Theorem 4.16. For any $A \in \mathbb{C}^{m\times n}$ we have

$$\|A\|_2^2 = \|A^*\|_2^2 = \|A^*A\|_2 = \|AA^*\|_2 = \lambda_{\max}$$

where $\lambda_{\max}$ is the largest eigenvalue of both A^*A and AA^*.

In the proof, $\|\cdot\|$ will always denote the 2-norm.

Lemma. For every vector $z \in \mathbb{C}^n$ we have $\|z\| = \sup_{\|y\|=1} |\langle y, z\rangle|$.

Proof. Indeed, by the Cauchy-Schwarz inequality

$$|\langle y, z\rangle| \le \langle y, y\rangle^{1/2} \langle z, z\rangle^{1/2} = \|z\|$$

and the equality is attained whenever y is parallel to z, we can set $y = \pm \frac{z}{\|z\|}$. $\square$

Step 1. To prove that $\|A\| = \|A^*\|$ we write

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle y, Ax\rangle| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle A^*y, x\rangle|$$

$$= \sup_{\|y\|=1} \sup_{\|x\|=1} |\langle x, A^*y\rangle| = \sup_{\|y\|=1} \|A^*x\| = \|A^*\|$$

Step 2. To prove that $\|A\|^2 = \|A^*A\|$ we write

$$\|A^*A\| = \sup_{\|x\|=1} \|A^*Ax\| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle y, A^*Ax\rangle| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle Ay, Ax\rangle|$$

Then again by the Cauchy-Schwarz inequality

$$|\langle Ay, Ax\rangle| \le \|Ax\|\,\|Ay\| \le \|A\|\,\|A\| = \|A\|^2$$

hence $\|A^*A\| \le \|A\|^2$. On the other hand, setting $x = y$ gives

$$\sup_{\|x\|=1} \sup_{\|y\|=1} |\langle Ay, Ax\rangle| \le \sup_{\|x\|=1} |\langle Ax, Ax\rangle| = \|A\|^2,$$

hence $\|A^*A\| \ge \|A\|^2$. Therefore, $\|A^*A\| = \|A\|^2$.

Step 3. Using an obvious symmetry we conclude that $\|A^*\|^2 = \|AA^*\|$

Lemma. Let B be a Hermitian matrix. Then

$$\|B\| = \max_{1\le i\le n} |\lambda_i|,$$

where $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of B.

Proof. By the Spectral Theorem, $B = Q^*\Lambda Q$, where Q is a unitary matrix and Λ a diagonal matrix whose diagonal entries are λ_i's. We know (from earlier homework assignments) that

$$\|B\| = \|Q^*\Lambda Q\| = \|\Lambda\|.$$

Now for any vector $x = (x_1, \dots, x_n)$ we have $\Lambda x = (\lambda_1 x_1, \dots, \lambda_n x_n)$, hence

$$\|\Lambda x\|^2 = |\lambda_1|^2|x_1|^2 + \cdots + |\lambda_n|^2|x_n|^2$$

Now if $\|x\| = 1$, then

$$\|\Lambda x\|^2 \le \max_{1\le i\le n} |\lambda_i|^2$$

On the other hand, if $|\lambda_j| = \max_{1\le i\le n} |\lambda_i|$ then we choose $x = e_j$ and obtain $\|\Lambda x\|^2 = |\lambda_j|^2$. Lemma is proven. □

This completes the proof of 4.16. Note that A^*A and AA^* are positive-semidefinite matrices, so their eigenvalues are ≥ 0, so $\max_{1\le i\le n} |\lambda_i|$ is simply the largest eigenvalue, we denote it by $\lambda_{\max}$.

A little modification of the previous Lemma:

Lemma. Let B be a Hermitian positive-semidefinite matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Then

$$\sup_{\|x\|=1} \langle Bx, x\rangle = \lambda_{\max} = \max_{1\le i\le n} \lambda_i,$$

and if x is a vector such that

$$\|x\| = 1 \qquad \text{and} \qquad \langle Bx, x\rangle = \lambda_{\max},$$

then x is a corresponding eigenvector: $Bx = \lambda_{\max} x$.

Proof. Again, we use the Spectral Theorem to reduce the problem to a diagonal matrix Λ, then the proof is just a direct inspection. □

Corollary. If $\lambda_{\max}$ again denotes the largest eigenvalue of A^*A, then

$$\|Ax\|_2 = \|A\|_2\|x\|_2 \qquad \Longleftrightarrow \qquad A^*Ax = \lambda_{\max} x.$$

Proof. On the one hand

$$\|Ax\|_2^2 = \langle Ax, Ax\rangle = \langle A^*Ax, x\rangle$$

and on the other hand

$$\|A\|^2 = \lambda_{\max},$$

so for any vector x with $\|x\| = 1$ we have

$$\|Ax\|_2^2 = \|A\|_2^2 \qquad \Longleftrightarrow \qquad \langle A^*Ax, x\rangle = \lambda_{\max}.$$

Then we use the previous lemma. □