HAROLD BELL AND THE PLANE FIXED POINT PROBLEM

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Dedicated to Harold Bell

Abstract. In this paper we try to present in a coherent fashion proofs of basic results developed so far by H. Bell for the plane fixed point problem. Some of these results have been announced much earlier but without accessible proofs. We define the concept of the variation of a map on a simple closed curve and relate it to the index of the map on that curve: Index = Variation + 1. We develop a prime end theory through geometric chords in maximal balls contained in the complement of a non-separating plane continuum. We define the concept of an *outchannel* for a fixed point free map which carries a non-separating plane continuum into itself and prove that such a map has a unique outchannel, and that outchannel must have variation $= -1$.

Cartwright and Littlewood showed that each non-separating invariant continuum, under an orientation preserving homeomorphism of the plane, contains a fixed point. Bell announced in 1984 an extension of this result to holomorphic mappings of the plane. In this paper we introduce the class of oriented maps of the plane. We show that among perfect maps of the plane the oriented maps are exactly the compositions of open maps and monotone maps. It follows that all such compositions satisfy the Maximum Modulus Theorem. We also extend the above fixed point theorems to the class of positively-oriented perfect maps of the plane.

1. Introduction

By $\mathbb{C}$ we denote the plane and by $\mathbb{C}_\infty$ the Riemann sphere. Let X be a plane continuum. By $T(X)$ we denote the *topological hull* of X consisting of X union all of its bounded complementary domains. Thus, $\mathbb{C}_\infty \setminus T(X)$ is a simply-connected domain containing ∞. The following is a long-standing question in topology.

Fixed Point Question: *"Does a continuous function taking a non-separating plane continuum into itself always have a fixed point?"*

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We study the slightly more more general question, "Is there a plane continuum Z and a continuous function $f : \mathbb{C} \to \mathbb{C}$ taking Z into $T(Z)$ with no fixed points in $T(Z)$?" A Zorn's Lemma argument shows that if one assumes the answer is "yes," then there is a subcontinuum $X \subset Z$ minimal with respect to these properties. Therefore, we will assume the following throughout this paper:

1.1. **Standing Hypotheses.** *We assume that $f : \mathbb{C} \to \mathbb{C}$ is a map and X is a plane continuum such that $f(X) \subset T(X) = Y$, f has no fixed points in Y, and X is minimal with respect to these properties.*

It will follow from Theorem 4.5 that for such a minimal continuum, $f(X) = X = \partial Y$ (though it may not be the case that $f(Y) \subset Y$).

By results of Bell [3] (see also Sieklucki [20], and Iliadis [11]), the only unsolved general case (with no special assumptions on the map) is where the boundary of X is indecomposable (with a dense channel, explained later).

In this paper we use tools first developed by Bell to elucidate the action of a fixed point free map (should one exist). This paper was written in close cooperation with Bell and most of the results were first obtained by him. Unfortunately many of these results have been inaccessible. We believe that they deserve to be developed in order to be useful to the mathematical community. We have also made an effort to restate many of these results using existing language such as prime ends. We are very much indebted to Bell for his help while writing this paper. Theorem 5.1 (Unique Outchannel) is a new result due to Bell. Complete proofs of Theorems 2.13 and 2.14 appear in print for the first time.

The classical fixed point question asks whether each map of a non-separating plane continuum into itself must have a fixed point. Cartwright and Littlewood [7] showed that the answer is yes if the map can be extended to an orientation-preserving homeomorphism of the plane. It took over 20 years until Bell [5] extended this to the class of all homeomorphisms of the plane. Our ultimate goal is to extend these results to a natural, but larger, class of plane maps. Bell announced in 1984 (see also Akis [2]) that the Cartwright-Littlewood Theorem can be extended to the class of all holomorphic maps of the plane. These maps behave like orientation-preserving homeomorphisms in the sense that they preserve local orientation. We will show that compositions of open and of monotone maps of the plane are oriented and naturally decompose into two classes, one of which preserves and the other of which reverses local orientation. Moreover, any map in either of these classes is itself a composition of a monotone and a light open map. We will also show that such maps induce a map from the circle of prime ends of a saturated invariant subcontinuum to the circle of prime ends of its image. Finally we will show that each invariant non-separating plane continuum, under a positively-oriented map of the plane, must contain a fixed point.

2. Tools

Let S^1 denote the unit circle in the complex plane and let $p : \mathbb{R} \to S^1$ denote the covering map $p(x) = e^{2\pi ix}$. Let $g : S^1 \to S^1$ be a map. By the *degree* of the map g, denoted by degree(g), we mean the number $\hat{g}(1) - \hat{g}(0)$, where $\hat{g} : \mathbb{R} \to \mathbb{R}$ is a lift of the map g to the universal covering space $\mathbb{R}$ of S^1 (i.e., $p \circ \hat{g} = g \circ p$). It is well-known that degree(g) is independent of the choice of the lift.

2.1. **Index.** Let $g : S^1 \to \mathbb{C}$ be a map and let $S = g(S^1)$. Suppose $f : S \to \mathbb{C}$ has no fixed points on S. Then for all $z \in S$, the vector $f(z) - z \neq 0$. Hence the unit vector $v(z) = \frac{f(z)-z}{|f(z)-z|}$ always exists. Define the map $\overline{v} = v \circ g : S^1 \to S^1$ by

$$\overline{v}(t) = v(g(t)) = \frac{f(g(t)) - g(t)}{|f(g(t)) - g(t)|}.$$

Then the map $\overline{v} : S^1 \to S^1$ *lifts* to a map $\widehat{v} : \mathbb{R} \to \mathbb{R}$ such that $p \circ \widehat{v} = \overline{v} \circ p$. Define the *index of f with respect to g*, denoted ind(f, g) by

$$\mathrm{ind}(f, g) = \widehat{v}(1) - \widehat{v}(0) = \mathrm{degree}(\overline{v}).$$

More generally, for any parameters $0 \leq a < b \leq 1$ in $S^1 = \mathbb{R}/\mathbb{Z}$, define the *fractional index* of f on the path $g|_{[a,b]}$ in S by

$$\mathrm{ind}(f, g|_{[a,b]}) = \widehat{v}(b) - \widehat{v}(a).$$

While necessarily, the index of f with respect to g is an integer, the fractional index of f on $g|_{[a,b]}$ need not be. We shall have occasion to use fractional index in the proof of Theorem 2.13. Note that (fractional) index is the net change in argument of the vector $f(g(t)) - g(t)$ as t runs along S^1 from a to b. Observe that if $c : S^1 \to \mathbb{C}$ is a constant map, then ind$(f, c) = 0$ and if, $c(S^1) = \{w\} \in T(S^1) \setminus S^1$, then ind$(c, id|_{s^1}) = 1$.

Proposition 2.1. *Let $g : S^1 \to \mathbb{C}$ be a map with $g(S^1) = S$, and suppose $f : S \to \mathbb{C}$ has no fixed points on S. Let $a \neq b \in S^1$ with $[a, b]$ denoting the counterclockwise subarc on S^1 from a to b (so $S^1 = [a, b] \cup [b, a]$). Then* $\mathrm{ind}(f, g) = \mathrm{ind}(f, g|_{[a,b]}) + \mathrm{ind}(f, g|_{[b,a]})$.

2.2. **Stability of Index.** The following standard theorems and observations about the stability of index under fixed-point-free homotopy are consequences of the fact that index is continuous and integer-valued.

Theorem 2.2. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a map and $g_1 : S^1 \to \mathbb{C}$ and $g_2 : S^1 \to \mathbb{C}$ are homotopic maps in $\mathbb{C}$ such that the homotopy misses the fixed point set of f. Then* $\mathrm{ind}(f, g_1) = \mathrm{ind}(f, g_2)$.

An embedding $g : S^1 \to S \subset \mathbb{C}$ is *orientation preserving* if g is isotopic to the indentity map $id|_{S^1}$. In particular, the index of f on a simple closed curve S missing the fixed point set of f is independent of choice of parameterizations of S with the same orientation. If g_1 and g_2 are orientation-preserving embeddings of S^1 with the same image set $g_1(S^1) = S = g_2(S^1)$, then we have a well-defined index of f on S, namely $\operatorname{ind}(f,S) = \operatorname{ind}(f,g_1) = \operatorname{ind}(f,g_2)$.

Theorem 2.3. *Suppose $g : S^1 \to \mathbb{C}$ is a map with $g(S^1) = S$, and $f_1, f_2 : S \to \mathbb{C}$ are homotopic maps such that each level of the homotopy is fixed-point-free on S. Then* $\operatorname{ind}(f_1, g) = \operatorname{ind}(f_2, g)$.

In particular, if S is a simple closed curve and $f_1, f_2 : S \to \mathbb{C}$ are maps such that there is a homotopy $h_t : S \to \mathbb{C}$ from f_1 to f_2 with h_t fixed-point free on S for each $t \in [0,1]$, then $\operatorname{ind}(f_1, S) = \operatorname{ind}(f_2, S)$.

Corollary 2.4. *Suppose $g : S^1 \to \mathbb{C}$ is an an orientation preserving embedding with $g(S^1) = S$, and $f : T(S) \to \mathbb{C}$ is a map such that f has no fixed points on S and $f(S) \subset T(S)$. Then* $\operatorname{ind}(f, g) = 1$.

Proof. Since $f(S) \subset T(S)$ which is a disk with boundary S and f has no fixed point on S, there is a fixed point free homotopy of $f|_S$ to a constant map $c : S \to \mathbb{C}$ taking S to a point in $T(S) \setminus S$. By Theorem 2.3, $\operatorname{ind}(f, g) = \operatorname{ind}(c, g) = 1$. □

Theorem 2.5. *Suppose $g : S^1 \to \mathbb{C}$ is a map with $g(S^1) = S$, and $f : T(S) \to \mathbb{C}$ is a map such that* $\operatorname{ind}(f, g) \neq 0$, *then f has a fixed point in $T(S)$.*

Proof. Notice that $T(S)$ is a locally connected non-separating plane continuum and, hence, contractible. Suppose f has no fixed point in $T(S)$. Choose point $q \in T(S)$. Let $c : S^1 \to \mathbb{C}$ be the constant map to q. Let H be a homotopy from g to c with image in $T(S)$. Since H misses the fixed point set of f, Theorem 2.2 implies $\operatorname{ind}(f, g) = \operatorname{ind}(f, c) = 0$. □

2.3. **Variation.** In this section we introduce the notion of variation of a map on an arc and relate it to winding number.

Definition 2.6 (Junctions)**.** *The* standard junction J_0 *is the union of the three rays $R_i = \{z \in \mathbb{C} \mid z = re^{i\pi/2},\ r \in [0,\infty)\}$, $R_+ = \{z \in \mathbb{C} \mid z = re^0,\ r \in [0,\infty)\}$, $R_- = \{z \in \mathbb{C} \mid z = re^{i\pi},\ r \in [0,\infty)\}$, having the origin 0 in common. By U we denote the lower half-plane $\{z \in \mathbb{C} \mid z = x + iy,\ x \in \mathbb{R},\ y < 0\}$. A* junction J_v *is the image of J_0 under any orientation-preserving homeomorphism $h : \mathbb{C} \to \mathbb{C}$ where $v = h(0)$.*

We will often suppress h and refer to $h(R_i)$ as R_i, and similarly for the remaining rays and the region $h(U)$. When needed we will write R_{v^+} etc. when we want to refer to a particular $h(R_+)$ of a junction J_v based at $v = h(0)$.

Suppose S is a simple closed curve and $A \subset S$ is a subarc of S with endpoints a and b, with $a < b$ in the counter-clockwise orientation on S. We will usually denote such a subarc by $A = [a, b]$ and by (a, b) its interior in S^1.

Definition 2.7 (Variation on an arc)**.** *Let S be a simple closed curve and $A = [a, b]$ a subarc of S such that $f(a), f(b) \in T(S)$ and $f(A) \cap A = \emptyset$. We define the* variation of f on A with respect to S, *denoted* $\mathrm{var}(f, A, S)$, *by the following algorithm:*

(1) *Choose an orientation-preserving homeomorphism h of $\mathbb{C}$ such that $h(0) = v \in A$ and $T(S) \subset h(U) \cup \{v\}$.*
(2) *As always we assume that $a < b$ in the counterclockwise order.*
(3) Counting crossings: *Consider the set $M = f^{-1}(J_v) \cap [a, b]$. Each time a point of $f^{-1}(h(R^+)) \cap [a, b]$ is immediately followed in M, in the natural order on $[a, b]$, by a point of $f^{-1}(h(R^i))$ count $+1$ and each time a point of $f^{-1}(h(R^i)) \cap [a, b]$ is immediately followed in M, in the natural order on $[a, b]$, by a point of $f^{-1}(h(R^+))$ count -1. Count no other crossings.*
(4) *The sum of the crossings found above is the variation, denoted* $\mathrm{var}(f, A, S)$.

Note that $f^{-1}(h(R^+)) \cap [a, b]$ and $f^{-1}(h(R^i)) \cap [a, b]$ are disjoint closed sets in $[a, b]$. Hence, in (3) in the above definition, we count only a finite number of crossings and $\mathrm{var}(f, A, S)$ is a finite integer.

Let $g : S^1 \to \mathbb{C}$ be a map and $w \in \mathbb{C} \setminus g(S^1)$ be a point. By the *winding number of g about the point w*, denoted by $\mathrm{win}(g, S^1, w)$, we mean the number $\mathrm{ind}(c, g)$, where $c : \mathbb{C} \to \mathbb{C}$ is the constant map $c(z) = w$. It is well-known that the winding number is invariant under homotopies of g in $\mathbb{C} \setminus w$ and independent of the choice of the point w in a particular component of $\mathbb{C} \setminus g(S^1)$. Note that if S is a simple closed curve, $A \subset S$ is an arc and B is the closure of $S \setminus A$ and $\alpha : S \to \mathbb{C}$ is any map such that $\alpha|_A = f|_A$ and $\alpha(B) \subset T(S) \setminus \{v\} \subset U$, then $\mathrm{var}(f, A, S) = \mathrm{win}(\alpha, S, v)$.

In case A is an open arc $(a, b) \subset S$ such that $\mathrm{var}(f, \overline{A}, S)$ is defined, it will be convenient to denote $\mathrm{var}(f, \overline{A}, S)$ by $\mathrm{var}(f, A, S)$

The following Lemma follows immediately from the definition.

Lemma 2.8. *Let S be a simple closed curve. Suppose that $a < c < b$ are three points in S such that $\{f(a), f(b), f(c)\} \subset T(S)$ and $f([a, b]) \cap [a, b] = \emptyset$. Then* $\mathrm{var}(f, [a, b], S) = \mathrm{var}(f, [a, c], S) + \mathrm{var}(f, [c, b], S)$.

2.4. **Stability of Variation.** By the above remark that variation is a winding number, the invariance of winding number under suitable homotopies implies that the variation $\mathrm{var}(f, A, S)$ also remains invariant under such homotopies. That is, even though the specific crossings in (3) in the algorithm may change, the sum remains invariant. We will state the required results about variation below without proof. Proofs can also be obtained directly by using the fact that $\mathrm{var}(f, A, S)$ is integer valued and continuous under suitable homotopies.

Proposition 2.9 (Junction Straightening)**.** *Any two junctions h_1, h_2 with $v_1 = h_1(0) \in A$, $v_2 = h_2(0) \in A$, $T(S) \subset h_1(U) \cup \{v_1\}$, and $T(S) \subset h_2(U) \cup \{v_2\}$ give the same variation.*

Proposition 2.10. *Variation* $\mathrm{var}(f, A, S)$ *is an integer, well-defined, and independent of* h*.*

Since U is open for a given junction J_v for $A = [a, b]$ with $T(S) \subset U \cup \{v\}$, the computation of $\mathrm{var}(f, A, S)$ depends only upon the crossings of the junction coming from a proper compact subarc of the open arc (a, b). Consequently, $\mathrm{var}(f, A, S)$ remains invariant under homotopies h_t of $f|_{[a,b]}$ such that $h_t(a)$ and $h_t(b)$ remain in U and $v \notin h_t([a, b])$ for all t. Moreover, the computation is stable under an isoptopy of the plane that moves the entire junction J_v (even off A), provided in the the isotopy v never crosses the image $f(A)$ and, $f(a)$ and $f(b)$ remain in the corresponding domain U_t.

Definition 2.11 (Variation on a finite union of arcs)**.** *Let* S *be a simple closed curve and* $A = [a, b]$ *a subcontinuum of* S *with partition a finite set* $F = \{a = a_0 < a_1, \ldots, a_n = b\}$*. For each* i *let* $A_i = [a_i, a_{i+1}]$*. Suppose that* f *satisfies* $f(a_i) \in T(S)$ *and* $f(A_i) \cap A_i = \emptyset$ *for each* i*. We define the* variation of f on A with respect to S, *denoted* $\mathrm{var}(f, A, S)$*, by*

$$\mathrm{var}(f, A, S) = \sum_{i=0}^{n-1} \mathrm{var}(f, [a_i, a_{i+1}], S).$$

In particular, we include the possibility that $a_n = a_0$ *in which case* $A = S$*.*

By considering a common refinement of two partitions F_1 and F_2 of an arc $A \subset S$ such that $f(F_1) \cup f(F_2) \subset T(S)$ and satisfying the conditions in Definition 2.11, it follows from Lemma 2.8 that we get the same value for $\mathrm{var}(f, A, S)$ whether we use the partition F_1 or the partition F_2. Hence, $\mathrm{var}(f, A, S)$ is well-defined. If $A = S$ we denote $\mathrm{var}(f, S, S)$ simply by $\mathrm{var}(f, S)$.

2.5. **Index and variation for finite partitions.** What links Theorem 2.5 with variation is Theorem 2.13 below, first obtained by Bell and announced in the mid 1980's, and later by Akis [2]. Our proof is a modification of Bell's unpublished proof. We first need a variant of Proposition 2.9. Let $r : \mathbb{C} \to \mathbb{D}$ be radial retraction: $r(z) = \frac{z}{|z|}$ when $|z| \geq 1$ and $r|_{\mathbb{D}} = id|_{\mathbb{D}}$.

Lemma 2.12 (Curve Straightening)**.** *Suppose* $f : S^1 \to \mathbb{C}$ *is a map with no fixed points on* S^1*. If* $[a, b] \subset S^1$ *is a proper subarc with* $f([a, b]) \cap [a, b] = \emptyset$*,* $f((a, b)) \subset \mathbb{C} \setminus T(S^1)$ *and* $f(\{a, b\}) \subset S^1$*, then there exists a map* $h : S^1 \to \mathbb{C}$ *homotopic to* f *in* $\mathbb{C} \setminus T(S)$ *relative to* $\{a, b\}$*, with each level of the homotopy fixed-point-free, such that* $r \circ h : [a, b] \to S^1$ *is locally one-to-one. Moreover,* $\mathrm{var}(f, [a, b], S^1) = \mathrm{var}(h, [a, b], S^1)$*.*

Note that if $\mathrm{var}(f, [a, b], S^1) = 0$, then $r \circ h$ carries $[a, b]$ one-to-one onto the arc in $S^1 \setminus [a, b]$ from $f(a)$ to $f(b)$. If the $\mathrm{var}(f, [a, b], S^1) = m > 0$, then $r \circ h$ wraps the arc $[a, b]$ counterclockwise about S^1 so that $h([a, b])$ meets each ray in J_v m times. A similar statement holds for negative variation.

Theorem 2.13 (Index = Variation + 1, Bell)**.** *Suppose $g : S^1 \to \mathbb{C}$ is an orientation preserving embedding onto a simple closed curve S and f has no fixed points on S. If $F = \{a_0 < a_1 < \cdots < a_n\}$ is a partition of S and $A_i = [a_i, a_{i+1}]$ for $i = 1, \ldots, n$ with $a_{n+1} = a_0$ such that $f(F) \subset T(S)$ and $f(A_i) \cap A_i = \emptyset$ for each i, then*

$$\mathrm{ind}(f, g) = \sum_{i=0}^{n} \mathrm{var}(f, A_i, S) + 1 = \mathrm{var}(f, S) + 1.$$

Note that it is possible for index to be defined yet variation not to be defined on a simple closed curve S. For example, consider the map $z \to 2z$ with S the unit circle.

Proof. By an appropriate conjugation of f and g, we may assume without loss of generality that $S = S^1$ and $g = id$. Let F and $A_i = [a_i, a_{i+1}]$ be as in the hypothesis. Consider the collection of arcs

$$\mathcal{K} = \{K \subset S \mid K \text{ is the closure of a component of } S \cap f^{-1}(f(S) \setminus T(S))\}.$$

For each $K \in \mathcal{K}$, there is an i such that $K \subset A_i$. Since $f(A_i) \cap A_i = \emptyset$, it follows from the remark after Proposition 2.10 that $\mathrm{var}(f, A_i, S) = \sum_{K \subset A_i, K \in \mathcal{K}} \mathrm{var}(f, K, S)$. In particular, we can compute $\mathrm{var}(f, K, S)$ using one fixed junction for A_i and it is now clear that there are at most finitely many such K with $\mathrm{var}(f, K, S) \neq 0$. Moreover, the images of the endpoints of K lie on S.

Let m be the cardinality of the set $\mathcal{K}_f = \{K \in \mathcal{K} \mid \mathrm{var}(f, K, S) \neq 0\}$. By the above remarks, $m < \infty$ and $\mathcal{K}_f$ is independent of F. We prove the theorem by induction on m.

Suppose for a given f we have $m = 0$. Observe that from the definition of variation and the fact that the computation of variation is independent of the choice of an appropriate partition, it follows that,

$$\mathrm{var}(f, S) = \sum_{K \in \mathcal{K}} \mathrm{var}(f, K, S) = 0.$$

We claim that there is a map $f_1 : S \to \mathbb{C}$ with $f_1(S) \subset T(S)$ and a homotopy H from $f|_S$ to f_1 such that each level H_t of the homotopy is fixed-point-free and $\mathrm{ind}(f_1, id|_S) = 1$.

To see the claim, first apply the Curve Straightening Lemma 2.12 to each $K \in \mathcal{K}$ (if there are infinitely many, they form a null sequence) to obtain a fixed-point-free homotopy of $f|_S$ to a map $h : S \to \mathbb{C}$ such that $r \circ h$ is locally one-to-one on each $K \in \mathcal{K}$, where r is radial retraction of $\mathbb{C}$ to $\mathbb{D}$, and $\mathrm{var}(h, K, S) = 0$ for each $K \in \mathcal{K}$. Let K be in $\mathcal{K}$ with endpoints x, y. Since $h(K) \cap K = \emptyset$, $r \circ h$ is one-to-one, and $\mathrm{var}(h, K, S) = 0$. Since $\mathcal{K}$ is a null family, we can do this for each $K \in \mathcal{K}$ so that we obtain the desired $f_1 : S \to \mathbb{C}$ as the end map of a fixed-point-free homotopy from f to f_1. Since f_1 carries S into $T(S)$, Corollary 2.4 implies $\mathrm{ind}(f_1, id|_S) = 1$.

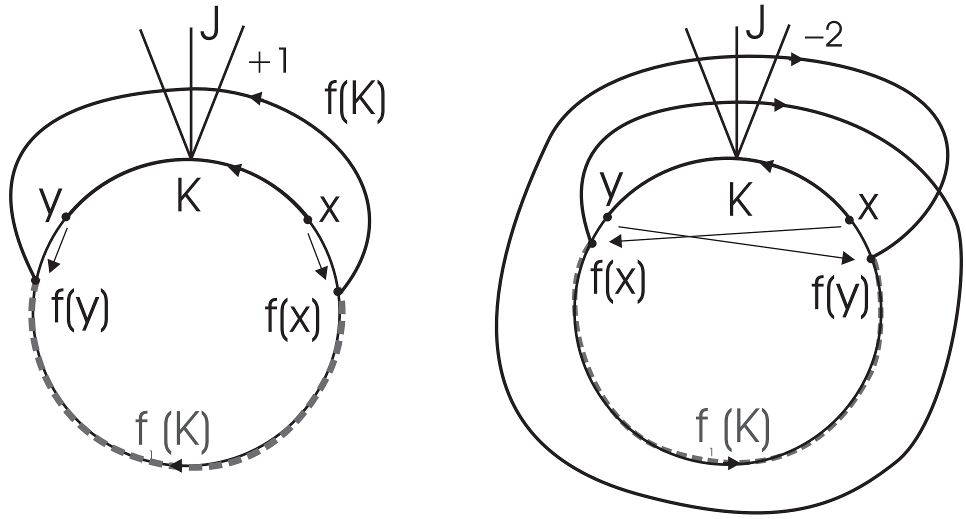

FIGURE 1. Replacing $f : S \to \mathbb{C}$ by $g : S \to \mathbb{C}$ with one less subarc of nonzero variation.

Since the homotopy $f \simeq f_1$ is fixed-point-free, it follows from Theorem 2.3 that $\operatorname{ind}(f, id|_S) = 1$. Hence, the theorem holds if $m = 0$ for any f and any appropriate partition F.

By way of contradiction, consider the collection of all maps f on S^1 which satisfy the hypotheses of the theorem, but not the conclusion. By the above $0 < |\mathcal{K}_f| < \infty$ for each. Let f and partition F be a counterexample for which $m = |\mathcal{K}_f|$ is minimal. By modifying f, we will show there exists another counterexample f' with $|\mathcal{K}_{f'}| < m$, a contradiction.

Choose $K \in \mathcal{K}$ such that $\operatorname{var}(f, K, S) \neq 0$. Then $K = [x, y] \subset A_i = [a_i, a_{i+1}]$ for some i. By the Curve Straightening Lemma 2.12 and Theorem 2.3, we may suppose $r \circ f$ is locally one-to-one on K. Define a new map $f_1 : S \to \mathbb{C}$ by setting $f_1|_{\overline{S \setminus K}} = f|_{\overline{S \setminus K}}$ and setting $f_1|_K$ equal to the linear map taking $[x, y]$ to the subarc $f(x)$ to $f(y)$ on S missing $[x, y]$. Figure 1 (left) shows an example of a (straightened) f and the corresponding f_1 for a case where $\operatorname{var}(f, K, S) = 1$, while Figure 1 (right) shows a case where $\operatorname{var}(f, K, S) = -2$.

Since on $\overline{S \setminus K}$, f and f_1 are the same map, we have

$$\operatorname{var}(f, S \setminus K, S) = \operatorname{var}(f_1, S \setminus K, S).$$

Likewise for the fractional index,

$$\operatorname{ind}(f, S \setminus K) = \operatorname{ind}(f_1, S \setminus K).$$

By definition (refer to the observation we made in the case $m = 0$),

$$\operatorname{var}(f, S) = \operatorname{var}(f, S \setminus K, S) + \operatorname{var}(f, K, S)$$

$$\operatorname{var}(f_1, S) = \operatorname{var}(f_1, S \setminus K, S) + \operatorname{var}(f_1, K, S)$$

and by Proposition 2.1,

$$\operatorname{ind}(f,S) = \operatorname{ind}(f,S\setminus K) + \operatorname{ind}(f,K)$$
$$\operatorname{ind}(f_1,S) = \operatorname{ind}(f_1,S\setminus K) + \operatorname{ind}(f_1,K).$$

Consequently,

$$\operatorname{var}(f,S) - \operatorname{var}(f_1,S) = \operatorname{var}(f,K,S) - \operatorname{var}(f_1,K,S)$$

and

$$\operatorname{ind}(f,S) - \operatorname{ind}(f_1,S) = \operatorname{ind}(f,K) - \operatorname{ind}(f_1,K).$$

We will now show that the changes in index and variation, going from f to f_1 are the same (i.e., we will show that $\operatorname{var}(f,K,S) - \operatorname{var}(f_1,K,S) = \operatorname{ind}(f,K) - \operatorname{ind}(f_1,K)$). We suppose first that $\operatorname{ind}(f,K) = n+\alpha$ for some nonnegative $n \in \mathbb{Z}$ and $0 < \alpha < 1$. That is, the vector $f(z) - z$ turns through n full revolutions counterclockwise and α part of a revolution counterclockwise as z varies from x to y in K. (See Figure 1 (left) for a case $n = 0$ and α about $\frac{2}{3}$.) Then as z varies from x to y, $f_1(z)$ goes along S from $f(x)$ to $f(y)$ in the clockwise direction, so $f_1(z) - z$ turns through the angle $-(1-\alpha) = \alpha - 1$. Hence, $\operatorname{ind}(f_1,K) = -(1-\alpha)$. It is easy to see that $\operatorname{var}(f,K,S) = n+1$ and $\operatorname{var}(f_1,K,S) = 0$. Consequently,

$$\operatorname{var}(f,K,S) - \operatorname{var}(f_1,K,S) = n+1-0 = n+1$$

and

$$\operatorname{ind}(f,K) - \operatorname{ind}(f_1,K) = n+\alpha-(\alpha-1) = n+1.$$

In Figure 1 on the left we assumed that $f(x) < x < y < f(y)$. The cases where $f(y) < x < y < f(x)$ and $f(x) = f(y)$ are treated similarly.

Thus when $n \geq 0$, in going from f to f_1, the change in variation and the change in index are the same. However, in obtaining f_1 we have removed one $K \in \mathcal{K}$, reducing the minimal m for f_1 by one, producing a counterexample for smaller m, a contradiction.

The cases where $\operatorname{ind}(f,K) = n+\alpha$ for negative n and $0 < \alpha < 1$ are handled similarly, and illustrated for $n = -2$ and α about $\frac{1}{2}$ in Figure 1 (right).

□

2.6. **Locating arcs of negative variation.** The principal tool in proving the main theorem in the next section is the following theorem first obtained by Bell. It provides a method for locating arcs of negative variation on a curve of index zero.

Theorem 2.14 (Lollipop Lemma, Bell). *Let $S \subset \mathbb{C}$ be a simple closed curve such that f has no fixed points on S. Suppose $F = \{a_0 < \cdots < a_n < a_{n+1} < \cdots < a_m\}$ is a partition of S, $a_{m+1} = a_0$ and $A_i = [a_i, a_{i+1}]$ such that $f(F) \subset T(S)$ and $f(A_i) \cap A_i = \emptyset$ for $i = 0, \dots, m$. Suppose I is an arc in $T(S)$ meeting S only at its endpoints a_0 and a_{n+1}. Let J_{a_0} be a junction in $(\mathbb{C} \setminus T(S)) \cup \{a_0\}$ and suppose that $f(I) \cap (I \cup J_{a_0}) = \emptyset$. Let*

$R = T([a_0, a_{n+1}] \cup I)$ and $L = T([a_{n+1}, a_{m+1}] \cup I)$. Then one of the following holds

(1) *If $f(a_{n+1}) \in R$, then*

$$\sum_{i \le n} \mathrm{var}(f, A_i, S) + 1 = \mathrm{ind}(f, I \cup [a_0, a_{n+1}]).$$

(2) *If $f(a_{n+1}) \in L$, then*

$$\sum_{i > n} \mathrm{var}(f, A_i, S) + 1 = \mathrm{ind}(f, I \cup [a_{n+1}, a_{m+1}].$$

(Note that in (1) in effect we compute $\mathrm{var}(f, \partial R)$ but technically, we have not defined $\mathrm{var}(f, A_i, \partial R)$ since the endpoints of A_i do not have to map inside R but they do map into $T(S)$. Similarly in Case (2).)

Proof. Without loss of generality, suppose $f(a_{n+1}) \in L$. Let $C = [a_{n+1}, a_{m+1}] \cup I$ (so $T(C) = L$). We want to construct a map $f' : C \to \mathbb{C}$, fixed-point-free homotopic to $f|_C$, that does not change variation on any arc A_i in C and has the properties listed below.

(1) $f'(a_i) \in L$ for all $n + 1 \le i \le m + 1$. Hence $\mathrm{var}(f', A_i, C)$ is defined for each $i > n$.
(2) $\mathrm{var}(f', A_i, C) = \mathrm{var}(f, A_i, S)$ for all $n + 1 \le i \le m$.
(3) $\mathrm{var}(f', I, C) = \mathrm{var}(f, I, S) = 0$.
(4) $\mathrm{ind}(f', C) = \mathrm{ind}(f, C)$.

Having such a map, it then follows from Theorem 2.13, that

$$\mathrm{ind}(f', C) = \sum_{i=n+1}^{m} \mathrm{var}(f', A_i, C) + \mathrm{var}(f', I, C) + 1.$$

By Theorem 2.5 $\mathrm{ind}(f', C) = \mathrm{ind}(f, C)$. By (2) and (3), $\sum_{i>n} \mathrm{var}(f', A_i, C) + \mathrm{var}(f', I, C) = \sum_{i>n} \mathrm{var}(f, A_i, S)$ and the Theorem would follow.

It remains to define the map $f' : C \to \mathbb{C}$ with the above properties. For each i such that $n + 1 \le i \le m + 1$, chose an arc I_i joining $f(a_i)$ to L as follows:

(a) If $f(a_i) \in L$, let I_i be the degenerate arc $\{a_i\}$.
(b) If $f(a_i) \in R$ and $n+1 < i < m+1$, let I_i be an arc in $R \setminus \{a_0, a_{n+1}\}$ joining $f(a_i)$ to I.
(c) If $f(a_0) \in R$, let I_0 be an arc joining $f(a_0)$ to L such that $I_0 \cap (L \cup J_{a_0}) \subset A_{n+1} \setminus \{a_{n+1}\}$.

Let $x_{n+1} = y_{n+1} = a_{n+1}$, $y_0 = y_{m+1} \in I \setminus \{a_0, a_{n+1}\}$ and $x_0 = x_{m+1} \in A_m \setminus \{a_m, a_{m+1}\}$. For $n + 1 < i < m + 1$, let $x_i \in A_{i-1}$ and $y_i \in A_i$ such that $y_{i-1} < x_i < a_i < y_i < x_{i+1}$. For $n + 1 < i < m + 1$ let $f'(a_i)$ be the endpoint of I_i in L, $f'(x_i) = f'(y_i) = f(a_i)$ and extend f' continuously from $[x_i, a_i] \cup [a_i, y_i]$ onto I_i and define f' from $[y_i, x_{i+1}] \subset A_i$ onto $f(A_i)$ by $f'|_{[y_i, x_{i+1}]} = f \circ h_i$, where $h_i : [y_i, x_{i+1}] \to A_i$ is a homeomorphism such that $h_i(y_i) = a_i$ and $h_i(x_{i+1}) = a_{i+1}$. Similarly, define f' on $[y_0, a_{n+1}] \subset I$ to

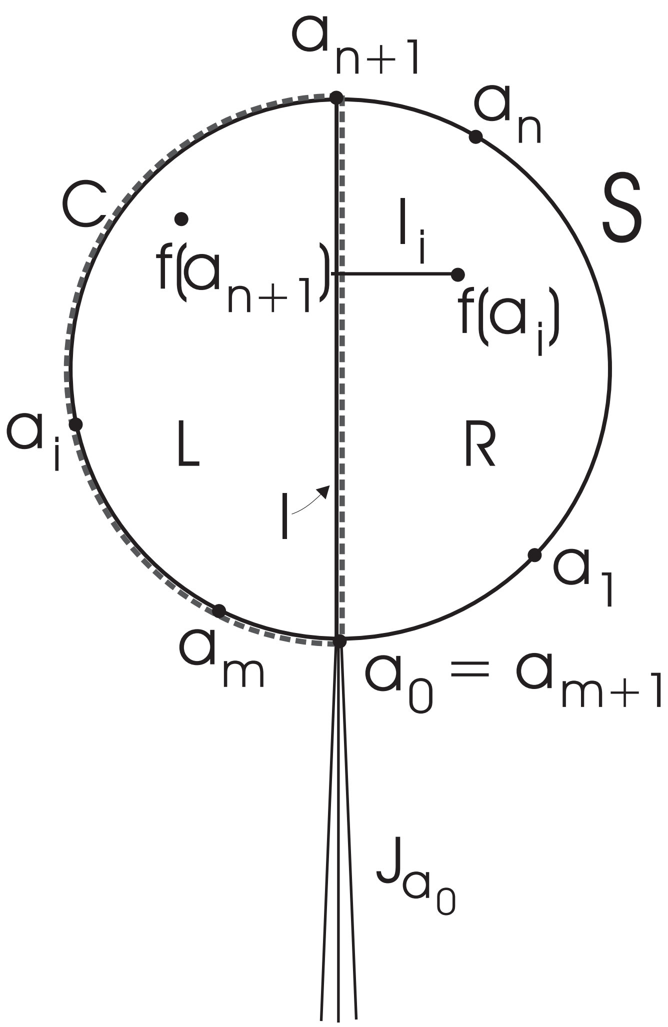

Figure 2. Bell's Lollipop.

$f(I)$ by $f|_{[y_0,a_{n+1}]} = f \circ h_0$, where $h_0 : [y_0, a_{n+1}] \to I$ is a homeomorphism such that $h(a_{n+1}) = a_{n+1}$ and extend extend f' from $[x_{m+1}, a_0] \subset A_m$ and $[a_o, y_0] \subset I$ onto I_0 such that $f'(x_{m+1}) = f'(y_0) = f(a_0)$ and $f'(a_0)$ is the endpoint of I_0 in L.

Note that $f'(A_i) \cap A_i = \emptyset$ for $i = n+1, \dots, m$ and $f'(I) \cap [I \cup J_{a_0}] = \emptyset$. To compute the variation of f' on each A_m and I we can use the junction J_{a_0} Hence $\mathrm{var}(f', I, C) = 0$ and, by the definition of f' on A_m, $\mathrm{var}(f', A_m, C) = \mathrm{var}(f(A_m, S)$. For $i = n+1, \dots, m-1$ we can use the same junction J_{v_i} to compute $\mathrm{var}(f', A_i, C)$ as we did to compute $\mathrm{var}(f, A_i, S)$. Since $I_i \cup I_{i+1} \subset T(S)$ we have that $f'([a_i, y_i]) \cup f'([x_{i+1}, a_{i+1}]) \subset I_i \cup I_{i+1}$ misses that junction and, hence, make no contribution to variation $\mathrm{var}(f', A_i, C)$.

Since $f'^{-1}(J_{v_i}) \cap [y_i, x_{i+1}]$ is isomorphic to $f^{-1}(J_{v_i}) \cap A_i$, $\mathrm{var}(f', A_i, C) = \mathrm{var}(f(A_i, S)$ for $i = n+1, \ldots, m$.

To see that f' is fixed-point-free homotopic to $f|_C$, note that we can pull the image of A_i back along the arcs I_i and I_{i+1} in R without fixing a point of A_i at any level of the homotopy. Since f' and $f|_C$ are fixed-point-free homotopic and f has no fixed points in $T(S)$, it follows from Theorems 2.3 and 2.5, that $\mathrm{ind}(f', C) = \mathrm{ind}(f, C)$. □

Note that if f is fixed point free on $T(S)$, then $\mathrm{ind}(f, S) = 0$ and the following Corollary follows.

Corollary 2.15. *Assume the hypotheses of Theorem 2.14. Suppose, in addition, f is fixed point free on $T(S)$. Then if $f(a_{n+1}) \in R$ there exists $i \leq n$ such that* $\mathrm{var}(f, A_i, S) < 0$. *If $f(a_{n+1}) \in L$ there exists $i > n$ such that* $\mathrm{var}(f, A_i, S) < 0$.

2.7. **Extensions to variation for infinite partitions.** Recall our Standing Hypotheses in 1.1: $f : \mathbb{C} \to \mathbb{C}$ takes continuum X into $T(X)$ with no fixed points in $T(X)$, and X is minimal with respect to these properties.

Definition 2.16 (Bumping Simple Closed Curve)**.** *A simple closed curve S in $\mathbb{C}$ which has the property that $S \cap X$ is nondegenerate and $T(X) \subset T(S)$ is said to be a* bumping simple closed curve for X. *A subarc A of a bumping simple closed curve, whose endpoints lie in X, is said to be a* bumping (sub)arc for X. *Moreover, if S' is any bumping simple closed curve for X which contains A, then S' is said to* complete A.

A *crosscut* of $O_\infty = \mathbb{C}_\infty \setminus T(X)$ is an open arc Q lying in O_∞ such that $\overline{Q}$ meets ∂O_∞ in two endpoints $a \neq b \in T(X)$. (As seems to be traditional, we use "crosscut of $T(X)$" interchangeably with "crosscut of O_∞.") If $S \cap X$ is nondegenerate and proper in S, then each component of $S \setminus X$ is a crosscut of $T(X)$. A similar statement holds for a bumping arc A.

Since f has no fixed points in $T(X)$ and X is compact, we can choose a bumping simple closed curve S so close pointwise to $T(X)$, with such small crosscuts, and with the domains cut off so close pointwise to $T(X)$, that f has no fixed points in $T(S)$. Thus, we obtain the following corollary to Theorem 2.5.

Corollary 2.17. *There is a bumping simple closed curve S for X such that $f|_{T(S)}$ is fixed point free; hence, by 2.5,* $\mathrm{ind}(f, S) = 0$. *Moreover, any bumping simple closed curve S' such that $S' \subset T(S)$ has* $\mathrm{ind}(f, S') = 0$. *Furthermore, any crosscut Q of X for which f has no fixed points in $T(X \cup Q)$ can be completed to a bumping simple closed curve S for which* $\mathrm{ind}(f, S) = 0$.

Theorem 2.18. *Suppose S is a bumping simple closed curve for X. Then there is a $\delta > 0$ such that if $A \subset S$ is a bumping subarc for X with* $\mathrm{diam}(A) \leq \delta$, *then* $\mathrm{var}(f, A, S) = 0$.

Proof. Suppose not. Then, without loss of generality, there is a sequence $\{A_i\}_{i=1}^{\infty}$ of bumping subarcs converging to a point $a \in X \cap S$ such that $\mathrm{var}(f, A_i, S) \neq 0$ for each i. Let J_a be a junction based at a. Since $f(a) \in X$, there are connected neighborhoods U of a and V of $f(a)$ such that $\overline{V} \cap J_a = \emptyset$ and $f(U) \subset V$. We may assume $U \cap S$ is connected. Since $A_i \to a$, there is a k such that for all $i \geq k$, $\overline{A_i} \subset U$. We may adjust the junction J_a to a junction J_{a_i}, keeping sufficiently close to S, so that for $i \geq k$, $a_i \in A_i$ and $f(\overline{A_i}) \cap J_{a_i} = \emptyset$. It follows that $\mathrm{var}(f, A_i, S) = 0$. This contradiction completes the proof. □

Corollary 2.19. *Suppose S is a bumping simple closed curve for X. Let $C \subset X$ be closed such that $S \setminus C = \bigcup_{i=1}^{\infty} A_i$, where the A_i are disjoint bumping subarcs (or crosscuts) such that $f(\overline{A_i}) \cap \overline{A_i} = \emptyset$ for each i. Then for all but finitely many A_i, $\mathrm{var}(f, A_i, S) = 0$.*

The following Theorem follows from 2.19 and the remark following Definition 2.11.

Theorem 2.20. *Suppose S is a bumping simple closed curve with A a bumping subarc in S such that $f(A) \cap A = \emptyset$. Suppose $A = \bigcup_{i \in I} A_i$ is a partition of A into possibly infinitely many bumping subarcs. Then* $\mathrm{var}(f, A, S) = \sum_{i \in I} \mathrm{var}(f, A_i, S)$.

Remark 2.21. *It follows from Corollary 2.19 and Theorem 2.20 that Theorems 2.13 and 2.14 hold for infinite partitions of bumping simple closed curves where the partition elements map off themselves.*

2.7.1. *Variation on a crosscut.* We show that variation is local by defining it for a single bumping subarc (or single crosscut).

Proposition 2.22. *Suppose A is a bumping subarc on X. If* $\mathrm{var}(f, A, S)$ *is defined for some bumping simple closed curve S completing A, then for any bumping simple closed curve S' completing A,* $\mathrm{var}(f, A, S) = \mathrm{var}(f, A, S')$.

Proof. Let A be a bumping subarc on X for which $f(A) \cap A = \emptyset$. Let S and S' be two bumping simple closed curves completing A for which variation is defined. Let J_a and $J_{a'}$ be junctions whereby $\mathrm{var}(f, A.S)$ and $\mathrm{var}(f, A, S')$ are respectively computed. Suppose first that both junctions lie (except for $\{a, a'\}$) in $\mathbb{C} \setminus (T(S) \cup T(S'))$. By the Junction Straightening Proposition 2.9, either junction can be used to compute either variation on A, so the result follows. Otherwise, at least one junction is not in $\mathbb{C} \setminus (T(S) \cup T(S'))$. But both junctions are in $\mathbb{C} \setminus T(X \cup A)$. Hence, we can find another simple closed curve S'' such that S'' completes Q and both junctions lie in $(\mathbb{C} \setminus T(S'')) \cup \{a, a'\}$. Then by the Propositions 2.9, 2.10 and the definition of variation, $\mathrm{var}(f, A, S) = \mathrm{var}(f, A, S'') = \mathrm{var}(f, A, S')$. □

It follows from Proposition 2.22 that variation on a crosscut of X is independent of the simple closed curve surrounding $T(X)$ of which Q is a subarc.

Definition 2.23 (Variation on a crosscut)**.** *Suppose Q is a crosscut of X such that $f(Q)\cap Q=\emptyset$. Let S be any bumping simple closed curve completing Q for which variation is defined. Define the* variation of f on Q with respect to X, *denoted* $\mathrm{var}(f,Q,X)$, *by* $\mathrm{var}(f,Q,X)=\mathrm{var}(f,Q,S)$.

We will need the following proposition in Section 4.

Proposition 2.24. *Suppose $Q=[a,b]$ is a crosscut of $T(X)$ such that f is fixed point free on $T(X\cup Q)$ and $f(Q)\cap Q=\emptyset$. Suppose Q is replaced by a bumping subarc A with the same endpoints such that Q separates $A\setminus\{a,b\}$ from ∞ and each component Q_i of $A\setminus X$ is a crosscut such that $f(\overline{Q_i})\cap\overline{Q_i}=\emptyset$. Then*
$$\mathrm{var}(f,Q,X)=\sum_i \mathrm{var}(f,Q_i,X).$$

Proof. Note that each of Q and A can be completed to a simple closed curve with the same bumping arc B such that on both $T(Q\cup B)$ and $T(A\cup B)$, f is fixed point free. By Corollary 2.17 and Remark 2.21 we have
$$\mathrm{var}(f,Q\cup B)+1=\mathrm{ind}(f,Q\cup B)=\mathrm{ind}(f,A\cup B)=\mathrm{var}(f,A\cup B)+1.$$
Thus,
$$\begin{aligned}\mathrm{var}(f,Q,Q\cup B)+\mathrm{var}(f,B,Q\cup B)&=\mathrm{var}(f,Q\cup B)=\mathrm{var}(f,A\cup B)\\&=\mathrm{var}(f,A,A\cup B)+\mathrm{var}(f,B,A\cup B).\end{aligned}$$
Consequently, by Theorem 2.20 and Proposition 2.22,
$$\begin{aligned}\mathrm{var}(f,Q,X)&=\mathrm{var}(f,Q,Q\cup B)=\mathrm{var}(f,A,A\cup B)\\&=\sum_i \mathrm{var}(f,Q_i,A\cup B)=\sum_i \mathrm{var}(f,Q_i,X).\end{aligned}$$
□

2.8. **Prime Ends.** Prime ends provide a way of studying the approaches to the boundary of a simply-connected plane domain with non-degenerate boundary. See [8] or [16] for an analytic summary of the topic and [21] for a more topological approach. We will be interested in the prime ends of $O_\infty=\mathbb{C}_\infty\setminus T(X)$. Recall $Y=T(X)$. Let $\Delta_\infty=\{z\in\mathbb{C}_\infty \mid |z|>1\}$ be the "unit disk about ∞." The Riemann Mapping Theorem guarantees the existence of a conformal map $\phi:\Delta_\infty\to O_\infty$ taking $\infty\to\infty$, unique up to the argument of the derivative at ∞. Fix such a map ϕ. We identify $S^1=\partial\Delta_\infty$ with $\mathbb{R}/\mathbb{Z}$ and identify points $e^{2\pi it}$ in $\partial\Delta_\infty$ by their argument (mod 2π). Crosscuts are defined in Section 2.7.

Definition 2.25 (Prime End)**.** *A* chain of crosscuts *is a sequence $\{Q_i\}_{i=1}^\infty$ of crosscuts of O_∞ such that for $i\neq j$, $Q_i\cap Q_j=\emptyset$,* $\mathrm{diam}(Q_i)\to 0$, *and for all $j>i$, Q_i separates Q_j from ∞ in O_∞. Two chains of crosscuts are said to be* equivalent *iff it is possible to form a sequence of crosscuts by selecting alternately a crosscut from each chain so that the resulting sequence of crosscuts is again a chain. A* prime end *$\mathcal{E}$ is an equivalence class of chains of crosscuts.*

If $\{Q_i\}$ is a chain of crosscuts of O_∞, it can be shown that $\{\phi^{-1}(Q_i)\}$ is a chain of crosscuts of Δ_∞ converging to a single point $t \in S^1 = \partial\Delta_\infty$, independent of the representative chain. Thus, we may name the prime end $\mathcal{E}$ defined by $\{Q_i\}$, where $\phi^{-1}(Q_i) \to t \in S^1$, by $\mathcal{E}_t$.

Let $\mathcal{E}_t$ be a prime end with defining chain of crosscuts $\{Q_i\}$. Let O_i denote the bounded complementary domain of $O_\infty \setminus Q_i$. We use $\{Q_i\}$ and $\{O_i\}$ to define two subcontinua of ∂O_∞ associated with $\mathcal{E}_t$.

Definition 2.26 (Impression and Principal Continuum)**.** *The set*

$$\mathrm{Im}(\mathcal{E}_t) = \bigcap_{i=1}^{\infty} \overline{O_i}$$

is a subcontinuum of ∂O_∞ called the impression *of $\mathcal{E}_t$. The set*

$$\mathrm{Pr}(\mathcal{E}_t) = \{z \in \partial O_\infty \mid \textit{for some chain } \{Q_i\} \textit{ defining } \mathcal{E}_t,\ Q_i \to z\}$$

is a continuum called the principal continuum *of $\mathcal{E}_t$.*

For a prime end $\mathcal{E}_t$, $\mathrm{Pr}(\mathcal{E}_t) \subset \mathrm{Im}(\mathcal{E}_t)$, possibly properly. We will be interested in the existence of prime ends $\mathcal{E}_t$ for which $\mathrm{Pr}(\mathcal{E}_t) = \mathrm{Im}(\mathcal{E}_t) = \partial O_\infty$.

Definition 2.27 (External Rays)**.** *Let $t \in [0, 1)$ and define*

$$R_t = \{z \in \mathbb{C} \mid z = \phi(re^{2\pi i t}), 1 < r < \infty\}.$$

We call R_t the external ray *at t. If $x \in R_t$ then the* (Y, x)-end *of R_t is the component K_x of $R_t \setminus \{x\}$ whose closure meets Y.*

The external rays R_t foliate O_∞ and it is easy to see that $\mathrm{Pr}(\mathcal{E}_t) = \overline{R_t} \setminus R_t$.

Definition 2.28 (Essential crossing)**.** *An external ray R_t is said to* cross *a crosscut Q* essentially *if and only if an (Y, x)-end of R_t is contained in the bounded complementary domain of $Y \cup Q$.*

The properties below may readily be established.

Proposition 2.29 ([8])**.** *Let $\mathcal{E}_t$ be a prime end of O_∞. Then* $\mathrm{Pr}(\mathcal{E}_t) = \overline{R_t} \setminus R_t$. *Moreover, for each $1 < r < \infty$ there is a crosscut Q_r at $\phi(re^{2\pi i t})$ on R_t with* $\mathrm{diam}(Q_r) \to 0$ *as $r \to 1$ and such that R_t crosses Q_r essentially.*

Definition 2.30 (Landing Points and Accessible Points)**.** *If* $\mathrm{Pr}(\mathcal{E}_t) = \{x\}$, *then we say R_t* lands *at $x \in T(X)$ and x is the* landing point *of R_t. A point $x \in \partial T(X)$ is said to be* accessible *(from O_∞) iff there is a arc in $O_\infty \cup \{x\}$ one endpoint of which is x.*

Proposition 2.31. *A point $x \in \partial T(X)$ is accessible iff x is the landing point of some external ray R_t.*

Definition 2.32 (Channels)**.** *A prime end $\mathcal{E}_t$ of O_∞ for which* $\mathrm{Pr}(\mathcal{E}_t)$ *is nondegenerate is said to be a* channel in ∂O_∞ *(or in $T(X)$). If moreover* $\mathrm{Pr}(\mathcal{E}_t) = \partial O_\infty = \partial T(X)$, *we say $\mathcal{E}_t$ is a* dense *channel. A crosscut Q with endpoints $\{a, b\}$ is said to* cross *the channel $\mathcal{E}_t$ iff R_t crosses Q essentially.*

When X is locally connected, there are no channels, as the following classical theorem proves. In this case, every prime end has degenerate principal set and degenerate impression.

Theorem 2.33 (Carathéodory)**.** *X is locally connected iff the Riemann map $\phi : \Delta_\infty \to O_\infty = \mathbb{C}_\infty \setminus T(X)$ taking $\infty \to \infty$ extends continuously to $S^1 = \partial\Delta_\infty$.*

2.9. **Index and Variation for Carathéodory Loops.** We extend the definitions of index and variation and the theorem relating index to variation to *Carathéodory loops.*

Definition 2.34 (Carathéodory Loop)**.** *Let $\phi : S^1 \to \mathbb{C}$ such that ϕ is continuous and has an extension $\psi : \mathbb{C} \setminus T(S^1) \to \mathbb{C} \setminus T(\phi(S^1))$ such that $\psi|_{\mathbb{C}\setminus T(S^1}$ is an orientation preserving homeomorphism from $\mathbb{C} \setminus T(S^1)$ onto $\mathbb{C} \setminus T(\phi(S^1))$. We call ϕ (and loosely, $S = \phi(S^1)$), a* Carathéodory loop.

In particular, if a Riemann map extends continuously to S^1, we have a Carathéodory loop. In order to define variation of f on a Carathéodory loop $S = \phi(S^1)$, we do the partitioning in S^1 and transport it to the Carathéodory loop $S = \phi(S^1)$. An *allowable* partition of S^1 is a set $\{a_0 < a_1 < \cdots < a_n\}$ in S^1 ordered counterclockwise, where $a_0 = a_n$ and A_i denotes the counterclockwise interval $[a_{i-1}, a_i]$, such that for each i, $f(\phi(a_i)) \in T(\phi(S^1))$ and $f(\phi(A_i)) \cap \phi(A_i) = \emptyset$. Variation on each path $\phi(A_i)$ is then defined exactly as in Definition 2.7, except that the junction (see Definition 2.6) is chosen so that the vertex $v \in \phi(A_i)$ and $T(\phi(S^1)) \subset h(U) \cup \{v\}$, and the crossings of the junction by $f(\phi(A_i))$ are counted (see Definition 2.7). Variation on the whole loop, or an allowable subarc thereof, is defined just as in Definition 2.11, by adding the variations on the partition elements. At this point in the development, variation is defined only relative to the given allowable partition F of S^1 and the parameterization ϕ of S: $\mathrm{var}(f, F, \phi(S^1))$.

Index on a Carathéodory loop S is defined exactly as in Section 2.1 with $S = \phi(S^1)$ providing the parameterization of S. Likewise, the definition of fractional index and Proposition 2.1 apply to Carathéodory loops.

Theorems 2.2, 2.3, Corollary 2.4, and Theorem 2.5 apply to Carathéodory loops. It follows that index on a Carathéodory loop S is independent of the choice of parameterization ϕ. It remains to extend Theorem 2.13 to Carathéodory loops. It then follows that variation on a Carathéodory loop S is independent of choice of parameterization $\phi(S^1) = S$ and allowable partition of S^1. Thus, $\mathrm{var}(f, S)$ is well-defined for any Carathéodory loop S that has some parameterization and some allowable partition.

Theorem 2.35. *Suppose $S = \phi(S^1)$ is a parameterized Carathéodory loop in $\mathbb{C}$ and f has no fixed points on S. Suppose variation of f on $S^1 = A_0 \cup \cdots \cup A_n$ with respect to ϕ is defined for some partition $A_0 \cup \cdots \cup A_n$ of S^1. Then*

$$\mathrm{ind}(f, \phi) = \sum_{i=0}^{n} \mathrm{var}(f, A_i, \phi(S^1)) + 1.$$

Proof. Let ψ be the homeomorphic extension of ϕ carrying $\mathbb{C} \setminus T(S^1)$ onto $\mathbb{C} \setminus T(S)$. Let $S_i = \{1 + \frac{1}{i})e^{2\pi i\theta} \mid \theta \in [0, 1)\}$ be the concentric circles of radius $1 + \frac{1}{i}$ converging to S^1. For the given partition $A_0 \cup \cdots \cup A_n$ of S^1, let $A_j = [a_{j-1}, a_j]$ with $a_n = a_0$, Then $a_j = e^{2\pi i\theta_j}$ for some $\theta_j \in [0, 1)$ with $\theta_0 < \theta_1 < \cdots < \theta_n = \theta_0$. For each j, let $R_{i,j} = \{re^{2\pi i\theta_j} \mid 1 \leq r \leq 1 + \frac{1}{i}\}$ be the radial arc from S_i to S^1 at a_j. Note $\operatorname{diam}(R_{i,j}) \to 0$ as $i \to \infty$. Let $C_{i,j} = \{(1 + \frac{1}{i})e^{2\pi i\theta} \mid \theta_{j-1} \leq \theta \leq \theta_j\}$ be the subarc of S_i between $c_{i,j-1} = S_i \cap R_{i,j-1}$ and $c_{i,j} = S_i \cap R_{i,j}$. Then $C_{i,j} = [c_{i,j-1}, c_{i,j}]$ approximates A_j as $i \to \infty$. Moreover, $c_{i,j} \to a_{i,j}$ as $i \to \infty$. Since ψ is a map, the same holds for the images.

For each j, choose a junction J_{v_j} with vertex $v_j \in \psi(A_j)$ so that $T(\psi(S^1)) \subset U_j \cup \{v_j\}$, where U_j is the usual complementary half-plane of the junction (see Definition 2.6).

Since $\psi(C_{i,j})$ approximates $\psi(A_j)$ with $\psi(R_{i,j})$ shrinking as $i \to \infty$, we may choose i sufficiently large so that the following conditions are satisfied:

(1) $\psi(C_{i,j}) \cap f(\psi(C_{i,j})) = \emptyset$.
(2) $\operatorname{var}(f, A_j, \psi(S^1)) = \operatorname{var}(f, \psi(C_{i,j}), \psi(S_i))$.
(3) There are no fixed points of f in $T(\psi(S_i)) \setminus T(\psi(S^1))$.
(4) $\operatorname{ind}(f, \psi(S_i)) = \operatorname{ind}(f, \psi(S^1))$.

Condition (1) holds because of continuity of ψ and the similar condition for A_j. To see condition (2), apply the observations about the stability of variation in Section 2.4. Condition (3) holds because there are no fixed points of f on S, and $\psi(S_i)$ approximates S. Condition (4) then follows from the stability of index under fixed-point-free homotopy, noted in Section 2.2. (Use ψ on S_k, $k \geq i$, to define the homotopy.)

By Theorem 2.13,

$$\operatorname{ind}(f, \psi(S_i)) = \sum_{j=0}^{n} \operatorname{var}(f, \psi(C_{i,j}), \psi(S_i)) + 1.$$

Hence, noting $\phi = \psi|_{S^1}$, it follows from conditions (2) and (4) that

$$\operatorname{ind}(f, \phi) = \sum_{j=0}^{n} \operatorname{var}(f, A_j, \phi(S^1)) + 1.$$

□

3. Geometric Prime Ends

We develop in this section special collections of geometric crosscuts of $T(X)$. We show that for any compact set K in the plane, the complement of K is partitioned into disjoint (and closed in K^c) nice geometric objects, contained in maximal round ball in K^c, whose boundaries consist of points in K and pieces of round circles in $\mathbb{C}_\infty \setminus K$ with endpoints in K, called *chords.* If K is a non-separating continuum, we can replace the chords by hyperbolic geodesics in the hyperbolic metric on $\mathbb{C}_\infty \setminus K$. We show that the family of all chords suffice for a satisfactory theory of prime ends of a non-separating plane continuum. These results, and their connection to the standard conformal theory of prime-ends, are more fully developed in [6]. In Bell's work these objects were (Euclidean) convex subsets of round balls and, hence the crosscuts were (Euclidean) straight line segments. Although all these choices of the crosscuts are essentially equivalent, it is sometimes easier to use circles or hyperbolic geodesics rather than straight line segments (see, for example [12, 17]).

3.1. **Kulkarni-Pinkall partition.** We are going to define a special class of *geometric crosscuts (chords)* of $O_\infty = \mathbb{C}_\infty \setminus Y$ and auxiliary nonseparating plane continua which contain Y as subsets, and in some sense have some of the same channels as Y, but have a nicer boundary than Y. To do this, we study closed round balls B such that the interior of B is in the complement of Y. More specifically, we study maximal balls B, that is balls whose boundary intersect Y in two points or more. Note that such balls are maximal since there exists no ball B', whose interior is contained in the complement of Y, which properly contains B. The geometric crosscuts are the hyperbolic chords in a maximal ball between points of $B \cap K$. Maximal balls were studied by Kulkarni and Pinkall and we closely follow their approach in [12].

If two balls B_1 and B_2 intersect and if $B_1 \cap B_2$ does not contain a diameter of either B_1 or B_2, then we say that $B_1 \cap B_2$ is the *lense of* B_1 and B_2.

Proposition 3.1. *The lense of two balls B_1 and B_2 is contained in a ball of radius strictly less than the radii of B_1 and B_2.*

Proof. Suppose that the circles ∂B_1 and ∂B_2 intersect in $\{s_1, s_2\}$. Then the ball around $(s_1 + s_2)/2$ of diameter $|s_1 - s_2|$ contains the lense. $\square$

Let K be any compact set. If B is a ball of minimal diameter that contains K, then we say that B is a *minimal ball*. Such a minimal ball is unique by the proposition on the lense. It exists, since any sequence of balls of decreasing diameter that contain K has a convergent subsequence.

We denote the *Euclidean convex hull of a planar set* K by $\mathrm{conv}_{\mathcal{E}}(K)$. It is the intersection of all closed half planes (a closed half plane is the closure of a component of the complement of a straight line) which contain K. Hence $p \in \mathrm{conv}_{\mathcal{E}}(K)$ if p cannot be separated from K by a straight line.

Every ball B is conformally equivalent with the unit disk and, hence, can be equipped with the hyperbolic metric. Geodesics Q in this metric are intersections of B with round circles C (or straight lines through the center of B) which perpendicularly cross the boundary ∂B. For every hyperbolic geodesic Q, $B \setminus Q$ has exactly two components U_Q and K_Q. Like in the Euclidean case, we call the closure of such components *half planes of B*. Given a compact set $K \subset \partial B$, the *hyperbolic convex hull of K in B* is the intersection of of all (closed) half planes which contain K and we denote it by $\mathrm{conv}_{\mathcal{H}}(B \cap K)$. Since we consider balls in the Riemann sphere, the maximal ball may be a half plane in $\mathbb{C}$ (corresponding to a circle in the

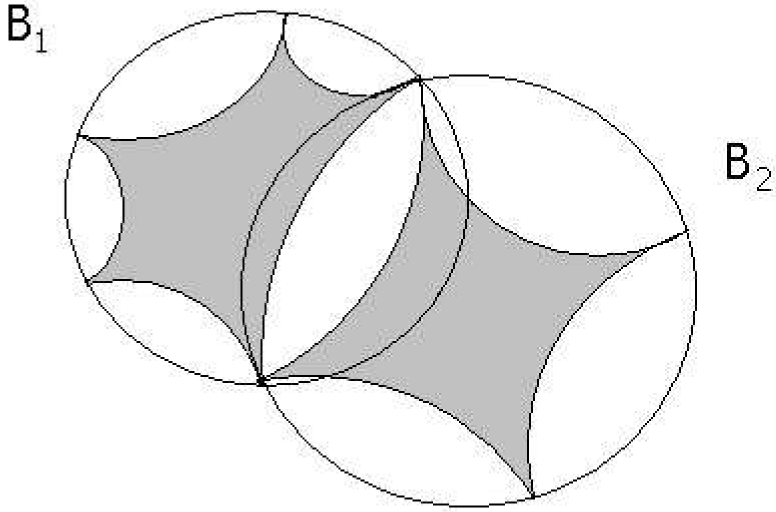

FIGURE 3. Maximal balls have disjoint hulls.

sphere containing ∞) and can be the exterior or interior of a round circle in the plane. Note if $B \subset \mathbb{C}$ and $K \subset \partial B$ has cardinality $|K| > 2$, then $\text{conv}_{\mathcal{H}}(B \cap K) \subset \text{conv}_{\mathcal{E}}(K)$.

Lemma 3.2. *Let $K \subset \mathbb{C}$ be compact. Suppose that B is the minimal ball containing K and let $c \in B$ be its center. Then $c \in \text{conv}_{\mathcal{H}}(K \cap \partial B)$.*

Proof. By contradiction. Suppose that there exists a circle that separates the center c from $K \cap \partial B$ and crosses ∂B perpendicularly. Then there exists a line ℓ through c such that a half plane bounded by ℓ contains $K \cap \partial B$ in its interior. Let $B' = B + v$ be a translation of B by a vector v that is orthogonal to ℓ and directed into the halfplane. If v is sufficiently small, then B' properly contains K hence it can be shrunk to a smaller ball, contradicting that B has minimal diameter. $\square$

We call a connected and open subset of the plane a *region.* We say that B is a *maximal ball in a region U* if its interior is contained in U and $|B \cap \partial U| \geq 2$. The exterior of the minimal ball around K is a "maximal" ball around ∞. Suppose from now on that that $\partial U = K$ is compact and contains at least two points.

Lemma 3.3. *Suppose that B_1 and B_2 are two distinct maximal balls in U. Then*

$$\text{conv}_{\mathcal{H}}(B_1 \cap \partial U) \cap \text{conv}_{\mathcal{H}}(B_2 \cap \partial U) \subset \partial U.$$

In particular, the intersection contains at most two points

Proof. A picture easily explains this, see figure 3. Note that ∂U intersects the boundary of $B_1 \cup B_2$ only. Therefore $B_1 \cap \partial U$ and $B_2 \cap \partial U$ share at most two points. The chords between these points in the respective balls are disjoint. □

It follows that any point in U can be contained in at most one hyperbolic convex hull. In the next lemma we see that each point is indeed contained in a convex hull, so the hulls of maximal balls partition the region U.

Since hyperbolic convex hulls are preserved by Möbius transformations, they are more easy to manipulate than Euclidean convex hulls, which are preserved only by Möbius transformations that fix ∞. This is illustrated by the proof of the following lemma.

Lemma 3.4 (Kulkarni-Pinkall inversion lemma)**.** *Suppose that $K \subset \mathbb{C}$ is compact and $U = \mathbb{C} \setminus K$ is connected. For any $p \in \mathbb{C}_\infty \setminus K$ there exists a maximal ball B in U such that p cannot be separated from $K \cap \partial B$ by transversal circles which cross the boundary of B perpendicularly.*

Proof. Möbius transformations preserve balls and they preserve inclusion so we may translate p to ∞ by the Möbius transformation $M(z) = \frac{1}{z-p}$. Let B' be the unique minimal round ball which contains $M(K)$ and let B^* be the complementary domain of B' in the sphere. By Lemma 3.2, there does not exist a circle C which separates the center c' of B' from $M(K) \cap B'$ and crosses $\partial B'$ perpendicularly. Hence ∞ cannot be separated from $M(K) \cap B'$, so $\infty \in \operatorname{conv}_{\mathcal{H}}(M(K) \cap \partial B^*)$ (any circle which separates K from infinity and croosses $\partial B'$ perpendicularly can also be used to separate the center of B' from K). Now let B be equal to $M^{-1}(B')$. It is a maximal ball and by the invariance of the hyperbolic convex hull under Möbius transformations, $p \in \operatorname{conv}_{\mathcal{H}}(K \cap \partial B)$. □

As a result we obtain following Theorem. It is due to Kulkarni and Pinkall [12] who established this result in much greater generality for compact n-manifolds.

Theorem 3.5. *Suppose that $K \subset \mathbb{C}$ is a compact set such that its complement U in the Riemann sphere is non-empty and connected. Then U is partitioned by the family*

$$\{U \cap \operatorname{conv}_{\mathcal{H}}(B \cap K) \colon B \text{ is a maximal ball in } U\}.$$

Proof. By the Kulkarni-Pinkall lemma there exists a maximal ball such that p cannot be separated from $K \cap \partial B$ by circles which cross ∂B perpendiculalrly. This means that p is contained in the convex hull of $K \cap \overline{B}$ in the hyperbolic metric in the ball B. By lemma 3.3, the ball B is unique. □

This Theorem is the linchpin of the theory of geometric crosscuts. It was known to Harold Bell and used by him implicitely since the early 1970's. Bell considered non-separating continua K and he used the equivalent notion of Euclidean convex hull of the sets $B \cap \partial U$ for all maximal balls (see Theorem 3.11 and the comment following it for a precise statement).

Example The following example may serve to illustrate Theorem 3.5. Let K be the unit square $\{x+yi\colon -1\le x,y\le 1\}$. There are five obvious maximal balls

$$\mathrm{Im}z\ge 1,\ \mathrm{Im}z\le -1,\ \mathrm{Re}z\ge 1,\ \mathrm{Re}z\le -1,\ |z|\ge\sqrt{2},$$

four of which are half planes. These are not all maximal balls as can be found from their hyperbolic hulls, but they are the only maximal balls which have a hull with nonempty interior. The hyperbolic hulls of the halfplanes are the semi-discs around ± 1 and $\pm i$. The hyperbolic hull of the circle $|z|\ge\sqrt{2}$ is bounded by four circle segments with radius $\sqrt{2}$ and centers ± 2 and $\pm 2i$. These hulls do not cover U as there are gaps between the hulls of the half planes and the hull of $|z|\ge\sqrt{2}$.

If C is a circle that circumscribes K and contains two of its vertices, such as $1\pm i$, then the exterior ball B bounded by C is maximal. Now $\mathrm{conv}_{\mathcal{H}}(B\cap K)$ is a single chord and the union of all such chords fills the remaining gaps.

3.2. **Hyperbolic stratification.** Theorem 3.5 applies to general compact subsets. It is particularly useful if it is applied to non-separating continua, for then the complement U in the Riemann sphere is simply connected and we can apply the Riemann mapping theorem and the theory of prime ends. Since we endow the hyperbolic metric, we define the Riemann map on the unit disc $\mathbb{D}$ rather than the exterior disk at infinity Δ_∞.

From now on $K\subset\mathbb{C}$ is a non-separating continuum. Its complement U in the sphere is the image of the unit disc $\mathbb{D}$ under a Riemann map $\phi\colon\mathbb{D}\to U$. We endow $\mathbb{D}$ by the hyperbolic metric, which U inherits by the Riemann map. In this section we show how the Kulkarni-Pinkall hulls induce a closed subset of $\mathbb{D}$ that is similar to a geodesic lamination in U. As a result we obtain Bell's original foliation of U by Euclidean convex hulls.

If $B\cap\partial U$ contains two points, then its hyperbolic hull is a single circle segment S with endpoints $s_1,s_2\in K$. We will call the crosscut $S\setminus\{s_1,s_2\}$ a *geometric crosscut* or simply a *chord*. If $B\cap\partial U$ contains three or more points, then we say that the hull $\mathrm{conv}_{\mathcal{H}}(B\cap\partial U)$ is a *gap*. A gap has nonempty interior. It's boundary in B is a union of open circle segments (with endpoints in K), which we also call geometric crosscuts or chords, and points in K. The end point of a chord is accessible in K, therefore the preimage of a chord is an arc in $\mathbb{D}$ with end points in the unit circle. Note that different chords may have common end points, so their preimages connect the same two points on the unit circle. We will replace such arcs by a single geodesic line in $\mathbb{D}$.

Lemma 3.6 (Jørgensen [18, p.31]). *Let B be a closed round ball such that its interior is in U. Let $\gamma\subset\mathbb{D}$ be a hyperbolic geodesic line. Then $\phi(\gamma)\cap B$ is connected. If $\phi(\gamma)$ intersects the interior of B then it divides B into two components.*

More briefly stated, this result says that the preimage of a ball $\phi^{-1}(B)$ is hyperbolically convex.

Let C be a chord and let γ be the geodesic line between the end points of $\phi^{-1}(C)$. Then we say that γ and $\phi(\gamma)$ are *hyperbolic chords.* If B is the maximal ball that belongs to C then by Jørgensen's lemma $\phi(\gamma) \subset B$.

Lemma 3.7. *Suppose that C_i is a sequence of chords and suppose that $x_i \in C_i$ is a sequence converging to $x \in U$. Then C_i converges to the chord that contains x.*

Proof. Let B_i be the sequence of associated maximal balls. The hyperspace of compact subsets with the Hausdorff metric is compact. It suffices to show that every convergent subsequence has the same limit. So we assume that C_i and B_i converge. It is not hard to see that the limit of C_i is circle segment C and the limit of B_i is a maximal ball B. The segment C is orthogonal to B. Since $x \in C$ the circle segment is not degenerate. The end points of C_i are in ∂U hence so are the end points of C. It follows that C is a chord and it must be the unique chord that contains x. □

Hence the Kulkarni-Pinkall partition of U has nice continuity properties, similar to a foliation.

Lemma 3.8. *For $e, f \in K$ define $C(e, f)$ as the union of all chords between e and f. Then $C(e, f) \cup \{e, f\}$, if not empty, is either the closure of a single chord, or a closed disk whose boundary consists of two chords contained in $C(e, f)$ and the set $\{e, f\}$.*

Proof. Suppose γ and δ are two chords between e and f. Then $\gamma \cup \delta$ is a simple closed curve and any element in its interior is in some $C(B)$. Since the hyperbolic hulls partition K^c, $C(B)$ can only intersect $\gamma \cup \delta$ in $\{e, f\}$. So $C(B) \cap K = \{e, f\}$ and it follows that the interior of $\gamma \cup \delta$ is contained in $C(e, f)$.

The rest of the Lemma follows from 3.7. □

From the viewpoint of prime ends, all chords in $C(e, f)$ are the same. That is why we pull back under the Riemann map ϕ and replace all these chords in $\mathbb{D}$ by a single hyperbolic chord. Note that chords do not intersect, except possibly in their end points, so the same is true for hyperbolic chords. Since we collapse chords in $C(e, f)$ to a single hyperbolic chord, we need to prove that hyperbolic chords preserve the continuity property. We will denote by Γ the union of all geodesics γ in $\mathbb{D}$ such that if γ joins the points z and w in the boundary of $\mathbb{D}$, then there exists a chord C such that $\phi^{-1}(C)$ is a crosscut in $\mathbb{D}$ also joining z and w.

Lemma 3.9. *Suppose that γ_i is a sequence of hyperbolic chords in Γ and suppose that $x_i \in \gamma_i$ is a sequence converging to $x \in \mathbb{D}$. Then γ_i converges to the hyperbolic chord in Γ that contains x.*

Proof. Consider a convergent subsequence of γ_{i_n}. Its limit necessarily is a geodesic line that contains x. Now choose a convergent subsequence of the associated chords C_{i_n}. Its limit C is a chord and its end points correspond to the end points of γ. Hence γ is a hyperbolic chord. It must be the hyperbolic chord that contains x, since chords do not cross. □

So we have stratified the Kulkarni-Pinkall partition to a family of geodesic lines in $\mathbb{D}$. By Jørgensen's lemma, a chord C and its hyperbolic chord $\phi(\gamma)$ are contained in the same maximal ball. Hence there is a deformation of U that maps one onto the other, which suggests that the complement of the hyperbolic chords $\phi(\gamma)$ corresponds to the interiors of the gaps of the Kulkarni-Pinkall partition. This is indeed the case.

Lemma 3.10. *There is a $1-1$ correspondence between complementary domains $D \subset \mathbb{D} \setminus \Gamma$ and Kulkarni-Pinkall gaps* $\mathrm{conv}_{\mathcal{H}}(\partial B \cap K)$.

Proof. Consider a gap $\mathrm{conv}_{\mathcal{H}}(\partial B \cap K)$ bounded by chords C_i. The corresponding hyperbolic chords γ_i bound a domain $D' \subset \mathbb{D}$ in the complement of Γ. The boundary $\bar{D}' \cap \partial\mathbb{D}$ corresponds to $\partial B \cap K$ since these points are accessible. Suppose that there exist a hyperbolic chord $\gamma \in \Gamma$ that connects points in $\bar{D}' \cap \partial\mathbb{D}$. Let C be the corresponding chord and let B' be its maximal ball. Since γ connects points in $\bar{D}' \cap \partial\mathbb{D}$, C connects points in $\partial B \cap K$. These points are in $\partial B \cap \partial B'$ so C connects the same end points as one of the chords in B. Hence γ is a hyperbolic chord on the boundary of D'. It follows that D' is in the complement of Γ.

Consider a complementary domain $D \subset \mathbb{D} \setminus \Gamma$, let γ be one of its bounding hyperbolic chords, and let $\mathbb{D} \setminus \gamma = L \cup R$ be the union of two disjoint and connected sets. Since D is located on one side of γ we can choose a point $b \in \partial\mathbb{D}$ such that D separates b from γ in $\mathbb{D}$. Let e, f be the end points of $\phi(\gamma)$. Then $C(e,f)$ is a lense (or a single chord) and $\phi^{-1}(C(e,f))$ is a compact subset that has the same end points as γ. The lense $C(e,f)$ is bounded by two chords P and C and one of $\phi^{-1}(\{P, C\})$ separates the other from b. Denote by C the chord such that $\phi^{-1}(C)$ separates $\phi^{-1}(P)$ from b. Let B be the maximal ball that contains C. We prove that $\mathrm{conv}_{\mathcal{H}}(\partial B \cap K)$ is a gap and that its chords correspond to the hyperbolic chords that bound D.

Suppose that C is not on the boundary of a gap $\mathrm{conv}_{\mathcal{H}}(\partial B \cap K)$. Then $\mathrm{conv}_{\mathcal{H}}(\partial B \cap K)$ is a chord and there exists a sequence $x_i \in B$ such that $x_i \notin C(e,f)$ that converges to an internal point of $x \in C$. We may choose each x_i on a chord C_i, so we know that C_i converges to C and the end points $e_i, f_i \in C_i$ converge to e, f. The hyperbolic chords γ_i with end points $\phi^{-1}(e_i)$ and $\phi^{-1}(f_i)$ converge to γ. Since the C_i are not contained in $C(e,f)$, the end points of γ_i are not equal to the endpoints of γ. In $\partial\mathbb{D}$ the end points of $\phi^{-1}(C_i)$ must converge to the end points of $\phi^{-1}(C)$ and by the choice of C, they must do so from the side of γ that contains D. But since D is complementary to Γ there exist no hyperbolic chord that is arbitrarily close to γ and has end points in this side of γ. Hence C is on the boundary of the gap $\mathrm{conv}_{\mathcal{H}}(\partial B \cap K)$. A similar argument shows that every hyperbolic chord in the boundary of D corresponds to a chord in the boundary of $\mathrm{conv}_{\mathcal{H}}(\partial B \cap K)$. So D must be the complementary domain that corresponds to $\mathrm{conv}_{\mathcal{H}}(\partial B \cap K)$. □

So if the complement of K is endowed with the hyperbolic metric, then there exists a family of geodesic lines that share the same end points as the Kulkarni-Pinkall partition. The complementary domains of the geodesic lines corresponds to the Kulkarni-Pinkall gaps. By Jørgensen's lemma, these complementary domains are contained in the same maximal ball as the gaps. We summarize the results:

Theorem 3.11. *Suppose that $K \subset \mathbb{C}$ is a non separating continuum and let O_∞ be its complementary domain in the Riemann sphere. There exists a family of geodesic lines $\mathcal{G}$ in the hyperbolic metric on O_∞ such that for each $\gamma \in \mathcal{G}$ there exist a maximal ball B such that $\gamma \subset B$ connects points in $B \cap \partial O_\infty$. Let $\Gamma = \cup\{\gamma \in \mathcal{G}\}$. Each domain in $O_\infty \setminus \Gamma$ is contained in a unique maximal ball B and its bounding geodesic lines in O_∞ correspond to the chords of* $\mathrm{conv}_{\mathcal{H}}(B \cap \partial O_\infty)$.

Instead of replacing the chords in $C(e, f)$ by a geodesic in the hyperbolic metric on O_∞, we may just as well replace them by a straight line segment; i.e, the geodesic in the euclidean metric. Then we obtain a family of straight lines. In doing so, we replace the gaps $\mathrm{conv}_{\mathcal{H}}(B \cap \partial O_\infty)$ by $\mathrm{conv}_{\mathcal{E}}(B \cap \partial O_\infty)$, which is the way in which Bell originally foliated $\mathrm{conv}_{\mathcal{E}}(K) \setminus K$. Note that in the latter two cases the elements of the foliation are not necessarily disjoint (hence we use the word "foliate" rather then "partition"). However, the intersection of any two elements is at most a common boundary leaf (i.e., either a hyperbolic chord or a straight line segment) and this is in most cases sufficient. So there are three closely related ways to foliate U: by Kulkarni-Pinkall chords, by hyperbolic chords or by Bell's straight line segments (the latter applies only to $\mathrm{conv}_{\mathcal{E}}(K) \setminus K$ but this is sufficient for the purpose of this paper).

3.3. **Geometric crosscuts and prime ends.** Recall our Standing Hypotheses in 1.1: $f : \mathbb{C} \to \mathbb{C}$ takes continuum X into $Y = T(X)$ with no fixed points in Y, and X is minimal with respect to these properties. We apply the Kulkarni-Pinkall partition to O_∞, the complementary domain of X that contains ∞. We show that the geometric crosscuts of Kulkarni and Pinkall are sufficient for a satisfactory prime-end theory and are convenient to compute variation.

The Kulkarni-Pinkall partition of the complementary domain O_∞ is denoted by $\mathcal{F}$. The maximal ball B_∞ that contains ∞ is the complement of the minimal ball that contains $T(X)$. We denote the points in its hull that are exterior to the minimal ball by $V(B_\infty) \in \mathcal{F}$ and more generally given the maximal ball B, then we denote the elements of $\mathrm{conv}_{\mathcal{H}}(B \cap \partial O_\infty)$ that are in the interior of B by $V(B)$. So $V(B)$ is a closed subset of O_∞. As before we use balls on the sphere. In particular, straight lines in the plane correspond to circles on the sphere containing the point at infinity. The subset of hulls of diameter $\le \delta$ in the spherical metric is denoted by $\mathcal{F}_\delta$. The chords of the partition (i.e. all the chords in the boundary of $\mathrm{conv}_{\mathcal{H}}(\partial B \cap K)$ for all maximal balls B) are denoted by $\mathcal{G}$. The subset of chords of diameter $\le \delta$ is denoted by $\mathcal{G}_\delta$.

By Lemma 3.7 we know that chords and hulls have nice continuity properties. However, $\mathcal{G}$ and $\mathcal{F}$ are not closed in the hyperspace of compact subsets of $\mathbb{C}_\infty$: a sequence of chords or hulls may converge to a point (and, hence, must be a null sequence).

Proposition 3.12 (Compactness). *If $\{Q_i\}$ is a convergent sequence of distinct element in $\mathcal{G}_\delta$ or in $\mathcal{F}_\delta$, then either Q_i converges to a chord in $\mathcal{G}_\delta$ or Q_i converges to a point of X. Moreover, if $Q_i \in \mathcal{G}_\delta$ and $\{Q_i\}$ converges to a chord C, then for sufficiently large i, $\mathrm{var}(f, Q, Y) = \mathrm{var}(f, Q_i, Y)$.*

Proof. By Lemma 3.7, we know that this is true if the limit Q contains a point in O_∞. Hence we only need to consider the case when $\lim Q_i = Q \subset \partial O_\infty = X$. If the diameter of Q_i converged to zero, then Q is a point as desired. Assume that this is not the case and let B_i be the maximal ball that contains Q_i. Under our assumption, the diameter of B_i does not decay to zero. Then $\lim B_i = B_\infty$ is a maximal ball in O_∞ and it follows easily that $\lim Q_i \cap B_\infty \neq \emptyset$, contradicting the fact that $Q \subset \partial O_\infty$. The last statement in the Lemma follows from stability of variation (see Section 2.4). □

Corollary 3.13. *For each $\varepsilon > 0$, there exist $\delta > 0$ such that for all $Q \in \mathcal{G}$ with $Q \subset B(Y, \delta)$, $\mathrm{diam}(Q) < \varepsilon$.*

Proof. Suppose not, then there exist $\varepsilon > 0$ and a sequence Q_i in $\mathcal{G}$ such that $\lim Q_i \subset X$ and $\mathrm{diam}(Q_i) \ge \varepsilon$ a contradiction to Proposition 3.12. □

The proof of the following well known proposition is included for completeness.

Proposition 3.14. *For each $\varepsilon > 0$ there exists $\delta > 0$ such that for each open arc A such that $\overline{A} \cap Y = \{a, b\}$, with $a \neq b$, and $\mathrm{diam}(A) < \delta$, $T(Y \cup A) \subset B(Y, \varepsilon)$.*

Proof. Suppose that A_i is a sequence of crosscuts such that $\mathrm{diam}(A_i) \to 0$ and $T(A_i \cup Y) \setminus B(Y, \varepsilon) \neq \emptyset$. Then there exist $z_i \in T(A_i \cup Y)$ such that $d(z_i, Y) \ge \varepsilon$. Without loss of generality we may assume $\varepsilon \le d(z_i, Y) \le 2\varepsilon$ and the the sequence z_i converges to $z_\infty \in \mathbb{C} \setminus Y$. Since Y is non-separating, there exists a ray $R \subset \mathbb{C} \setminus Y$ joining z_∞ to infinity. Since $d(R, Y) > 0$ there exist i such that if B is the straight line segment joining z_i to z_∞, then $A_i \cap [R \cup B] = \emptyset$ contradicting the fact that $z_i \in T(Y \cup A_i)$. □

A maximal ball B centered around $z \in \mathbb{C}$ consists of all $w \in \mathbb{C}$ such that $d(w, z) \le d(z, Y)$. Therefore, the diameter of $V(B)$ is bounded by $2d(z, Y)$.

Proposition 3.15. *Let ε, δ be as in Proposition 3.14 above with $\delta < \varepsilon/2$ and let A be a crosscut of Y such that* $\operatorname{diam}(A) < \delta$. *If $x \in T(A \cup Y) \cap V(B)$ and $d(x, A) \geq \varepsilon$, then* $\operatorname{diam}(V(B)) < 2\varepsilon$.

Proof. Let z be the center of B. If $d(z, Y) < \varepsilon$ then $\operatorname{diam}(V(B)) < 2\varepsilon$ and we are done. Hence we may assume that $d(z, Y) \geq \varepsilon$, which implies that $z \notin T(A \cup Y)$ by Proposition 3.14 and our choice of δ. We will show that this leads to a contradiction. The straight line segment ℓ from x to z must cross $Y \cup A$ at some point w. Since the segment is in the interior of the maximal ball B, it is disjoint from Y, so $w \in A$. Hence $d(x, w) \geq \varepsilon$ and, since $x \in B$, $B(w, \epsilon) \subset B$. This is a contradiction since $A \subset B(w, \delta)$ and $\delta < \epsilon/2$ so $\overline{A}$ would be contained in the interior of B which is impossible since A is a crosscut. □

Proposition 3.16. *Let C be a crosscut of Y and let A and B be disjoint closed sets in Y such that $\overline{C} \cap A \neq \emptyset \neq \overline{C} \cap B$. Let $F_x \in \mathcal{F}$ be the hull that contains $x \in C$. If each F_x intersects $A \cup B$, then there exists an $F_\infty \in \mathcal{F}$ that intersects A, B and $\overline{C}$.*

Proof. Let $a \in A, b \in B$ be the end points of $\overline{C}$. Let $C_a, C_b \subset C$ be the set of points such that F_x intersects A or B, respectively. Then C_a and C_b are closed subsets by Proposition 3.12. Note that $d(A, B) > 0$. If $C_a = \emptyset$, choose $x_i \in C$ converging to $a \in A \cap \overline{C}$. Then $F_{x_i} \cap B \neq \emptyset$ and $\lim F_{x_i} = F_\infty \subset V(B_\infty) \in \mathcal{F}$, where $B_\infty = \lim B_i$ is the limit of the maximal balls B_i whose hulls contain F_{x_i}. Then $F_\infty = V(B_\infty)$ meets both A and B and contains a and we are done. So we may assume that $C_a \neq \emptyset \neq C_b$ and $C_a \cap C_b$ is nonempty by the connectedness of C. Hence if $x \in C_a \cap C_b$, then $F_x \cap A \neq \emptyset \neq F_x \cap B$ and $F_x \cap C \neq \emptyset$ as desired. □

Proposition 3.16 allows us to replace small crosscuts which cross a prime end $\mathcal{E}_t$ with non-trivial principal continuum essentially by small nearby chords which also cross $\mathcal{E}_t$ essentially. For if C is a small crosscut in $\operatorname{Convexhull}(Y)$ with endpoints a and b which crosses the external ray R_t essentially, let A and B be the closures of the sets in Y accessible from a and b, respectively by small arcs missing R_t.

Fix a Riemann map $\phi : \Delta_\infty \to O_\infty = \mathbb{C}_\infty \setminus Y$ taking $\infty \to \infty$.

Proposition 3.17. *Suppose the external ray R_t lands on $x \in Y$, and $\{Q_i\}_{i=1}^\infty$ is a sequence of crosscuts converging to x with $\phi^{-1}(Q_i) \to t \in \partial\Delta_\infty$. Then for sufficiently large i,* $\operatorname{var}(f, Q_i, Y) = 0$.

Proof. Since f is fixed point free on Y and $f(x) \in Y$, we may choose a connected neighborhood W of x such that $f(\overline{W}) \cap (\overline{W} \cup R_t) = \emptyset$. For sufficiently large i, $Q_i \subset W$. For each such i, let J_i be a junction starting from a point in Q_i, staying in W until it reaches R_t, then following R_t to ∞. By our choice of W, $\operatorname{var}(f, Q_i, Y) = 0$. □

Proposition 3.18. *Suppose that for an external ray $R_t \cap \operatorname{Int}(\operatorname{conv}_{\mathcal{E}}(Y)) \neq \emptyset$. Then there exists $x \in R_t$ such that the (Y, x)-end of R_t is contained in*

$\text{conv}_{\mathcal{E}}(Y)$. *In particular there exists a chord $Q \in \mathcal{G}$ such that R_t crosses Q essentially.*

Proof. External rays correspond to geodesic half lines starting from the origin of Δ_∞. Half planes are conformally equivalent to disks. Therefore, Jørgensen's lemma applies: the intersection of R_t with a halfplane is connected, so it is a half line. Since the convexhull of Y is the intersection of all half planes containing Y, $R_t \cap \text{conv}_{\mathcal{E}}(Y)$ is connected. □

Proposition 3.19. *If R_t is an external ray of Y. Then one of the following must hold:*

(1) *R_t lands on a point of $B \cap \partial Y$ for some maximal ball B,*
(2) *There is a defining sequence Q_i of chords for R_t.*

Note that if in case (1) the maximal ball B is the exterior of the minimal ball that contains K, then R_t lands on a point of $Y \cap \partial\text{conv}_{\mathcal{E}}(Y)$. The proposition says that either a ray gets trapped in a maximal ball, or it keeps forever crossing chords.

Proof. If the external ray is trapped in a maximal ball B it must land on one of the accessible points in $B \cap \partial Y$. So suppose it never gets trapped. Without loss of generality we may assume that the external ray is $R_0 = \phi(1, \infty)$ and by our assumption there exists a decreasing sequence of reals $r_1 > r_2 > \dots$ converging to 1 such that $\phi(r_n)$ is not contained in any of the maximal balls that contain $\phi(r_i)$ for $i < n$. Let $F_n \in \mathcal{F}$ be the hull that contains $\phi(r_n)$. By choosing a subsequence if necessary we may assume that F_n converges. It cannot converge to a hull, since the sequence $\phi(r_n)$ never gets trapped. So by Proposition 3.12 it converges to a singleton in ∂Y. For each F_n there exists a minimal $s_n > 1$ such that $\phi(s_n) \in F_n$. Then $\phi(s_n)$ is contained in an open chord $C_n \subset F_n$. The sequence of chords C_n defines R_t. □

Lemma 3.20. *Let $\mathcal{E}_t$ be a channel in Y such that $\Pr(\mathcal{E}_t)$ is non-degenerate. Then for each $x \in \Pr(\mathcal{E}_t)$, for every $\delta > 0$, there is a chain $\{Q_i\}_{i=1}^\infty$ of chords defining $\mathcal{E}_t$ selected from $\mathcal{G}_\delta$ with $Q_i \to x \in \partial Y$.*

Proof. Let $x \in \Pr(\mathcal{E}_t)$ and let $\{C_i\}$ be a defining chain of crosscuts for $\Pr(\mathcal{E}_t)$ with $\{x\} = \lim C_i$. By Proposition 3.16, in particular by the remark following the proof of that proposition, there is a sequence $\{Q_i\}$ of chords such that $Q_i \cap C_i \neq \emptyset$ and $\Pr(\mathcal{E}_t)$ crosses each Q_i essentially. By Proposition 3.15, the sequence Q_i converges to $\{x\}$. □

Lemma 3.21. *Suppose an external ray R_t lands on $a \in Y$ with $\{a\} = \Pr(\mathcal{E}_t) \neq \text{Im}(\mathcal{E}_t)$. Suppose $\{x_i\}_{i=1}^\infty$ is a collection of points in O_∞ with $x_i \to x \in \text{Im}(\mathcal{E}_t) \setminus \{a\}$ and $\phi^{-1}(x_i) \to t$. Then for sufficiently large i, there is a sequence of chords $\{Q_i\}_{i=1}^\infty$ such that Q_i separates x_i from ∞, $Q_i \to a$ and $\phi^{-1}(Q_i) \to t$.*

Proof. The existence of the chords Q_i again follows from the remark following proposition 3.16. It is easy to see that $\lim \varphi^{-1}(Q_i) \to t$. □

3.4. **Auxiliary Continua.** We use chords to form Carathéodory loops around the continuum.

Definition 3.22. *Fix $\delta > 0$. Define the following collections of chords:*

$$\mathcal{G}_\delta^+ = \{Q \in \mathcal{G}_\delta \mid \mathrm{var}(f, Q, Y) \geq 0\}$$

$$\mathcal{G}_\delta^- = \{Q \in \mathcal{G}_\delta \mid \mathrm{var}(f, Q, Y) \leq 0\}$$

To each collection of chords above, there corresponds an auxiliary continuum defined as follows:

$$Y_\delta = T(Y \cup (\cup \mathcal{G}_\delta))$$
$$Y_\delta^+ = T(Y \cup (\cup \mathcal{G}_\delta^+))$$
$$Y_\delta^- = T(Y \cup (\cup \mathcal{G}_\delta^-))$$

Proposition 3.23. *Let $Z \in \{Y_\delta, Y_\delta^+, Y_\delta^-\}$, and correspondingly $\mathcal{W} \in \{\mathcal{G}_\delta, \mathcal{G}_\delta^+, \mathcal{G}_\delta^-\}$. Then the following hold:*

1. *Z is a nonseparating plane continuum.*
2. *$\partial Z \subset Y \cup (\cup \mathcal{W})$.*
3. *Every accessible point p in ∂Z is either a point of Y or a point interior to a chord $A \in \mathcal{W}$.*
4. *If p is an accessible point of ∂Z and in the interior of the chord $A \in \mathcal{W}$, then every point of A is accessible in ∂Z.*

Proof. By Proposition 3.12, $Y \cup (\cup \mathcal{W})$ is compact. Moreover, Y is connected and each crosscut $A \in \mathcal{W}$ has endpoints in Y. Hence, the topological hull $T(Y \cup (\cup \mathcal{W}))$ is a nonseparating plane continuum, establishing (1).

Since Z is the topological hull of $Y \cup (\cup \mathcal{W})$, no boundary points can be in complementary domains of $Y \cup (\cup \mathcal{W})$. Hence, $\partial Z \subset Y \cup (\cup \mathcal{W})$, establishing (2). Conclusion (3) follows immediately. Conclusion (4) follows from the disjointness of the chords. ☐

To simplify notation we write $e^{it} \in \partial \Delta_\infty$ simply as t. Given a non-separating continuum Y and a crosscut A of Y (i.e. an open arc in $\mathbb{C} \setminus Y$ whose closure is a closed arc with distinct endpoints in Y) we denote by $\mathrm{Sh}(A)$, the *shadow of* A, the bounded component of $\mathbb{C} \setminus [Y \cup A]$.

Proposition 3.24. *Y_δ is locally connected; hence, ∂Y_δ is a Carathéodory loop.*

Proof. Let $\mathcal{G}$ be all the chords in the Kulkarni-Pinkall partition of $U = \mathbb{C} \setminus Y$. Suppose that Y_δ is not locally connected. Then there exists a non-trivial impression and there exist $0 < \varepsilon < \delta/2$ and a chain A_i of crosscuts of Y_δ such that $\mathrm{diam}(\mathrm{Sh}(A_i)) > 5\varepsilon$ for all i. We may assume that $\lim A_i = y \in Y_\delta$. By Proposition 3.12 Y_δ is locally connected in the interior of each chord G of $\mathcal{G}$ which is contained in the boundary of Y_δ. Hence $y \in Y$. Choose $z_i \in \mathrm{Sh}(A_i)$ such that $d(z_i, y) > 2\varepsilon$. We can enlarge the crosscut A_i of Y_δ to a crosscut B_i of Y as follows. Suppose that A_i joins the points a_i^+ and a_i^- in Y_δ. If $a_i^+ \in Y$, put $y_i^+ = a_i^+$. Otherwise a_i^+ is contained in a chord $G_i^+ \in \mathcal{G}$, with endpoints in Y, which is contained in the boundary of Y_δ. Since $\lim A_i = y$ we can select one of these endpoints and call it y_i^+ such that $d(y_i^+, a_i^+) \to 0$. Define y_i^-, which is an endpoint of the chord G_i^-, similarly. Then $G_i^+ \cup A_i \cup G_i^-$ contains a chord B_i of Y joining the points y_i^+ and y_i^- such that $\lim B_i = y$. We claim that $z_i \in \mathrm{Sh}(B_i)$. To see this note that, since $z_i \in Sh(A_i)$, there exists a halfray $R_i \subset \mathbb{C} \setminus Y_\delta$ joining z_i to infinity such that $|R_i \cap A_i|$ is an odd number and each intersection is transverse. Since $R_i \cap B_i = R_i \cap A_i$ it follows that $z_i \in \mathrm{Sh}(B_i)$. Let $V(B_i)$ be the unique hull of the Kulkarni-Pinkall partition $\mathcal{F}$ which contains z_i. Since $\mathrm{diam}(B_i) \to 0$ and $d(z_i, y) > 2\varepsilon$, it follows from Proposition 3.15 that $\mathrm{diam}(V(B_i)) < 2\varepsilon < \delta$. This contradicts the fact that $z_i \in \mathbb{C} \setminus Y_\delta$ and completes the proof.

□

4. Outchannels

In this section we will show that X has at least one *negative outchannel*, which is defined as follows.

Definition 4.1 (Outchannel)**.** *An* outchannel *of the nonseparating plane continuum Y is a prime end $\mathcal{E}_t$ of $O_\infty = \mathbb{C}_\infty \setminus Y$ such that for some chain $\{Q_i\}$ of crosscuts defining $\mathcal{E}_t$, $\mathrm{var}(f, Q_i, Y) \neq 0$ for every i. We call an outchannel $\mathcal{E}_t$ of Y* a geometric outchannel *iff for sufficiently small δ, every chord in $\mathcal{G}_\delta$, which crosses $\mathcal{E}_t$ essentially, has nonzero variation. We call a geometric outchannel* negative *(respectively,* positive*) iff every chord in $\mathcal{G}_\delta$, which crosses $\mathcal{E}_t$ essentially, has negative (respectively, positive) variation.*

Lemma 4.2. *Let $Z \in \{Y_\delta^+, Y_\delta^-\}$. Fix a Riemann map $\phi : \Delta_\infty \to \mathbb{C}_\infty \setminus Z$ such that $\phi(\infty) = \infty$. Suppose R_t lands at $x \in \partial Z$. Then there is an open interval $M \subset \partial\Delta_\infty$ containing t such that ϕ can be extended continuously over M.*

Proof. As in the proof of Proposition 3.24 put $\phi(t) = x$. Again let t_i converge to t in $\partial\Delta_\infty$ such that R_{t_i} lands on x_i in Z and x_i converges to x. By Lemma 3.21 there exist crosscuts Q_i such that R_{t_i} crosses Q_i essentially. By Proposition 3.17, $\mathrm{var}(f, Q_i, X) = 0$ so eventually $Q_i \in \mathcal{G}_\delta^+ \cap \mathcal{G}_\delta^-$. So $Q_i \subset Z$ and ϕ extends continuously over some interval $M \subset \partial\Delta_\infty$. □

Lemma 4.3. *If there is a chord Q of Y of negative (respectively, positive) variation, such that there is no fixed point in $T(Y \cup Q)$, then there is a negative (respectively, positive) geometric outchannel $\mathcal{E}_t$ of Y for which a defining chain begins with Q.*

Proof. Without loss of generality, assume $\mathrm{var}(f, Q, Y) < 0$. Choose $\delta > 0$ so small that $Q \notin \mathcal{G}_\delta$, no chord in $\mathcal{G}_\delta$ separates Q from ∞, so, since there are no fixed points in $T(Y \cup Q)$, every chord in $\mathcal{G}_\delta$ separated from ∞ by Q moves off itself under f and variation on it is defined.

Let $\phi : \Delta_\infty \to \mathbb{C} \setminus Y_\delta^+$ be the Riemann map. Let a, b be the end points of C. Since $a, b \in Y$ are accessible and since chords are disjoint, there exists a ray R_t that lands on a and a ray $R_{t'}$ that lands on b. By Lemma 4.2, ϕ extends continuously over intervals in $\partial\Delta_\infty$ that contain t, t'. Consider the interval $I \subset \partial\Delta_\infty$ from t to t' that is separated from ∞ by $\phi^{-1}(C)$. Suppose that all chords in G_δ that is separated from ∞ by Q has positive or zero variation. Then the boundary of Y_δ^+ that is separated from ∞ by Q coincides with the boundary of Y_δ, which is locally connected. Hence is it is possible to extend ϕ over this entire interval I from t to t'. Then $\phi(I)$ is an arc in Y_δ^+ joining a to b which is separated by Q from ∞. Any junction from $\phi(I)$ defines a junction for Q so

$$\operatorname{var}(f, Q, X) = \operatorname{var}(f, \phi(I), X) = \sum_{C \in \mathcal{G}_\delta,\ C \subset \tilde{\phi}(I)} \operatorname{var}(f, C, X).$$

This is a contradiction since $\operatorname{var}(f, Q, X) < 0$ and all $\operatorname{var}(f, C, X) \geq 0$. Hence ϕ does not extend over I and it follows that a chord in G_δ that is separated by Q has negative variation. $\square$

4.1. **Invariant Channel in X.** We are now in a position to prove Bell's principal result on any possible counter-example to the fixed point property, under our standing hypothesis.

Lemma 4.4. *Suppose $\mathcal{E}_t$ is a geometric outchannel of $Y = T(X)$ under f. Then the principal continuum $\operatorname{Pr}(\mathcal{E}_t)$ of $\mathcal{E}_t$ is invariant under f. So $\operatorname{Pr}(\mathcal{E}_t) = X$.*

Proof. Let $x \in \operatorname{Pr}(\mathcal{E}_t)$. Then for some chain $\{Q_i\}_{i=1}^\infty$ of crosscuts defining $\mathcal{E}_t$ selected from $\mathcal{G}_\delta$, we may suppose $Q_i \to x \in \partial T(X)$ and $\operatorname{var}(f, Q_i, X) \neq 0$ for each i. The external ray R_t meets all Q_i and is equivalent to a junction; any junction from Q_i "parallels" R_t. Since $\operatorname{var}(f, Q_i, X) \neq 0$, each $f(Q_i)$ intersects R_t. Since $\operatorname{diam}(f(Q_i)) \to 0$, we have $f(Q_i) \to f(x)$ and $f(x) \in \operatorname{Pr}(\mathcal{E}_t)$. We conclude that $\operatorname{Pr}(\mathcal{E}_t)$ is invariant. $\square$

Theorem 4.5 (Dense channel, Bell)**.** *Under our standing Hypothesis, $Y = T(X)$ contains a negative geometric outchannel; hence, $\partial O_\infty = \partial T(X) = X = f(X)$ is an indecomposable continuum.*

Proof. Recall that the map $f : \mathbb{C} \to \mathbb{C}$ taking X into $Y = T(X)$ has no fixed points in Y, and X is minimal with respect to these properties. Choose $\delta > 0$ so that each crosscut $Q \in \mathcal{G}_\delta$ is sufficiently close to Y so that f has no fixed points in $T(Y \cup Q)$, and so that for any geometric crosscut $Q \in \mathcal{G}_\delta$, $f(Q) \cap Q = \emptyset$. By Lemma 3.24 ∂Y_δ is a Carathéodory loop. Since f is fixed point free on ∂Y_δ, $\operatorname{ind}(f, \partial Y_\delta) = 0$. Consequently, by Theorem 2.13 for Carathéodory loops, $\operatorname{var}(f, \partial Y_\delta) = -1$. By the summability of variation on ∂Y_δ, it follows that on some chord $Q \subset \partial Y_\delta$, $\operatorname{var}(f, Q, Y) < 0$. By Lemma 4.3, there is a negative geometric outchannel $\mathcal{E}_t$ under the crosscut Q.

Since $\operatorname{Pr}(\mathcal{E}_t)$ is invariant under f by Lemma 4.4, it follows that $\operatorname{Pr}(\mathcal{E}_t)$ is an invariant subcontinuum of $\partial O_\infty \subset \partial Y \subset X$. So by the minimality condition in our Standing Hypothesis, $\Pr(\mathcal{E}_t)$ is dense in ∂O_∞. Hence, $\partial O_\infty = \partial Y = X$ and $\Pr(\mathcal{E}_t)$ is dense in X. It then follows from a theorem of Rutt [19] that X is an indecomposable continuum. □

Theorem 4.6. *The boundary of Y_δ is a simple closed curve. The set of accessible points in the boundary of each of Y_δ^+ and Y_δ^- is a countable union of continuous one-to-one images of $\mathbb{R}$.*

Proof. By Theorem 4.5, X is indecomposable, so it has no cut points. By Proposition 3.24, ∂Y_δ is a Carathéodory loop. Since X has no cut points, neither does Y_δ. A Carathéodory loop without cut points is a simple closed curve.

Let $Z \in \{Y_\delta^+, Y_\delta^-\}$. Fix a Riemann map $\phi : \Delta_\infty \to \mathbb{C}_\infty \setminus Z$ such that $\phi(\infty) = \infty$. Corresponding to the choice of Z, let $\mathcal{W} \in \{\mathcal{G}_\delta^+, \mathcal{G}_\delta^-\}$. Apply Lemma 4.2 and find the maximal collection $\mathcal{J}$ of disjoint open subarcs of $\partial\Delta_\infty$ over which ϕ can be extended continuously. The collection $\mathcal{J}$ is countable. Since X has no cutpoints the extension is one-to-one over $\cup\mathcal{J}$. Since angles that correspond to accessible points are dense in $\partial\Delta_\infty$, so is $\cup\mathcal{J}$. If $Z = Y_\delta^+$, then it is possible that $\cup\mathcal{J}$ is all of $\partial\Delta_\infty$ except one point, but it cannot be all of $\partial\Delta_\infty$ since there is at least one negative geometric outchannel by Theorem 4.5. □

Theorem 4.6 still leaves open the possibility that $Z \in \{Y_\delta^+, Y_\delta^-\}$ has a very complicated boundary. The set $C = \partial\Delta_\infty \setminus \cup\mathcal{J}$ is compact and zero-dimensional. Note that ϕ is discontinuous at points in C, we may call C the set of outchannels of Z. In principle, there could be an uncountable set of outchannels, each dense in X. The one-to-one continuous images of $\mathbb{R}$ lying in ∂Z are the "sides" of the outchannels. If two elements J_1 and J_2 of the collection $\mathcal{J}$ happen to share a common endpoint t, then the prime end $\mathcal{E}_t$ is an outchannel in Z, dense in X, with $\phi(J_1)$ and $\phi(J_2)$ as its sides. It seems possible that an endpoint t of $J \in \mathcal{J}$ might have a sequence of elements J_i from $\mathcal{J}$ converging to it. Then the outchannel $\mathcal{E}_t$ would have only one (continuous) "side." Such exotic possibilities are eliminated in the next section.

In the lemma below we show that pieces of the boundary of Y_δ^- which correspond to arc components in the set of accessible points, are well behaved and do not contain large unnecessary "wiggles."

Lemma 4.7. *Assume that ∂Y_δ^- is not a simple closed curve. Let K be an arc component of the set of accessible points of Y_δ^-. Then for each ε, $0 < \varepsilon < \delta/2$, there exists $\xi > 0$ such that for any two points $x, y \in K \cap X$, if Q is any crosscut of Y_δ^- joining x to y, and* $\operatorname{diam}(Q) < \xi$, *Then there exists an arc $B \subset K$, joining x to y such that* $\operatorname{diam}(B) < 8\varepsilon$.

Proof. Let $\varepsilon > 0$ be fixed and choose ξ as in Proposition 3.15. Let B be the unique arc in K joining x to y. By Theorem 4.6, K is a one-to-one continuous image of $\mathbb{R}$. We will denote the unique subarc of K which joins two points $p, q \in K$ by $\langle p, q\rangle$. Hence $B = \langle x, y\rangle$. Suppose there exist $z \in B$ such that $d(z, Q) \geq 8\varepsilon$. Let $b \in T(Q \cup Y_\delta^-) \setminus Y_\delta^-$ such that $d(b, z) \leq \varepsilon/2$. Let $P = \langle x, z\rangle \setminus B(b, \varepsilon)$ and $M = \langle z, y\rangle \setminus B(b, \varepsilon)$. Then P and Q are disjoint closed sets in Y_δ^-. Let N be a component of $B(Q, 3\varepsilon) \setminus Y_\delta^-$ which separates b from ∞ in $\mathbb{C} \setminus Y_\delta^-$ and such that N is contained in the bounded component of $T(Q \cup Y_\delta^-)$. By Proposition 3.15, each point of N lies in an element $F_x \in \mathcal{F}$ with diameter at most 2ε. Since F_x is small and meets Y, $F_x \cap (P \cup M) \neq \emptyset$ for each $x \in N$. It follows from Proposition 3.16 that there exists $x \in N$ such that $F_x = F$ meets both P and M and, hence, F separates b from ∞ in $\mathbb{C} \setminus Y_\delta^-$. We may assume that F is a chord, since if it is a gap $F = V(B)$ then we can replace it by one of the chords of $\partial V(B)$.

Now, $\mathrm{var}(f, F, X) \leq 0$. For if $\mathrm{var}(f, F, X) > 0$, there exists a positive geometric outchannel $\mathcal{E}_s$ for which a defining chain starts with F. But, if the end points of F are x', y', then R_s would cross some chord $G \subset \langle x', y'\rangle \subset K$ essentially. This is a contradiction since K contains no crosscuts of positive variation. So $\mathrm{var}(f, F, X) \leq 0$. It follows that $F \subset Y_\delta^-$ and $T(Y \cup F) \subset Y_\delta^-$. This contradicts that $b \in \mathbb{C} \setminus Y_\delta^-$. □

5. Uniqueness of the Outchannel

Theorem 4.5 asserts the existence of at least one negative geometric outchannel which is dense in X. We show below that there is exactly one geometric outchannel, and that its variation is -1. Of course, X could have other dense channels, but they are "neutral" as far as variation is concerned.

Theorem 5.1 (Unique Outchannel)**.** *Assume the standing hypothesis 1.1. Then there exists a unique geometric outchannel $\mathcal{E}_t$ for X, which is dense in $X = \partial Y$. Moreover, for any sufficiently small chord Q in any chain defining $\mathcal{E}_t$,* $\mathrm{var}(f, Q, X) = -1$, *and for any sufficiently small chord Q' not crossing R_t essentially,* $\mathrm{var}(f, Q', X) = 0$.

Proof. Suppose by way of contradiction that X has a positive outchannel. Let $\delta > 0$ such that $T(B(Y, 2\delta))$ contains no fixed points of f and such that, if $M \subset B(Y, 2\delta)$ with $\mathrm{diam}(M) < 2\delta$, then $f(M) \cap M = \emptyset$. Since X has a positive outchannel, ∂Y_δ^- is not a simple closed curve. By Theorem 4.6 ∂Y_δ^- contains an arc component K which is the one-to-one continuous image of $\mathbb{R}$. Note that each point of K is accessible.

Let $\varphi : \Delta_\infty \to U_\infty = \mathbb{C} \setminus Y_\delta^-$ a conformal map. By Theorem 4.6, and its proof, φ extends continuously and injectively to a map $\tilde{\varphi} : \tilde{\Delta}_\infty \to \tilde{U}_\infty$, where $\tilde{\Delta}_\infty \setminus \Delta_\infty$ is a dense and open subset of S^1 which contains K in its image. Then $\tilde{\varphi}^{-1}(K) = (t', t) \subset S^1$ is an open arc with $t' < t$ in the counterclockwise order on S^1. Let $<$ denote the order in K induced by $\tilde{\varphi}$ and for $x < y$ in K, denote the arc in K with endpoints x and y by $\langle x, y\rangle$. Let $\langle x, \infty\rangle = \cup_{y > x}\langle x, y\rangle$

Let $\mathcal{E}_t$ be the prime-end corresponding to t. Then $\Pr(\mathcal{E}_t)$ is a positive geometric outchannel and, hence, by Lemma 4.4, $\Pr(\mathcal{E}_t) = X$. Let $R_t =$

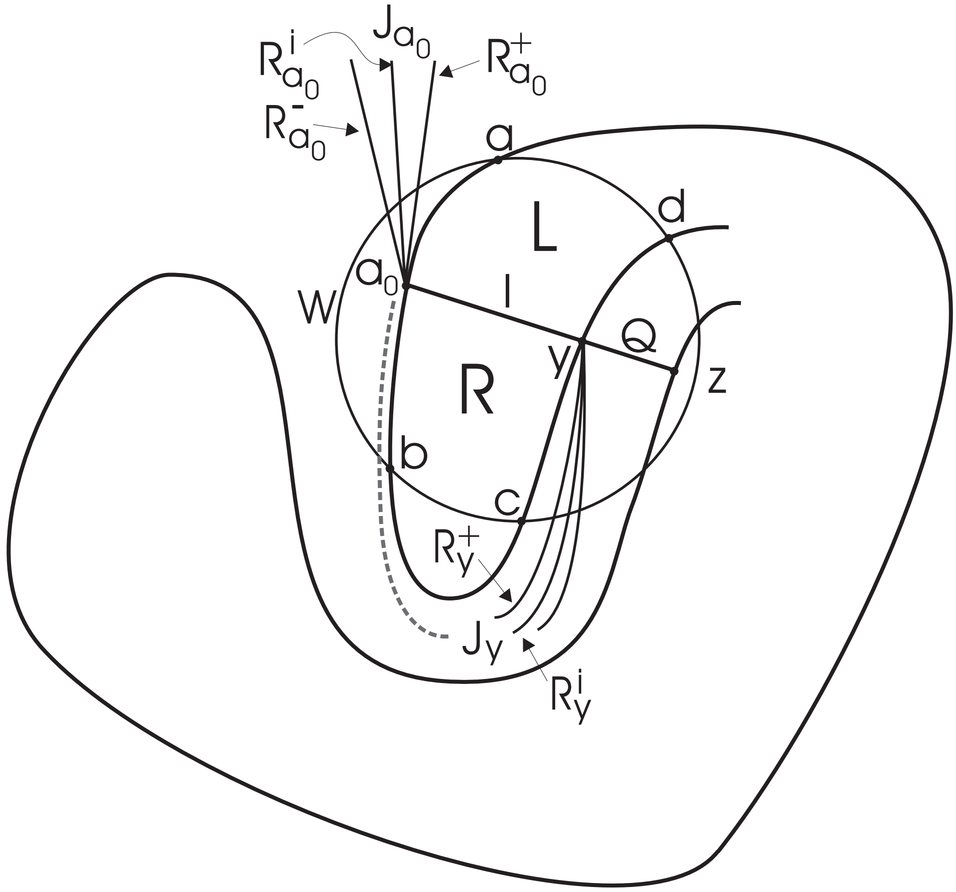

FIGURE 4. Uniqueness of the negative outchannel.

$\varphi(re^{it})$, $r > 1$, be the external conformal ray corresponding to the prime-end $\mathcal{E}_t$. Since $\overline{R_t} \setminus R_t = X$ and the small chords which define $\Pr(\mathcal{E}_t)$ have one end point in K (c.f., Proposition 3.15), $\overline{\langle x, \infty\rangle} \cap Y_\delta^- = X$.

Let $\varepsilon > 0$ such that $T(B(Y,\varepsilon)) \subset B(Y,\delta)$ (by Proposition 3.14). It follows from Propositions 3.14, 3.16 and 3.19, there exists $x \in K$ such that in each arc $M \subset \langle x, \infty\rangle$ with $\operatorname{diam}(M) > \varepsilon/4$, there exists $y \in M$ and a chord $G \in \mathcal{G}_\delta$ with end point y which crosses R_t essentially.

Let $a_0 \in K \cap X$ so that $a_0 > x$ and J_{a_0} a junction of Y_δ^-. Let W be a topological disk about a_0 with simple closed curve boundary of diameter less than ε so that the component of $K \cap W$ containing a_0 has closure $\langle a, b\rangle$, $a < a_0 < b$ in K and $f(\overline{W}) \cap (\overline{W} \cup J_{a_0}) = \emptyset$. We may suppose that $(K \cap W) \setminus \langle a, b\rangle$ is contained in one component of $W \setminus \langle a, b\rangle$ since one side of K is accessible from ∞ in $\mathbb{C} \setminus Y_\delta^-$. Since $X \subset \overline{\langle a_0, \infty\rangle}$, there are components of $W \cap \langle b, \infty\rangle$ which pass arbitrarily close to a_0. Choose $\langle c, d\rangle$ to be the closure of a component of $W \cap \langle b, \infty\rangle$ such that:

(1) a and d lie in the same component of $\partial W \setminus \{b, c\}$,
(2) there exists $y \in \langle c, d\rangle \cap X \cap W$ and an arc $I \subset (W \setminus \langle a, d\rangle) \cup \{a_0, y\}$ joining a_0 to y, and
(3) there is a chord $Q \subset W$ with y and z as endpoints which crosses R_t essentially.

To see the above, note that there are small chords or simplexes which cross R_t essentially through each point of R_t. By Proposition 3.16 given an arc A that crosses R_t essentially and is sufficiently close to X, there is a small diameter chord that essentially crosses R_t and meets A. By Lemma 4.7, it follows that if two small chords both cross R_t essentially and both meet a small diameter arc in R_t, then they both meet a small diameter arc in K. Thus we can satisfy (3) on any arc in $W \cap K$ which gets close to a_0. Note that (2) holds for any point of $\langle c, d\rangle$ for which (3) holds.

Let B be the arc in $\partial W \setminus \{b\}$ with end points a and d. Let A be a bumping arc in $(\mathbb{C} \setminus [J_{a_0} \cup T(\langle a, d\rangle \cup B)]) \cup \{a, d\}$ with end-points a and d such that $Y \setminus T(\langle a, d\rangle \cup B) \subset T(A \cup B)$. Hence, $S = \langle a, d\rangle \cup A$ is a simple closed curve and $Y \subset T(S)$. We may suppose that $A \subset B(Y, \varepsilon)$ so that f is fixed point free on $T(S)$ and each component of $A \setminus X$ has diameter less than δ so that variation is defined on each such component.

Since $\overline{Q} \cap Y = \{y, z\}$ we may suppose that $A \cap \overline{Q} = \{z\}$. Note that I is an arc in $T(S)$ which meets S only at its end points a_0 and y. Since $I \subset W$, $f(I) \cap J_{a_0} = \emptyset$. Let $R = T(\langle a_0, y\rangle \cup I)$ and let $L = T(\langle y, d\rangle \cup I \cup A \cup \langle a, a_0\rangle)$. Let J_y be a junction for S such that $J_y \cap \overline{Q} = \{y\}$, $J_{a_0} \setminus W \subset J_y$ so that $R^*_{a_0} \setminus W \subset R^*_y$ for each $* \in \{+, i, -\}$ and J_y runs very close to $\langle a_0, y\rangle \cup J_{a_0}$.

Note that the order $<$ on K coincides with the counterclockwise order on S. It follows that $W \cup R^i_y$ separates $L \cup R^-_y \setminus J_{a_0}$ from $R \cup R^+_y \setminus J_{a_0}$. Since Q crosses R_t essentially, we know that $\mathrm{var}(f, Q, Y) > 0$. We will use this information to show that $f(y) \in R$. To compute $\mathrm{var}(f, Q, Y) = \mathrm{var}(f, Q, S)$ we will use the fact that the variation is invariant under a homotopy which keeps y and z in $h(U_y)$ (see Proposition 2.10 and the remark following that proposition). Hence, if we homotope $f|_{\overline{Q}}$ to a map $f' : \overline{Q} \to \mathbb{C} \setminus W$ such that $f|_{f^{-1}(T(S))\cap\overline{Q}} = f'|_{f^{-1}(T(S))\cap\overline{Q}}$, then $\mathrm{var}(f', Q, S) = \mathrm{var}(f, Q, S)$. Moreover, we can choose f' such that the number of components of $f'^{-1}(\mathbb{C} \setminus T(S))$ is minimal (the set of components of $\overline{Q} \cap f^{-1}(\mathbb{C} \setminus T(S))$ whose closures meet both $f^{-1}(T(L))$ and $f^{-1}(T(R))$ is finite since $f(\overline{Q}) \cap W = \emptyset$). Then to compute $\mathrm{var}(f', Q, S)$ we use the following recipe: As we go along Q from y to z, each time the image of f' goes from R to L count $+1$. Each time the image goes from L to R count -1. Make no other counts. Then it follows that if $f'(y) = f(y) \in R$ and $f(z) \in L$, then $\mathrm{var}(f', Q, S) = +1$, if $f'(y) \in L$ and $f'(z) \in R$, then $\mathrm{var}(f', Q, S) = -1$. Otherwise $\mathrm{var}(f', Q, S) = 0$. Since $\mathrm{var}(f', Q, S) > 0$, $f(y) \in R$.

The Lollipop Lemma, Theorem 2.14, applies to S and the arc I. Also, since $Y \subset T(S)$, $\mathrm{var}(f, C, S) = \mathrm{var}(f, C, Y)$ for each chord C contained in S. Hence, there exists a chord $Q_1 \subset \langle a_0, y\rangle$ such that $\mathrm{var}(f, Q_1, Y) < 0$. Since there are no chords of positive variation on $\langle a_0, y\rangle$ and

$$0 = \operatorname{ind}(f, I \cup \langle a_0, y\rangle) = \sum_{C\in\mathcal{G},\ C\subset\langle a_0,y\rangle} \operatorname{var}(f, C, Y) + 1,$$

we know that $\operatorname{var}(f, Q_1, Y) = -1$.

We repeat the above argument starting with $y \in K$ in place of a_0 and J_y in place of J_{a_0} and an open disk $V \subset W$ about y to find a second chord $Q_2 \subset \langle y, \infty\rangle$ with $\operatorname{var}(f, Q_2, Y) = -1$.

We will now show that the existence of chords Q_1 and Q_2 in K with variation -1 on each leads to a contradiction. Choose $c' < d' \in K$ such that $\langle c', d'\rangle$ is the closure of a component of $K \cap W$ satisfying the following conditions.

(1) $Q_1, Q_2 \subset \langle a_0, c'\rangle \subset K$,
(2) $\{a, d'\}$ is contained in one component of $\partial W \setminus \{b, c'\}$,
(3) there exist $y' \in \langle c', d'\rangle \cap Y \cap W$ and an arc I' from a_0 to y' in $\{a_0, y'\} \cup (W \setminus \langle a, d'\rangle)$, and
(4) there exists a chord $Q' \subset W$ with endpoints y' and z' such that Q' crosses R_t essentially.

Let B' be the arc in $\partial W \setminus \{b\}$ with endpoints $\{a\}$ and $\{d'\}$. Let A' be an arc in $\{a, d'\} \cup \mathbb{C} \setminus T(\langle a, d'\rangle \cup B')$ such that $Y \setminus T(\langle a, d'\rangle \cup B') \subset T(A' \cup B')$ and such that the components of $A' \setminus X$ have diameter less than δ. We may suppose that $\overline{Q}' \cap A' = \{z'\}$. We can prove, as above, that $f(y') \in R' = T(\langle a_0, y'\rangle \cup I')$ and, hence all conditions of the Lollipop Lemma 2.14 are again satisfied for S' and I' and

$$\operatorname{ind}(f, \langle a_0, y'\rangle \cup I') = \sum_{C\in\mathcal{G},\ C\subset\langle a_0,y'\rangle} \operatorname{var}(f, C, S') + 1.$$

Since $\langle a_0, y'\rangle$ contains Q_1 and Q_2, $\operatorname{var}(f, Q_i, S') = \operatorname{var}(f, Q_i, Y) < 0$ and contains no chords of positive variation, $\sum \operatorname{var}(f, C, S') + 1 \leq -1$.

Since f is fixed point free on R', $\operatorname{ind}(f, \langle a_0, y'\rangle \cup I') = 0$ by Theorem 2.5. This contradiction shows that X has no positive outchannels.

By Theorems 4.5 and 2.13, X has exactly one negative outchannel and its variation is -1. $\square$

6. Oriented maps

A *perfect map* is a closed continuous surjection, each of whose point inverses is compact. A map $f : X \to Y$ is *monotone* provided for each continuum $K \subset Y$, $f^{-1}(K)$ is connected. A map $f : X \to Y$ is *confluent* provided for each continuum $K \subset Y$ and each component C of $f^{-1}(K)$, $f(C) = K$. A map $f : X \to Y$ is *light* provided for each point $y \in Y$, $f^{-1}(y)$ is totally disconnected.

It is well know that each homeomorphism of the plane is either orientation-preserving or orientation-reversing. In this section we will establish an appropriate extension of this result for confluent perfect mappings of the plane

(Theorem 6.7) by showing that such maps either preserve or reverse local orientation. As a consequence it follows that all perfect and confluent maps of the plane satisfy the Maximum Modulus Theorem. We will call such maps positively- or negatively-oriented maps, respectively. For perfect mappings of the plane, Lelek and Read have shown that confluent is equivalent to the composition of open and monotone maps [14]. Holomorphic maps are prototypes of positively-oriented maps but positively-oriented maps, unlike holomorphic maps, do not have to be light. A non-separating plane continuum is said to be *acyclic*.

Definition 6.1 (Degree of a map)**.** *Let $f : U \to \mathbb{C}$ be a map from a simply connected domain U into the plane. Let S be a simple closed curve in U, and $p \in U \setminus f^{-1}(f(S))$ a point. Define $f_p : S \to \mathbb{S}^1$ by*

$$f_p(x) = \frac{f(x) - f(p)}{|f(x) - f(p)|}.$$

Then f_p has a well-defined degree*, denoted degree(f_p). Note that degree(f_p) is the winding number* $\mathrm{win}(f, S, f(p))$ *of $f|_S$ about $f(p)$),*

Definition 6.2 (Bell)**.** *A map $f : U \to \mathbb{C}$ from a simply connected domain U is* positively-oriented *(respectively,* negatively-oriented*) provided for each simple closed curve S in U and each point $p \in T(S) \setminus f^{-1}(f(S))$, degree$(f_p) \geq 0$ (degree$(f_p) \leq 0$, respectively). If for each $p \in T(S) \setminus f^{-1}(f(S))$, the degree of f_p, degree$(f_p) > 0$ (degree$(f_p) < 0$, respectively), we say that f is* strictly positively-oriented *(respectively,* strictly negatively-oriented*).*

Definition 6.3. *A perfect map $f : \mathbb{C} \to \mathbb{C}$ is* oriented *provided for each simple closed curve S and each $x \in T(S)$, $f(x) \in T(f(S))$.*

Clearly every strictly positively- or strictly negatively-oriented map is oriented. The definition immediately implies that all oriented maps satisfy the Maximum Modulus Theorem.

It is well known that both open and monotone maps (and hence compositions of such maps) of continua are confluent. It will follow (Lemma 6.6) from a result of Lelek and Read [14] that each perfect mapping of the plane is the composition of a monotone map and a light open map. The following Lemmas are in preparation for the proof of Theorem 6.7.

Lemma 6.4. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a perfect map. It follows that f is confluent if and only if f is oriented.*

Proof. Suppose that f is oriented. Let A be an arc in $\mathbb{C}$ and let C be a component of $f^{-1}(A)$. Suppose that $f(C) \neq A$. Let $a \in A \setminus f(C)$. Since $f(C)$ does not separate a from infinity, we can choose a simple closed curve S with $C \subset T(S)$, $S \cap f^{-1}(A) = \emptyset$ and $f(S)$ so close to $f(C)$ that $f(S)$ does not separate a from ∞. Then $a \notin T(f(S))$. Since f is oriented, $f(C) \subset T(f(S))$. Hence there exists a $y \in A \cap \partial T(f(S)) \subset A \cap f(S)$. This contradicts the fact that $A \cap f(S) = \emptyset$.

Now suppose that K is an arbitrary continuum in $\mathbb{C}$ and let L be a component of $f^{-1}(K)$. Let $x \in L$ and let A_i be a sequence of arcs in $\mathbb{C}$ such that $\lim A_i = K$ and $f(x) \in A_i$ for each i. Let M_i be the component of $f^{-1}(A_i)$ containing the point x. By the previous paragraph $f(M_i) = A_i$. Since f is perfect, $M = \limsup M_i \subset L$ is a continuum and $f(M) = K$. Hence f is confluent.

Suppose next that $f : \mathbb{C} \to \mathbb{C}$ is not oriented. Then there exists a simple closed curve S in $\mathbb{C}$ and $p \in T(S) \setminus f^{-1}(f(S))$ such that $f(p) \notin T(f(S))$. Let L be a half-line with end-point $f(p)$ running to infinity in $\mathbb{C} \setminus f(S)$. Let L^* be an arc in L with endpoint $f(p)$ and diameter greater than the diameter of the continuum $f(T(S))$. Let K be the component of $f^{-1}(L^*)$ which contains p. Then $K \subset T(S)$, since $p \in T(S)$ and $L \cap f(S) = \emptyset$. Hence, $f(K) \neq L^*$, and so f is not confluent. $\square$

Lemma 6.5. *Let $f : \mathbb{C} \to \mathbb{C}$ be a light open perfect map. Then there exists an integer k and a finite subset $B \subset \mathbb{C}$ such that f is a local homeomorphism at each point of $\mathbb{C} \setminus B$, and for each point $y \in \mathbb{C} \setminus f(B)$, $|f^{-1}(y)| = k$.*

Proof. Let $\mathbb{C}_\infty$ be the one point compactification of $\mathbb{C}$. Since f is perfect, we can extend f to a map of $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$ so that $f^{-1}(\infty) = \infty$. By abuse of notation we also denote the extended map by f. Then f is a light open mapping of the compact 2-manifold $\mathbb{C}_\infty$. The result now follows from a theorem of Whyburn [22, X.6.3]. $\square$

The following is the special case for confluent perfect maps of the monotone-light factorization theorem.

Lemma 6.6. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a confluent perfect map. It follows that $f = g \circ h$, where $h : \mathbb{C} \to \mathbb{C}$ is a monotone perfect map with acyclic fibers and $g : \mathbb{C} \to \mathbb{C}$ is a light open perfect map.*

Proof. By the monotone-light factorization theorem [15, Theorem 13.3], $f = g \circ h$, where $h : \mathbb{C} \to X$ is monotone, $g : X \to \mathbb{C}$ is light, and X is the quotient space obtained from $\mathbb{C}$ by identifying each component of $f^{-1}(y)$ to a point for each $y \in \mathbb{C}$. Let $y \in \mathbb{C}$ and let C be a component of $f^{-1}(y)$. If C were to separate $\mathbb{C}$, then $f(C) = y$ would be a point while $f(T(C))$ would be a non-degenerate continuum. Choose an arc $A \subset \mathbb{C} \setminus \{y\}$ which meets both $f(T(C))$ and its complement and let $x \in T(C) \setminus C$ such that $f(x) \in A$. If K is the component of $f^{-1}(A)$ which contains x, then $K \subset f(T(C))$. Hence $f(K)$ cannot map onto A contradicting the fact that f is confluent. Thus for each $y \in \mathbb{C}$, each component of $f^{-1}(y)$ is acyclic.

By Moore's Plane Decomposition Theorem [9], X is homeomorphic to $\mathbb{C}$. Since f is confluent, it is easy to see that g is confluent. By a theorem of Lelek and Read [14] g is open since it is confluent and light (also see [15, Theorem 13.26]). Since h and g factor the perfect map f through a Hausdorff space $\mathbb{C}$, both h and g are perfect [10, 3.7.5]. $\square$

Theorem 6.7 (Maximum Modulus Theorem)**.** *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a perfect map. Then the following are equivalent:*

(1) *f is either strictly positively or strictly negatively oriented.*
(2) *f is oriented.*
(3) *f is confluent.*

Proof. It is clear that (1) implies (2). By Lemma 6.4 every oriented map is confluent. Hence suppose that $f : \mathbb{C} \to \mathbb{C}$ is a perfect confluent map. By Lemma 6.6, $f = g \circ h$, where $h : \mathbb{C} \to \mathbb{C}$ is a monotone perfect map with acyclic fibers and $g : \mathbb{C} \to \mathbb{C}$ is a light open perfect map. By Stoilow's Theorem [23] there exists a homeomorphism $j : \mathbb{C} \to \mathbb{C}$ such that $g \circ j$ is an analytic map. Then $f = g \circ h = (g \circ j) \circ (j^{-1} \circ h)$. Since $k = j^{-1} \circ h$ is a monotone surjection of $\mathbb{C}$ with acyclic fibers, it is a near homeomorphism. That is, there exists a sequence k_i of homeomorphisms of $\mathbb{C}$ such that $\lim k_i = k$ [9, Theorem 25.1]. We may assume that all of the k_i have the same orientation. Let $f_i = (g \circ j) \circ k_i$, S a simple closed curve and $p \in T(S) \setminus f^{-1}(f(S))$. Note that $\lim f_i^{-1}(f_i(S)) \subset f^{-1}(f(S))$. Hence $p \in T(S) \setminus f_i^{-1}(f_i(S))$ for i sufficiently large. Moreover, since f_i converges to f, $f_i|_S$ is homotopic to $f|_S$ in the complement of $f(p)$ for i large. Thus for large i, degree$((f_i)_p) =$ degree(f_p), where

$$(f_i)_p(x) = \frac{f_i(x) - f_i(p)}{|f_i(x) - f_i(p)|} \text{ and } f_p(x) = \frac{f(x) - f(p)}{|f(x) - f(p)|}.$$

Since $g \circ j$ is an analytic map, it is positively oriented and degree$((f_i)_p) =$ degree$(f_p) > 0$ if k_i is orientation preserving and degree$((f_i)_p) =$ degree$(f_p) < 0$ if k_i is orientation reversing. Thus, f is positively-oriented if each k_i is orientation-preserving and f is negatively-oriented if each k_i is orientation-reversing. $\square$

Let X be an acyclic plane continuum. We shall need the following three results in the next section.

Lemma 6.8. *Let X and Y be non-degenerate acyclic plane continua and $f : \mathbb{C} \to \mathbb{C}$ a perfect map such that $f^{-1}(Y) = X$ and $f|_{\mathbb{C}\setminus X}$ is confluent. Then for each $y \in \mathbb{C} \setminus Y$, each component of $f^{-1}(y)$ is acyclic.*

Proof. Suppose there exists $y \in \mathbb{C}\setminus Y$ such that some component C of $f^{-1}(y)$ is not acyclic. Then there exists $z \in T(C) \setminus f^{-1}(y) \cup X$. By unicoherence of $\mathbb{C}$, $Y \cup \{y\}$ does not separate $f(z)$ from infinity in $\mathbb{C}$. Let L be a ray in $\mathbb{C} \setminus [Y \cup \{y\}]$ from $f(z)$ to infinity. Then $L = \cup L_i$, where each $L_i \subset L$ is an arc with end-point $f(z)$. For each i the component M_i of $f^{-1}(L_i)$ containing z maps onto L_i. Then $M = \cup M_i$ is a connected closed subset in $\mathbb{C} \setminus f^{-1}(y)$ from z to infinity. This is a contradiction since $z \in T(f^{-1}(y))$. $\square$

Theorem 6.9. *Let X and Y be non-degenerate acyclic plane continua and $f : \mathbb{C} \to \mathbb{C}$ a perfect map such that $f^{-1}(Y) = X$ and $f|_{\mathbb{C}\setminus X}$ is confluent. If A and B are crosscuts of X such that $B \cup X$ separates $A \setminus f^{-1}(f(B))$ from ∞ in $\mathbb{C}$, then $f(B) \cup Y$ separates $f(A) \setminus f(B)$ from ∞.*

Proof. Suppose not. Then there exists a half-line L joining $f(A)$ to infinity in $\mathbb{C} \setminus (f(B) \cup Y)$. As in the proof of Lemma 6.8, there exists a closed and connected set $M \subset \mathbb{C} \setminus (B \cup X)$ joining A to infinity, a contradiction. $\square$

Corollary 6.10. *Under the conditions of Theorem 6.9, if L is a ray irreducible from Y to infinity, then each component of $f^{-1}(L)$ which meets $\mathbb{C} \setminus X$ is a closed and connected set from X to infinity.*

7. Induced maps of prime ends

Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an oriented perfect map and $f^{-1}(Y) = X$, where X and Y are acyclic continua. We will show that in this case the map f induces a confluent map F of the circle of prime ends of X to the circle of prime ends of Y. This result was announced by Mayer in the early 1980's but never appeared in print. It was also used (for homeomorphisms) by Cartwright and Littlewood in [7]. There are easy counterexamples that show if f is not confluent then it may not induce a continuous function between the circles of prime ends. We denote by Δ the closed unit ball in $\mathbb{C}$. Then $\Delta \subset \mathbb{C} \subset \mathbb{C}_\infty$. Note that by the Riemann Mapping Theorem, if $X \subset \mathbb{C}_\infty$ is a non-degenerate acyclic continuum, then there exists a conformal surjection $\phi : \mathbb{C}_\infty \setminus X \to \Delta$.

Theorem 7.1. *Let X and Y be non-degenerate acyclic plane continua and $f : \mathbb{C} \to \mathbb{C}$ a perfect map such that:*

1. *Y has no cut point,*
2. *$f^{-1}(Y) = X$ and*
3. *$f|_{\mathbb{C}\setminus X}$ is confluent.*

Let $\varphi : \mathbb{C}_\infty \setminus X \to \mathbb{C}_\infty \setminus \Delta$ and $\psi : \mathbb{C}_\infty \setminus Y \to \mathbb{C}_\infty \setminus \Delta$ be conformal mappings. Define $\hat{f} : \mathbb{C}_\infty \setminus \Delta \to \mathbb{C}_\infty \setminus \Delta$ by $\hat{f} = \psi \circ f \circ \varphi^{-1}$.

Then $\hat{f}$ extends to a map $\bar{f} : \overline{\mathbb{C}_\infty \setminus \Delta} \to \overline{\mathbb{C}_\infty \setminus \Delta}$. Moreover, $\bar{f}^{-1}(S^1) = S^1$ and $F = \bar{f}|_{S^1}$ is a confluent map.

Proof. Note that f takes accessible points of X to accessible points of Y. For if P is a path in $[\mathbb{C} \setminus X] \cup \{p\}$ with end point $p \in X$, then by (2), $f(P)$ is a path in $[\mathbb{C} \setminus Y] \cup \{f(p)\}$ with endpoint $f(p) \in Y$.

Let A be a crosscut of X such that the diameter of $f(A)$ is less than half of the diameter of Y and let U be the bounded component of $\mathbb{C}\setminus(X \cup A)$. Let the endpoints of A be $x, y \in X$ and suppose that $f(x) = f(y)$. If x and y lie in the same component of $f^{-1}(f(x))$ then each crosscut $B \subset U$ of X is mapped to a generalized return cut of Y based at $f(x)$ (i.e., the endpoints of B map to $f(x)$). Note that in this case by Theorem 6.9, $\partial f(U) \subset f(A) \cup \{f(x)\}$.

Now suppose that $f(x) = f(y)$ and x and y lie in distinct components of $f^{-1}(f(x))$. Then by unicoherence of $\mathbb{C}$, $\partial U \subset A \cup X$ is a connected set and $\partial U \not\subset \bar{A} \cup f^{-1}(f(x))$. Now $\partial U \setminus (\bar{A} \cup f^{-1}(f(x))) = \partial U \setminus f^{-1}(f(\bar{A}))$ is an open set in ∂U. Thus there is a crosscut $B \subset U \setminus f^{-1}(f(\bar{A}))$ of X with $\bar{B} \setminus B \subset \partial U \setminus f^{-1}(f(\bar{A}))$. Now $f(B)$ is contained in a bounded component of $\mathbb{C} \setminus (Y \cup f(A)) = \mathbb{C} \setminus (Y \cup f(\bar{A}))$ by Theorem refconfeq. Since

$Y \cap f(\bar{A}) = \{f(x)\}$ is connected and Y does not separate $\mathbb{C}$, it follows by unicoherence that $f(B)$ lies in a bounded component of $\mathbb{C} \setminus f(\bar{A})$. Since $Y \setminus \{f(x)\}$ meets $f(\bar{B})$ and misses $f(\bar{A})$ and $Y \setminus f(x)$ is connected, $Y \setminus \{f(x)\}$ lies in a bounded complementary component of $f(\bar{A})$. This is impossible as the diameter of $f(A)$ is smaller than the diameter of Y. It follows that there exists a $\delta > 0$ such that if the diameter of A is less than δ and $f(x) = f(y)$, then x and y must lie in the same component of $f^{-1}(f(x))$.

In order to define the extension $\bar{f}$ of f over the boundary S^1 of $\overline{\mathbb{C}_\infty \setminus \Delta}$, let C_i be a chain of crosscuts of $\mathbb{C}_\infty \setminus \Delta$ which converge to a point $p \in S^1$ such that $A_i = \varphi^{-1}(C_i)$ is a null chain of crosscuts of X with end points a_i and b_i which converge to a point $x \in X$. There are three cases to consider:

Case 1. f identifies the end points of A_i for some A_i with diameter less than δ. In this case the chain of crosscuts is mapped by f to a chain of generalized return cuts based at $f(a_i) = f(b_i)$. Hence $f(a_i)$ is an accessible point of Y which corresponds (under ψ) to a unique point $q \in S^1$. Define $\bar{f}(p) = q$.

Case 2. Case 1 does not apply and there exists an infinite subsequence A_{i_j} of crosscuts such that $f(\bar{A}_{i_j}) \cap f(\bar{A}_{i_k}) = \emptyset$ for $j \neq k$. In this case $f(A_{i_j})$ is a chain of generalized crosscuts which converges to a point $f(x) \in Y$. This chain corresponds to a unique point $q \in S^1$ since Y has no cut points. Define $\bar{f}(p) = q$.

Case 3. Cases 1 and 2 do not apply. Without loss of generality suppose there exists an i such that for $j > i$ $f(\bar{A}_i) \cap f(\bar{A}_j)$ contains $f(a_i)$. In this case $f(A_j)$ is a chain of generalized crosscuts based at the accessible point $f(a_i)$ which corresponds to a unique point q on S^1 as above. Define $\bar{f}(p) = q$.

It remains to be shown that $\bar{f}$ is a continuous extension of $\hat{f}$ and F is confluent. For continuity it suffices to show continuity at S^1. Let $p \in S^1$ and let C be a small crosscut whose endpoints are on opposite sides of p such that $A = \varphi^{-1}(C)$ has diameter less than δ and such that the endpoints of A are two accessible points of X. Since f is uniformly continuous near X, the diameter of $f(A)$ is small and since ψ is uniformly continuous with respect to connected sets in the complement of Y ([21]), the diameter of $B = \psi \circ f \circ \varphi^{-1}(C)$ is small. Also B is either a generalized crosscut or generalized return cut. Since $\hat{f}$ preserves separation of crosscuts, it follows that the image of the domain U bounded by C which does not contain ∞ is small. This implies continuity of $\bar{f}$ at p.

To see that F is confluent let $K \subset S^1$ be a subcontinuum and let H be a component of $\bar{f}^{-1}(K)$. Choose a chain of crosscuts C_i such that $\varphi^{-1}(C_i) = A_i$ is a crosscut of X meeting X in two accessible points a_i and b_i, $C_i \cap \bar{f}^{-1}(K) = \emptyset$ and $\lim C_i = H$. It follows from the preservation of crosscuts (see Theorem 6.9) that $\hat{f}(C_i)$ separates K from ∞. Hence $\hat{f}(C_i)$ must meet S^1 on both sides of K and $\lim \bar{f}(C_i) = K$. Hence $F(H) = \lim \bar{f}(C_i) = K$ as required. □

Corollary 7.2. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a perfect, confluent and onto mapping of the plane, $X \subset \mathbb{C}$ is a subcontinuum without cut points and $f(X) = X$. Let $\hat{X}$ be the component of $f^{-1}(f(X))$ containing X. Let $\varphi : \mathbb{C}_\infty \setminus T(\hat{X}) \to \mathbb{C}_\infty \setminus \Delta$ and $\psi : \mathbb{C}_\infty \setminus T(f(X)) \to \mathbb{C}_\infty \setminus \Delta$ be conformal mappings. Define $\hat{f} : \mathbb{C}_\infty \setminus [\Delta \cup \varphi(f^{-1}(X))] \to \mathbb{C}_\infty \setminus \Delta$ by $\hat{f} = \psi \circ f \circ \varphi^{-1}$. Put $S^1 = \overline{\mathbb{C}_\infty \setminus \Delta} \setminus [\mathbb{C}_\infty \setminus \Delta]$.*

Then $\hat{f}$ extends over S^1 to a map $\bar{f} : \overline{\mathbb{C}_\infty \setminus \Delta} \to \overline{\mathbb{C}_\infty \setminus \Delta}$. Moreover $\bar{f}^{-1}(S^1) = S^1$ and $F = \bar{f}|_{S^1}$ is a confluent map.

Proof. By Lemma 6.6 $f = g \circ m$ where m is a monotone perfect and onto mapping of the plane with acyclic point inverses, and g is an open and perfect surjection of the plane to itself. By Lemma 6.5, $f^{-1}(X)$ has finitely many components. Let S in $\mathbb{C} \setminus f^{-1}(f(X))$ be a simple closed curve separating $\hat{X}$ from infinity and all other components of $f^{-1}(X)$ and let U be the component of $\mathbb{C} \setminus S$ which contains $\hat{X}$. Then U is simply connected and hence homeomorphic to $\mathbb{C}$. By [13] $f(U)$ is also simply connected. Then $f|_U : U \to f(U)$ is a locally confluent map. By [14], $f|_{U \setminus \hat{X}}$ is confluent. The result now follows from Theorem 7.1 applied to f restricted to U. □

8. Fixed points for positively oriented maps

In this section we will consider a positively oriented map of the plane. As we shall see below, a straight forward application of the tools developed above will give us the desired fixed point result. We will assume by way of contradiction that $f : \mathbb{C} \to \mathbb{C}$ is a positively oriented map, X is a non-separating plane continuum such that $f(X) \subset X$ and X contains no fixed points of f.

Lemma 8.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a map and X a non-separating continuum such that $f(X) \subset X$. Suppose $C = (a, b)$ is a crosscut of the continuum X. Let $v \in (a, b)$ be a point and J_v be a junction such that $J_v \cap (X \cup C) = \{v\}$. Then there exists an arc I such that $S = I \cup C$ is a simple closed curve, $X \subset T(S)$ and $f(I) \cap J_v = \emptyset$.*

Proof. Since $f(X) \subset X$ and $J_v \cap X = 0$, it is clear that there exists an arc I with endpoints a and b sufficiently close to X such that $I \cup C$ is a simple closed curve, $X \subset T(I \cup C)$ and $f(I) \cap J_v = \emptyset$. This completes the proof. □

Corollary 8.2. *Suppose X is an invariant continuum for a positively oriented map $f : \mathbb{C} \to \mathbb{C}$. Then for each crosscut C such that $f(\overline{C}) \cap \overline{C} = \emptyset$,* $\mathrm{var}(f, C) \geq 0$

Proof. Suppose that $C = (a, b)$ is a crosscut of X such that $f(\overline{C}) \cap \overline{C} = \emptyset$ and $\mathrm{var}(f, C) \neq 0$. Choose a junction J_v such that $J_v \cap (X \cup C) = \{v\}$ and $v \in C \setminus X$. By Lemma 8.1, there exists an arc I such that $S = I \cup C$ is a simple closed curve and $f(S) \cap J_v = f(C) \cap J_v$. Moreover, by choosing I sufficiently close to X, we may assume that $v \in \mathbb{C} \setminus f(S)$. Hence $\mathrm{var}(f, C) = \mathrm{Win}(f, S, v) \geq 0$. □

Theorem 8.3. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a positively oriented map and X is a non-separating continuum such that $f(X) \subset X$. Then there exists a point $x_0 \in X$ such that $f(x_0) = x_0$.*

Proof. Suppose we are given a non-separating continuum X and $f : \mathbb{C} \to \mathbb{C}$ a positively oriented map such that $f(X) \subset X$. Assume that $f|_X$ is fixed point free. Choose a simple closed curve S such that $X \subset T(S)$ and points $a_0 < a_1 < \ldots < a_n$ in $S \cap X$ such that for each i $C_i = (a_i, a_{i+1})$ is a sufficiently small crosscut of X, $f(C_i) \cap C_i = \emptyset$ and $f|_{T(S)}$ is fixed point free. By Corollary 8.2, $\mathrm{var}(f, C_i) \geq 0$ for each i. Hence, $\mathrm{ind} f S = \sum \mathrm{var}(f, C_i) + 1 \geq 1$. This contradiction completes the proof. □

Corollary 8.4. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a perfect and onto confluent map, and X is a non-separating continuum such that $f(X) \subset X$. Then there exists a point $x_0 \in X$ of period 2.*

Proof. By Theorem 6.7, f is either positively or negatively oriented. In either case, the second iterate f^2 is positively oriented and must have a fixed point in X by Theorem 8.3. □

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