\alpha > 0$. Then fix a small $\varepsilon > 0$. At any point $x \in M$ the ε-ball in the tangent space $\mathcal{T}_x M$ can be identified with a small neighborhood $\mathcal{U}(x)$ of x via the exponential map. Now consider all local manifolds of dimension d^u in $\mathcal{U}(x)$ passing through x whose tangent vectors lie within the cone C^u_x.

Under the map T, each of these manifolds stretches out in all directions by a factor λ^{-1}. It is easy to see that the image of each such manifold covers one of the manifolds constructed similarly at the point $T(x)$. Removing the excesses (the parts of the images stretching beyond $\mathcal{U}(Tx)$), we obtain a transformation of manifolds in $\mathcal{U}(x)$ onto those in $\mathcal{U}(Tx)$. Now it is easy to check that the images get closer together: their variation in the direction parallel to E^s_{Tx} shrinks by a factor $\sim \lambda^2$. Also, the variation of their tangent planes shrinks by a factor of $\sim \lambda^2$. Now we take all those manifolds constructed at the point $T^{-n}(x)$ and map them under T^n. Their images make a very thin "pancake". It is easy to see that in the limit $n \to \infty$ they converge to a single C^1 manifold, call it $W^u_\varepsilon(x)$. It is also easy to check that $\mathcal{T}_x W^u_\varepsilon(x) = E^u_x$ (the "pancake" always contains some manifolds whose tangent plane at x is E^u_x).

An extra technical work (we skip it) is required to show that $W^u_\varepsilon(x)$ is C^p smooth, i.e. the stable and unstable manifolds are as smooth as the Anosov diffeomorphism T itself. Thus we obtain (a). The claim (b) can be verified directly. The claim (c) holds since the same construction can be applied to the point y. The claim (d) follows from (c) and the continuity of the spaces $E^{u,s}_x$.

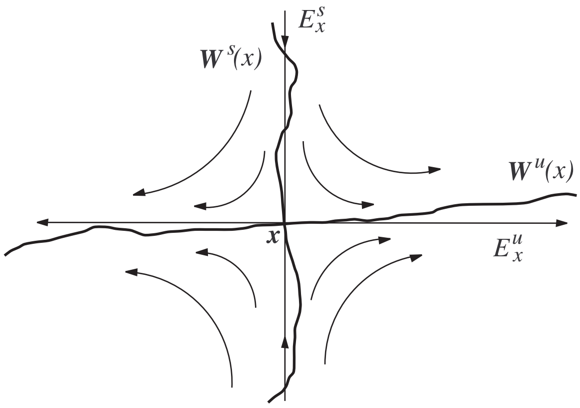

Figure 3: Stable and unstable manifolds.

1.10 Remark

We will denote local stable and unstable manifolds by $W^u(x)$ and $W^s(x)$ suppressing their size ε. Note that local stable and unstable manifolds are thus not unique. However they are essentially unique. In particular, if $z \in W^u(x) \cap W^u(y)$, then the intersection $W^u(x) \cap W^u(y)$ contains an open neighborhood of z in both manifolds (a similar statement holds for stable manifolds).

1.11 Definition (global stable and unstable manifolds)

Global stable and unstable manifolds are defined by

$$\mathcal{W}^s(x) = \cup_{n\geq 0} T^{-n} W^s(T^n x) \quad \text{and} \quad \mathcal{W}^u(x) = \cup_{n\geq 0} T^{n} W^u(T^{-n} x)$$

1.12 Remarks

We have

$$\mathcal{W}^u(x) = \{y \in M : \ \mathrm{dist}(T^{-n}x, T^{-n}y) \to 0 \ \text{ as } \ n \to \infty\}$$

(a similar property holds for $\mathcal{W}^s(x)$).

Also, $T^n(\mathcal{W}^u(x)) = \mathcal{W}^u(T^n x)$ and $T^n(\mathcal{W}^s(x)) = \mathcal{W}^s(T^n x)$ for all $n \in \mathbb{Z}$. Global stable (and unstable) manifolds are unique. In particular, if $\mathcal{W}^u(x) \cap \mathcal{W}^u(y) \neq \emptyset$, then $\mathcal{W}^u(x) = \mathcal{W}^u(y)$ (a similar statement holds for stable manifolds).

The global stable and unstable manifolds are "infinitely large", so that they never terminate. However, since the entire manifold M is compact, the global stable and unstable manifold have to wrap around M and, typically, become dense in M.

1.13 Examples

In the course of Dynamical Systems, MA 760, we have studied hyperbolic toral automorphisms. Those are Anosov diffeomorphisms. Their stable and unstable manifolds are straight lines parallel to the eigenvectors of the corresponding matrix. (Recall that we proved that each such line is dense on the torus.)

By Proposition 1.6, small C^1 perturbations of hyperbolic toral automorphisms are Anosov diffeomorphisms, too. It is rather strange that there are no more simple examples...

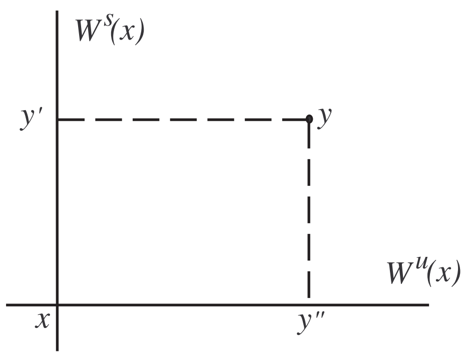

Figure 4: Canonical coordinates.

1.14 Proposition (Canonical coordinates)

There exists a small $\varepsilon_0 > 0$ such that if $\mathrm{dist}(x, y) < \varepsilon_0$, then the intersection

$$[x, y] := W^s(x) \cap W^u(y)$$

consists exactly of one point. Furthermore, the point $[x, y]$ depends continuously on x and y.

As a result, one can represent every point y in the ε_0-neighborhood of x by two points

$$y' = [x, y] \in W^s(x) \quad \text{and} \quad y'' = [y, x] \in W^u(x)$$

The points y', y'' play the role of coordinates of the point y, and the manifolds $W^s(x)$ and $W^u(x)$ are coordinate axes.

Proof. The intersection $W^s(x) \cap W^u(x)$ is transversal, and such intersections are preserved under small perturbations.

1.15 Corollary

The map $T : M \to M$ is expansive, i.e. for any $x \neq y$ there is an $n \in \mathbb{Z}$ such that $\mathrm{dist}(T^n x, T^n y) > \varepsilon_0$.

Proof. Otherwise $y \in W^u(x) \cap W^s(x)$ by 1.10. However, this is impossible by 1.14.

2 Axiom A Diffeomorphisms

2.1 Definition (Hyperbolic Set)

A closed subset $\Lambda \subset M$ is *hyperbolic* if $T(\Lambda) = \Lambda$ and each tangent space $\mathcal{T}_x M$ with $x \in \Lambda$ is a direct sum

$$\mathcal{T}_x M = E_x^u \oplus E_x^s$$

of subspaces so that

(a) $D_x T(E_x^u) = E_{Tx}^u$ and $D_x T(E_x^s) = E_{Tx}^s$

(b) there exist constants $C > 0$ and $\lambda \in (0, 1)$ such that

$$\|D_x T^n(v)\| \le C\lambda^n \|v\| \quad \text{when} \quad v \in E_x^s, \quad n \ge 0$$

$$\|D_x T^{-n}(v)\| \le C\lambda^n \|v\| \quad \text{when} \quad v \in E_x^u, \quad n \ge 0$$

(c) The spaces E_x^u and E_x^s vary continuously with x, and the angle between E_x^u and E_x^s is uniformly bounded away from zero.

Note that the conditions (a)–(c) are identical to those in Definition 1.1, except now we require them only for $x \in \Lambda$.

Observe that if $T : M \to M$ is an Anosov diffeomorphism, then the entire manifold M is a hyperbolic set.

2.2 Definition (Nonwandering Points)

A point $x \in M$ is nonwandering if

$$\mathcal{U} \cap \cup_{n>0} T^n(\mathcal{U}) \neq \emptyset$$

for every open neighborhood $\mathcal{U}$ of x. The set of all nonwandering points $x \in M$ is denoted by $\Omega(T)$.

2.3 Exercise

Show that $\Omega(T)$ is nonempty, closed and T-invariant. Show that $\Omega(T)$ contains all periodic points of T.

2.4 Definition (Axiom A Diffeomorphism)

$T : M \to M$ is an *Axiom A diffeomorphism* if $\Omega(T)$ is hyperbolic and periodic points are dense in $\Omega(T)$.

2.5 Example: Smale's horseshoe

Let R be a rectangle and $T : R \to \mathbb{R}^2$ be a map that stretches R in the horizontal direction, bends it into the shape of a "horseshoe" (see Figure 5) so that $R \cap T(R)$ consists of two narrow horizontal rectangles. We assume that T restricted to $R \cap T^{-1}(R)$ is linear and its derivative is

$$DT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

for some $\lambda_1 > 2$ and $\lambda_2 < 1/2$.

The maximal invariant set

$$\Omega(T) = \cap_{n=-\infty}^{\infty} T^n(R)$$

is a Cantor set with a natural product structure (it is the product of a Cantor set on the x axis and a Cantor set on the y axis).

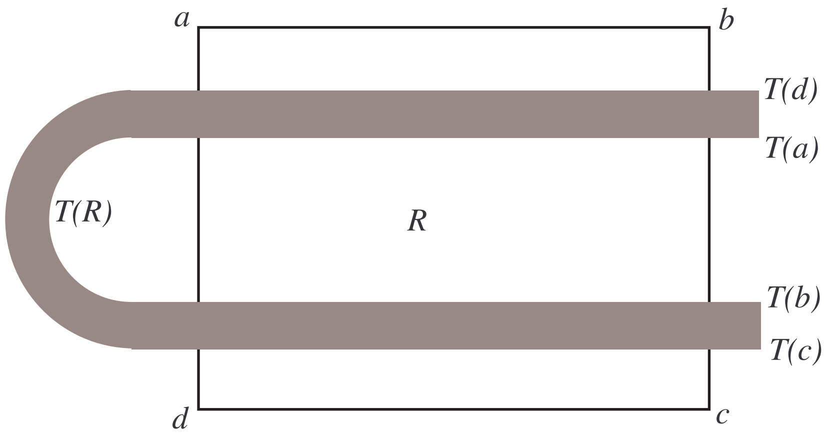

Figure 5: Smale's horseshoe.

Note: the map T on R is not exactly a diffeomorphism of a compact manifold. It can be extended to such, but this is rather difficult, and the details of such an extension would be completely irrelevant. We will simply pretend that we deal with a diffeomorphism of a compact manifold, and in fact all our considerations will be restricted to R.

2.6 Exercise

(a) Show that the conditions (a)–(c) of Definition 2.1 are satisfied for $\Lambda = \Omega(T)$ in the horseshoe example. Describe the spaces E_x^u and E_x^s, find the constants C and λ. Deduce that $\Omega(T)$ is a hyperbolic set.

(b) [This is optional, it might be difficult!] Show that periodic points are dense in $\Omega(T)$. Conclude that T is an Axiom A diffeomorphism. [Note: this fact will actually follow from some statements that we prove below.]

2.7 Remark

Smale's horseshoe is a canonical example of an Axiom A diffeomorphism, which is essentially different from an Anosov diffeomorphism. One can easily extend the horseshoe construction to any dimension.

In a sense, all Axiom A diffeomorphisms are either Anosov or horseshoes or combinations thereof.

2.8 Extension of Anosov theory

Nearly all the results we obtained in Section 1 for Anosov diffeomorphisms extend to Axiom A diffeomorphisms with obvious modifications:

- The claims in Exercise 1.2, the existence of an adapted metric 1.4, and Proposition 1.5 apply word for word.
- Definition 1, Remark 1.8, Theorem 1.9, and Remark 1.10 apply to all the points $x \in \Omega(T)$. Note, however: stable and unstable manifolds may not completely belong to $\Omega(T)$ (recall the horseshoe, for example!).
- Theorem 1.9 applies to all points $x \in \Omega(T)$ and $y \in W^{u,s}(x) \cap \Omega(T)$.
- Remark 1.10, Definition 1.11, and Remark 1.12 remain unchanged.
- Proposition 1.14 needs an important addition: if $x, y \in \Omega(T)$ and $\text{dist}(x, y) < \varepsilon_0$, then $[x, y] \in \Omega(T)$ (this follows from Lemma 2.9 below).
- Corollary 1.15 applies to the map $T : \Omega(T) \to \Omega(T)$.

2.9 Lemma

Let T be an Axiom A diffeomorphism and $x, y \in \Omega(T)$. Assume that $\exists z \in \mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ and $\exists w \in \mathcal{W}^s(x) \cap \mathcal{W}^u(y)$ and both intersections are transversal at the points z and w. Then $z, w \in \Omega(T)$.

Proof. Let $z \in \mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ and $w \in \mathcal{W}^s(x) \cap \mathcal{W}^u(y)$. We need to show that z is nonwandering. Let $\mathcal{U}$ be an open neighborhood of z. Since periodic points are dense in $\Omega(T)$, then x and y can be approximated by periodic points, say p and q, with periods m and n, respectively, so that $\mathcal{L} := \mathcal{W}^u(p) \cap \mathcal{U} \neq \emptyset$ and $\mathcal{L} \cap \mathcal{W}^s(q) \neq \emptyset$. We will show that $T^k(\mathcal{L}) \cap \mathcal{U} \neq \emptyset$ for some large k.

Note that p and q are fixed points for the map $S = T^{mn}$. Observe that the images $S^k(\mathcal{L})$ accumulate on $W^u(q)$ as k grows, i.e. $W^u(q) \subset \lim_{k\to\infty} S^k(\mathcal{L})$. Next, for large k those images stretch out along $\mathcal{W}^u(q)$ and cross $\mathcal{W}^s(p)$ near the point w. Hence, for even larger k they will come back to p along $\mathcal{W}^s(p)$, and therefore will accumulate on $\mathcal{W}^u(p)$. For still larger k, they will stretch out along $\mathcal{W}^u(p)$ and cross $\mathcal{U}$.

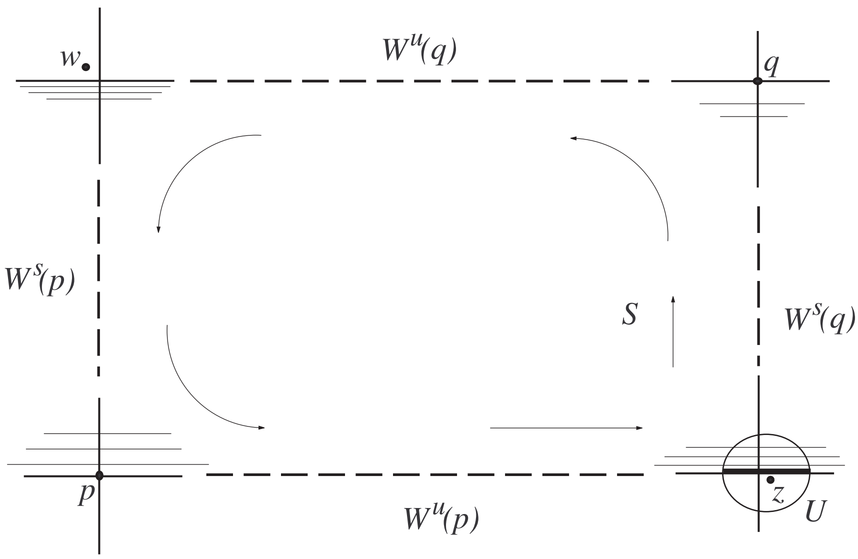

Figure 6: Proof of Lemma 2.9.

2.10 Definition (topological transitivity and mixing)

A map $T : X \to X$ of a topological space X is *topologically transitive* if for any open subsets $U, V \subset X$ there is an $n \geq 0$ such that $U \cap T^n(V) \neq \emptyset$.

T is *topologically mixing* if for any open subsets $U, V \subset X$ there is an $n_0 \geq 0$ such that $U \cap T^n(V) \neq \emptyset$ for all $n > n_0$.

From now on $T : M \to M$ will always be an Axiom A diffeomorphism.

2.11 Theorem (Spectral Decomposition)

(a) We have $\Omega(T) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s$, a disjoint union of closed sets, such that $T(\Omega_i) = \Omega_i$ and $T|_{\Omega_i}$ is topologically transitive for each i.

(b) Next, $\Omega_i = \Omega_{i,1} \cup \cdots \cup \Omega_{i,s_i}$, a disjoint union of closed sets, such that $T(\Omega_{i,j}) = \Omega_{i,j+1}$ (with identification $\Omega_{i,s_i+1} = \Omega_{i,1}$), and $T^{s_i}|_{\Omega_{i,j}}$ is topologically mixing for each j.

The sets Ω_i are called *basic sets*.

Proof. We will define an equivalence relation on $\Omega(T)$ and obtain Ω_{ij} as equivalence classes. Let $x \sim y$ iff $\mathcal{W}^u(x) \cap \mathcal{W}^s(y) \neq \emptyset$ and $\mathcal{W}^s(x) \cap \mathcal{W}^u(y) \neq \emptyset$. (Note: the reflexivity and symmetry are trivial, and transitivity follows from Lemma 2.9). The analogue of 1.14 implies that each equivalence class is open in $\Omega(T)$. By compactness, there are finitely many equivalence classes. If X is one of them, then for each $n \in \mathbb{Z}$ the set $T^n(X)$ is an equivalence class, too. Thus T permutes equivalence classes.

The topological transitivity in (a) follows from the topological mixing in (b). To prove the latter, let $U, V \subset \Omega_{ij}$ be open sets and pick two periodic points $p \in U$ and $q \in V$. Let $T_0 = T^{s_i}$ be the minimal iteration of T that leaves Ω_{ij} invariant. Denote by n and m the periods of p and q with respect to the map T_0 and set $S = T_0^{mn}$. An argument similar to the proof of Lemma 2.9 shows that $U \cap S^k(V) \neq \emptyset$ for all large enough k. Lastly, the topological mixing follows from the same argument applied to each pair of points $T_0^i(p)$, $1 \le i \le n$, and $T_0^j(q)$, $1 \le j \le m$.

2.12 Definition (Shadowing)

A sequence of points $\{x_n\}_{n=a}^b$ ($a = -\infty$ or $b = +\infty$ is permitted) in M is an *δ-pseudo-orbit* if

$$\mathrm{dist}(Tx_i, x_{i+1}) < \delta \quad \text{for all} \quad i \in [a, b)$$

A point $x \in \Omega(T)$ *ε-shadows* $\{x_n\}_{n=a}^b$ if

$$\mathrm{dist}(T^i(x), x_i) < \varepsilon \quad \text{for all} \quad i \in [a, b]$$

2.13 Lemma (Shadowing Lemma)

For any $\varepsilon > 0$ there is a $\delta > 0$ such that every *δ-pseudo-orbit* in $\Omega(T)$ (i.e., every $x_i \in \Omega(T)$) is ε-shadowed by a points $x \in M$.

Proof. For each $n \in [a, b]$ consider a neighborhood $\mathcal{U}_n \subset \Omega(T)$ of the point x_n (in the set $\Omega(T)$) obtained by taking a direct product, in the sense of 1.14, of the closed ε-balls on $W^u(x_n) \cap \Omega(T)$ and $W^s(x_n) \cap \Omega(T)$. By reducing ε if necessary we can make all those neighborhoods very close to regular cylindrical sets, i.e. make their distortions arbitrarily small (since our manifold is compact, this can be done uniformly over all n's).

Next we assume that $\delta \ll \varepsilon$. It is easy to verify that for all $y \in \mathcal{U}_n \cap T^{-1}\mathcal{U}_{n+1}$

$$T^{-1}(W^s(Ty) \cap \mathcal{U}_{n+1}) \supset W^s(y) \cap \mathcal{U}_n$$

and

$$T^{-1}(W^u(Ty) \cap \mathcal{U}_{n+1}) \subset W^u(y) \cap \mathcal{U}_n$$

Hence, $\mathcal{U}_n \cap T^{-1}\mathcal{U}_{n+1}$ is a direct product of two closed sets: one is $W^s(x_n) \cap \mathcal{U}_n$ and the other is $W^u(x_n) \cap T^{-1}\mathcal{U}_{n+1}$, which is a subset of $W^u(x_n) \cap \mathcal{U}_n$. By induction on k, it follows that for each $n+k \le b$ the set $\cap_{i\in[n,n+k]} T^{n-i}\mathcal{U}_i$ is a direct product of $W^s(x_n)\cap\mathcal{U}_n$ and a closed subset of $W^u(x_n)\cap\mathcal{U}_n$, which decreases as k grows.

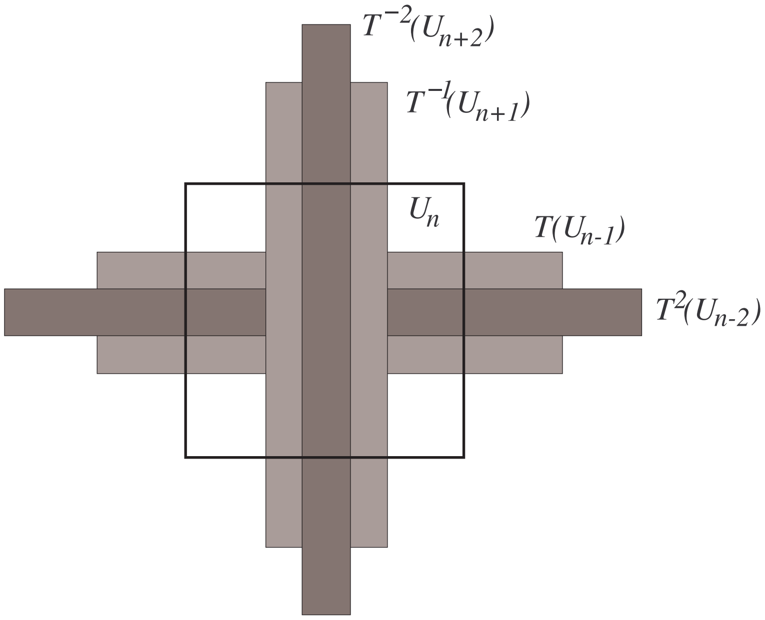

Figure 7: Proof of Lemma 2.13.

Similarly, for each $n-k \ge a$ the set $\cap_{i\in[n-k,n]} T^{n-i}\mathcal{U}_i$ is a direct product of $W^u(x_n)\cap\mathcal{U}_n$ and a closed subset of $W^s(x_n)\cap\mathcal{U}_n$, which decreases as k grows.

Now we fix $n \in [a,b]$ and define

$$K = \cap_{i\in[a,b]} T^{n-i}\mathcal{U}_i$$

This is a direct product of two closed subsets of $W^s(x_n)\cap\mathcal{U}_n$ and $W^u(x_n)\cap\mathcal{U}_n$. Any point $x \in T^{-n}(K)$ will now shadow our pseudo-orbit.

Note: if $a = -\infty$ and $b = +\infty$, then K is a singleton, and the point x is unique.

2.14 Corollary (Anosov's Closing Lemma)

For any $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in \Omega(T)$ and $\mathrm{dist}(T^n x, x) < \delta$, then there is an $x' \in \Omega(T)$ with $T^n(x') = x'$ and

$$\mathrm{dist}(T^i x, T^i x') < \varepsilon$$

for all $i \in [0,n]$.

Proof. Let $x_i = T^k(x)$ for $i \equiv k \pmod{n}$ and $0 \le k < n$. Then $\{x_i\}_{i=-\infty}^{\infty}$ is a δ-pseudo-orbit. By 2.13, it is ε-shadowed by some point $x' \in \Omega(T)$. Next,

$$\operatorname{dist}(T^i x', T^{i+n} x') < \operatorname{dist}(T^i x', x_i) + \operatorname{dist}(x_i, T^{i+n} x') < 2\varepsilon$$

for all $i \in \mathbb{Z}$. We can assume that ε is small enough and obtain $x' = T^n x'$ by expansivity, cf. 1.15.

2.15 Corollary

Every Anosov diffeomorphism is an Axiom A diffeomorphism.

Proof. Indeed, periodic points are dense in $\Omega(T)$ by 2.14.

2.16 Remark

If $T : M \to M$ is an Anosov diffeomorphism and $\Omega(T) = M$, then 2.11 implies that T is topologically transitive and mixing. Conversely, if $T : M \to M$ is a topologically transitive Anosov diffeomorphism, then it is easily seen that $\Omega(T) = M$.

Open Problem: It is unknown if $\Omega(T) = M$ for every Anosov diffeomorphism.