ON ALMOST ONE-TO-ONE MAPS

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ABSTRACT. A continuous map $f : X \to Y$ of topological spaces X, Y is said to be *almost 1-to-1* if the set of the points $x \in X$ such that $f^{-1}(f(x)) = \{x\}$ is dense in X; it is said to be *light* if pointwise preimages are 0-dimensional. We study almost 1-to-1 light maps of some compact and σ-compact spaces (e.g., n-manifolds or dendrites) and prove that in some important cases they must be homeomorphisms or embeddings. In a forthcoming paper we use these results and show that if f is a minimal self-mapping of a 2-manifold then point preimages under f are tree-like continua and either M is a union of 2-tori, or M is a union of Klein bottles permuted by f.

1. INTRODUCTION

A number of papers and even books are devoted to one-dimensional dynamics, i.e. to studying continuous maps of one-dimensional (branched) manifolds (interval, circle, "graphs"); as excellent sources we recommend the books [ALM01] and [BC92] and the references therein. However, with the exception of maps of one-dimensional manifolds, the dynamics of arbitrary continuous maps of manifolds is not extensively studied. This is quite understandable because continuity puts little restriction on maps of spaces of dimension higher than 1. Therefore any substantial study of continuous maps of manifolds is bound to begin with a list of restrictions which could be of smooth or topological nature. We are concerned with topological problems, so the former is not really applicable in our situation. The latter so far has been almost exclusively represented by the assumption that the map is a homeomorphism. This motivates us to study weaker conditions on continuous maps which would imply that a map is a homeomorphism or an embedding.

A natural choice of such conditions is given below. Let us say that a continuous map $f : X \to Y$ of topological spaces X, Y is *almost 1-to-1*

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if the set of the points $x \in X$ such that $f^{-1}(f(x)) = \{x\}$ is dense in X. Almost 1-to-1 maps have been studied before; by Whyburn ([Why42], VIII Theorem 10.2), a continuous onto map $f : X \to Y$ between compacta is almost 1-to-1 if and only if there are no closed proper subsets A of X with $f(A) \neq Y$ (Whyburn calls such maps *strongly irreducible*). Our interest in almost 1-to-1 maps is explained by the fact that they are natural candidates for being homeomorphisms/embeddings, but also by the fact that all minimal maps are almost 1-to-1 ([KST01]).

More precisely, a map $f : X \to X$ of a topological space X is said to be *minimal* if all orbits of points are dense (the *orbit* of a point x is the set $\{x, f(x), f^2(x), \dots\}$). Minimal maps are studied in a lot of papers and books (an excellent survey of this topic can be found in a 2001 paper in Fundamenta by Kolyada, Snoha and Trofimchuk, [KST01]). Until relatively recently it has been unknown whether there exist minimal maps which are not homeomorphisms. The first examples of non-invertible minimal maps are due to Auslander and Yorke [AY80] (see also [Ree79] as well as [KST01] in which ideas from [Ree79] are developed). Still, it was discovered in [KST01] that minimal (not necessarily invertible) maps resemble homeomorphisms in the sense that they are almost 1-to-1 thus justifying our interest in such maps.

The paper is arranged as follows. In Section 2 we prove basic results about almost 1-to-1 maps and study them for one-dimensional continua. Apart from being interesting by itself, this also serves as an important ingredient in the proof of our Main Theorem given in Section 3. Let us now state the Main Theorem; to do so recall that a map is said to be *light* if pointwise preimages are 0-dimensional.

Main Theorem. *Suppose that $f : M \to N$ is a light and almost 1-to-1 map from an n-manifold M into a connected n-manifold N. Then*

$$f|_{M \setminus \partial M} : M \setminus \partial M \to N$$

is an embedding. In particular, if M is a closed manifold, then f is a homeomorphism.

A map $f : X \to Y$ is called *nowhere 1-to-1* if for every open subset $U \subset X$ there exist $x_1 \neq x_2 \in U$ such that $f(x_1) = f(x_2)$. Phil Boyland asked the following question. Suppose that $f : M \to M$ is a nowhere 1-to-1 map of a closed 2-manifold. Does there exist a dense G_δ-set $D \subset M$ such that for every $d \in D$ the set $f^{-1}(d)$ is a Cantor set? We finish Section 3 by deducing from Theorem 3 the affirmative answer to this question.

To explain how these results apply to minimal maps we would like to mention that in a number of examples and theorems in [KST01], describing some classes of minimal maps of the 2-torus, point preimages are tree-like continua. A natural question then is whether this is a general property of minimal maps on the 2-torus, or more generally, on any 2-dimensional manifold. As it turns out, the Main Theorem together with some dynamical tools allow us to prove the following theorem which will appear in the forthcoming paper [BOT02].

Theorem 1.1. *Suppose that $f : M \to M$ is a minimal map of a 2-manifold. Then for every point $x \in M$ the set $f^{-1}(x)$ is a tree-like continuum and M is either a finite union of tori, or a finite union of Klein bottles which are cyclically permuted by f.*

Let us fix some general notation and terminology. All spaces are separable and metric. For a subset Y of a topological space X, we denote the *boundary* of Y by $\mathrm{Bd}(Y)$ and the *interior* of Y by $\mathrm{Int}(Y)$. A *continuum* is a compact and connected space. A locally connected continuum containing no subsets homeomorphic to the circle is called a *dendrite*. We rely upon the standard definition of a *closed n-manifold* (compact connected manifold without boundary); by a *manifold* we mean a manifold of any sort (closed, manifolds with boundary, open). In the case of a compact manifold M *with* boundary its boundary is called the *manifold boundary* of M and denoted by ∂M. Thus, if D is homeomorphic to the closed unit ball in a Euclidean space then $\mathrm{Bd}(D)$ is empty while ∂D is the corresponding unit sphere.

We thank the referees for making suggestions which helped improve our paper. For completeness we add proofs of some easy statements.

2. Light maps of one-dimensional continua

The central question about which our study revolves is the following one. Suppose that $f : X \to Y$ is a map of a σ-compact space X into a σ-compact space Y. What can be the set of points with unique preimages? Can we guarantee that if it is in some sense "big" in $f(X)$ then in fact it has to coincide with $f(X)$ and thus f has to be 1-to-1? Studying these questions for maps of one-dimensional continua leads to Theorem 2.4 which is later used in the proof of the Main Theorem.

Let us begin by studying properties of the set R_f of points of Y with unique preimages. For a map $f : X \to Y$ denote by D_f the set of points in X such that $f^{-1}(f(x)) = x$. Clearly $f(D_f) = R_f$ and $f^{-1}(R_f) = D_f$. The following lemma can be easily deduced from some well-known facts (see, e.g., [Why42], pp. 162–164).

Lemma 2.1. *Suppose that $f : M \to N$ is a continuous map of metric σ-compact spaces. Then the set R_f is a G_δ-subset of $f(M)$.*

By Lemma 2.1, D_f is always a G_δ subset of X. Lemma 2.2 shows in what way almost 1-to-1 maps (for which D_f is dense) are related to maps for which R_f is dense. To state it, we need the following definition: a map $f : X \to Y$ is said to be *quasi-interior* if for every non-empty open set $U \subset X$, the interior of $f(U)$ is not empty.

Lemma 2.2. *Suppose that $f : X \to Y$ is a closed map (e.g., this holds if X is compact). Then the following properties are equivalent:*

1. *D_f is dense in X,*
2. *R_f is dense in $f(X)$ and f is quasi-interior as a map from X to $f(X)$.*

In particular, if R_f is dense in Y then R_f is a dense G_δ-subset of Y.

Proof. Suppose that (1) holds. Since $R_f = f(D_f)$, then R_f is dense in $f(X)$. Suppose that U is an open subset of X, then there exists a point $d \in D_f \cap U$. Put $F = X \setminus U$. Then $f(d) \in Y \setminus f(F) \subset f(U)$ where $Y \setminus f(F)$ is open because $f(F)$ is closed (recall that f is a closed map). So the interior of $f(U)$ is non-empty and f is quasi-interior. Suppose next that (2) hold. Then, since f is quasi-interior and R_f is dense in $f(X)$, $D_f = f^{-1}(R_f)$ is dense in X. □

From now on we assume in this section that all topological spaces are metric and compact. We want to know when the fact that R_f is "big" implies that f is a homeomorphism/embedding. In general it is not always so. For instance, consider a non-strictly monotone surjective map $f : [0, 1] \to [0, 1]$ with one *flat spot* J. Then $f(J)$ is a point and $R_f = [0, 1] \setminus \{f(J)\}$ is very big, still the map f is far from being a homeomorphism. Judging from this example, a natural extra assumption on the map is that it is light. However, this is still not enough. Indeed, consider a map which identifies one pair of antipodal points of a circle and thus maps the circle onto a figure 8. This light map is of course 1-to-1 everywhere but at two points, but it is neither a homeomorphism nor an embedding. Hence, yet additional restrictions are needed. They could be of various nature; we begin by imposing more general ones and proving a useful Lemma 2.3.

Lemma 2.3. *Suppose that $f : X \to Y$ is a light map where X is a continuum. Suppose that the following holds:*

1. *for every $y \in f(X)$ there exists a sequence of sets K_i containing y whose diameters converge to 0 such that the boundary of every K_i consists of points of R_f (i.e. having a unique f-preimage);*
2. *for every K_i as above and for every component T of $f(X) \setminus \overline{K_i}$ the intersection $\overline{T} \cap \overline{K_i}$ consists of one point.*

Then f is an embedding (and so if R_f is dense in Y then f is a homeomorphism).

Proof. To prove the lemma assume by way of contradiction that there is a point $y \in Y$ which has two preimages u and v. Since the map f is light, there exists a positive ε such that any set of diameter less than ε containing y has a disconnected preimage. Choose a set K_i satisfying (1) for y; we may assume that K_i is of diameter less than ε. Consider its closure $\overline{K_i}$. Then the full preimage of $\overline{K_i}$ is disconnected and can be divided into two disjoint closed subsets R and S.

Consider the sets $R' = f(R) \cap \mathrm{Bd}(\overline{K_i})$ and $S' = f(S) \cap \mathrm{Bd}(\overline{K_i})$. Then by the assumptions all points of the boundary of $\overline{K_i}$ have unique preimages under f, and so R' and S' are disjoint. Now, the set $Y \setminus \mathrm{Int}(K_i)$ can be divided into components each of which either intersects R' at one point, or intersects S' at one point (this follows from (2)). The union of those components intersecting R' (resp. S') is denoted R'' (resp. S''), and for each such component its unique point of intersection with $R' \cup S' = \mathrm{Bd}(K_i)$ is called its *basepoint*.

Clearly the sets R'', S'' are disjoint. Let us show that R'' and S'' are closed. By way of contradiction suppose that there exists a sequence of components A_i from R'' with basepoints a_i such that some points $b_i \in A_i$ converge to a point $b \in S''$. Choose a subsequence of A_i such that $a_i \to a$. Since sets R', S' are closed and disjoint we see that $a \in R'$.

So, components A_i stretch between smaller and smaller neighborhoods of a and b. Hence a and b must belong to the same component of $Y \setminus \mathrm{Int}(K_i)$. Indeed, otherwise consider a separation of $Y \setminus \mathrm{Int}(K_i)$, i.e. two closed disjoint sets P and Q containing a and b respectively. Since A_i are connected, each of them must be contained in either P or Q. Choosing a subsequence we may assume that they all are contained in P, and, therefore, cannot approach $b \in Q$, a contradiction. However, $a \in R'$ and so all components of $Y \setminus \mathrm{Int}(()K_i)$ which have a as their basepoint are themselves contained in R'' while $b \in S''$ by the assumption. This contradiction implies that R'' and S'' are closed.

Consider now two subsets of X: the set $R \cup f^{-1}(R'')$ and the set $S \cup f^{-1}(S'')$. It follows that they are disjoint and closed while their union is X, a contradiction with the assumption that X is connected. □

The next theorem relies upon Lemma 2.3 and is later used in the proof of the Main Theorem. It is the main result of Section 2 which serves as a model for several forthcoming theorems, if not in terms of the method than at least in terms of the result. Lemma 2.3 also shows that for the topic discussed here the topology of the range is more important than that of the domain. To state it we need the following definition: a subset A of a topological space Y is said to be *con-dense* if any non-degenerate sub-continuum of Y contains a point of A.

Theorem 2.4. *Suppose that $f : X \to Y$ is a light map, X is a continuum and Y is a dendrite. Moreover, suppose that the set R_f is con-dense in $f(X)$. Then f is an embedding (so if R_f is also dense in Y then f is a homeomorphism of X onto Y).*

Proof. First observe that a set $A \subset Y$ is con-dense if and only if every subarc of Y contains a point of A. This follows from the fact that every non-degenerate continuum in Y must contain an arc.

Observe also that without the "con-density" assumption the conclusion of the theorem fails. Indeed, given a dendrite Y and an arc $I = [a, b]$ such that $Y \setminus I$ is dense in Y, let us construct a continuum X which is the union of Y and an arc J attached to Y at the point a. Then let us define the map f as the map $f : X \to Y$ which acts as the identity on Y and folds the arc J of X onto the arc I of Y. Then all points of Y but those of $I \setminus \{a\}$ have one preimage while the points of $I \setminus \{a\}$ have two preimages. In other words, although the set $R_f = (Y \setminus I) \cup \{a\}$ is dense in Y, the map f is not an embedding. Thus, the assumption that any arc contains a point of R_f is necessary, and we need to prove that it is sufficient.

The idea of the proof is to apply Lemma 2.3. To do so we need to verify its conditions. First we check that for every point $y \in Y$ there exists a sequence of sets K_i containing y whose diameters converge to 0 such that the boundary of every K_i consists of points having a unique preimage under f. Indeed, given $\varepsilon = 1/i$ choose a neighborhood W_i of y of diameter less than ε such that its boundary is a finite collection of points $y_1, \dots, y_n$ (this can be done because Y as a dendrite has a basis of neighborhoods each of which has finite boundary). Then for each j we have $[y, y_j] \subset W_i$ and by the assumption we can choose points $y'_j \in [y_j, y), 1 \le j \le n$, which have a unique f-preimage. Denote by K_i the component of $Y \setminus \{y'_1, \dots, y'_n\}$ containing y. Clearly, the diameter of $\overline{K_i}$ is at most ε and the boundary of K_i is the set $\{y'_1, \dots, y'_n\} \subset R_f$.

It remains to check that for every component T of $Y \setminus \overline{K_i}$ the intersection $\overline{T} \cap \overline{K_i}$ consists of one point. Indeed, by the Boundary Bumping Theorem the intersection $\overline{T} \cap \overline{K_i}$ is always non-empty. If it contains two points then there exists a path connecting them inside $\overline{T}$ as well as inside $\overline{K_i}$, a contradiction. Hence all the conditions of Lemma 2.3 are satisfied and so our Theorem 2.4 holds. □

For the sake of completeness and to pose some open problems we would like to finish this section by mentioning other types of one-dimensional continua for which similar questions can be raised. Indeed, the dendrites are a particular case *dendroids* defined as arcwise connected and hereditarily unicoherent continua (a continuum X is *hereditarily unicoherent* if the intersection of any two subcontinua is connected). Clearly, every dendrite is a dendroid but not otherwise. In the case of dendroids the situation with almost 1-to-1 maps is more complicated; Proposition 2.5 below is our only result in this direction.

Proposition 2.5. *Suppose that $f : X \to Y$ is a light map from an arcwise connected continuum X onto a dendroid Y. Moreover, suppose that the set R_f is con-dense. Then f is a homeomorphism.*

Proof. If there are points $x \neq z \in X$ with $f(x) = f(z) = y$ then we connect x and z in X by an arc $[x, z]$. Then $g = f|_{[x,z]} : [x, z] \to f([x, y])$ is a light map from a continuum onto a dendrite $g([x, y])$ such that D_g is con-dense. By Theorem 2.4 g is an embedding, a contradiction. □

We do not know if the assumption that X is arcwise connected can be omitted, or replaced by the assumption that X is hereditarily decomposable.

3. Light almost 1-to-1 maps of manifolds

The aim of this section is to prove the Main Theorem. We also answer the question of Phil Boyland, quoted in Introduction. To begin with observe that since light maps of manifolds do not lower dimension we have the following useful lemma.

Lemma 3.1. *A light map of an n-dimensional manifold X into an n-dimensional manifold Y is quasi-interior. In particular, the interior of $f(X)$ is dense in $f(X)$ and all relatively open subsets of $f(X)$ have non-empty interior in Y.*

Below we need the following definitions. A map $f : X \to Y$ from a Hausdorff space X to a Hausdorff space Y is *weakly-confluent* provided for every continuum $K \subset f(X)$ there exists a component C of $f^{-1}(K)$ such that $f(C) = K$. A map $f : X \to Y$ from a Hausdorff space X to a Hausdorff space Y is *perfect* if for every compact set $A \subset Y$ its preimage $f^{-1}(A)$ is compact too.

Let us now list some results of [Why42] concerning strongly irreducible maps and some results of [KST01] concerning minimal maps (these results were briefly mentioned in Introduction). In Theorem VIII 10.2 [Why42] it is shown that an onto map $f : A \to B$ on a compact space A is strongly irreducible if and only if D_f is dense in A. Moreover, a Corollary on the same page states that if f is also open then it is a homeomorphism. It is proven in [KST01] that any minimal map in a compact Hausdorff space is quasi-interior; if in addition the map is open then it is shown to be an onto homeomorphism. Moreover, it is proven in [KST01] that for any minimal map $f : X \to X$ of a compact metric space X into itself, the set R_f is a dense G_δ-subset of X.

The following lemma extends the above quoted corollary [Why42] according to which an open strongly irreducible map is a homeomorphism onto locally connected spaces and serves as a useful tool. Observe that instead of open maps we consider quasi-interior maps, but on the other hand we use some extra-assumptions.

Lemma 3.2. *Suppose that $g : X \to Y$ is a weakly confluent, light, perfect and almost 1-to-1 mapping of a locally compact space X onto a locally compact and locally connected space Y (in particular, this is so if X is a continuum and Y is a locally connected continuum). Then g is a homeomorphism.*

Proof. We show that g is closed. Indeed, let $F \subset X$ be closed. To see that $g(F)$ is closed it is enough to show that if $g(x_i) \to y$ with $x_i \in F$, then $y \in g(F)$. By the assumptions y has a compact neighborhood W whose full preimage is compact. Since $x_i \in g^{-1}(W)$ then we can choose a subsequence of x_i converging to some point x. Since F is closed then $x \in F$ and by continuity $f(x) = y$ as desired.

Hence it suffices to show that g is 1-to-1. Suppose that g is not 1-to-1. Then there exists a point $y \in Y$ such that $g^{-1}(y)$ contains at least two points. Since g is light and Y is locally connected and compact there exists a sufficiently small open and connected neighborhood V of y such that if $C = \overline{V}$, then $\overline{C}$ is compact and $g^{-1}(C)$ has at least two components which meet $g^{-1}(y)$. Since g is weakly confluent there exists a component K of $g^{-1}(C)$ such that $g(K) = C$. Then $g^{-1}(C \cap D_g) \subset K$. Let x be a point in $g^{-1}(y) \setminus K$ and let U be an open neighborhood of x such that $U \cap K = \emptyset$ and $g(U) \subset V$. Since g is quasi-interior, $g(U) \cap D_g \neq \emptyset$ contradicting the fact that $g^{-1}(V \cap D_g) \subset K$. □

Note that weak confluence is essential in Lemma 3.2 as there is a light, almost 1-to-1, quasi-interior map of the interval $[0, 1]$ onto the two-dimensional disk. In fact, a significant part of the proof of the claim (1) of the Main Theorem is to show that in the case of manifolds almost 1-to-1 and light maps are weakly confluent. So it makes sense to consider some examples in low dimensions which allow us to introduce all necessary techniques dealing with the property of weak confluence.

Let us show (the well-known fact) that every map f of a continuum X onto $[0,1]$ is weakly confluent (given $a<b$, consider the components of the set $F=f^{-1}([a,b])$ containing a, the components of F, containing b and observe that if no component is common for these two families, then X can be shown to be non-connected, a contradiction). In the case of n-manifolds weak confluence does not follow this easily. To show that in some case maps of n-manifolds are weakly confluent, we need the following technical definition. Let X be a closed subset of a connected n-manifold N and let $F\subset X$ be dense in X. Let B be a closed n-disk in the interior of N and let S^{n-1} be its boundary sphere. We say B is an (F,X)-*basic disk* and S^{n-1} is an (F,X)-*basic sphere* if S^{n-1} separates X and $F\cap S^{n-1}$ is dense in $X\cap S^{n-1}$.

In the Lemma below we show that basic (F,X)-disks are ubiquitous.

Lemma 3.3. *Suppose that N is a n-manifold, $X\subset N$ is a non-degenerate compact set and $F\subset X$ is dense in X. Then for any $\varepsilon>0$ and any n-disk $B\subset N\setminus\partial N$ such that $X\cap Int(B)\neq\emptyset\neq X\setminus B$, there exists a (F,X)-basic n-disk D such that the Hausdorff distance $H(B,D)<\varepsilon$.*

Proof. We may assume that X is a compact subset of an open subset $U\subset\mathbb{R}^n$ and S_0 is the boundary of a closed n-disk D_0 with $\mathrm{Int}(D_o)\cap X\neq\emptyset\neq X\setminus D_0$. We may assume that $X\cap S_0\neq\emptyset$ (otherwise we are done). Since F is dense in X, we may assume that actually $F\cap S_0\neq\emptyset$ by adjusting D_0 slightly if necessary.

Let P_0 be the closure of $F\cap S_0$ and assume that $(X\cap S_0)\setminus P_0\neq\emptyset$. Choose $x_1\in(X\cap S_0)\setminus P_0$ such that the distance $d(x_1,P_0)=r_1$ is maximal. Since F is dense in X, there exists arbitrarily close to x_1 a point $f_1\in F$. Hence we can slightly modify our disk D_0 only inside $B(x_1,r_1/2)$ to a disk D_1 such that $f_1\in S_1=\partial D_1$. Observe that all points in the set $B(x_1,r_1/2)\cap S_1$ are at most $.75r_1$-distant from $f_1\in F$. Also, the set $F\cap S_1\supset F\cap S_0$.

By inductively choosing $x_n,D_n,S_n=\partial(D_n),P_n$ and r_n as above we either have $S_n\cap X=P_n$ for some n or $\lim r_n=0$. Indeed, the sequence r_n is decreasing by the construction. Suppose that $\lim r_n=r>0$ and consider the sequence of points x_i. A subsequence of points x_{i_k} will converge to some point x. Then by the previous paragraph the points of the set $B(x_{i_k},r_{i_k}/2)\cap S_{i_k}$ are at most $.75r_{i_k}$-distant from $f_{i_k}\in F$. This leads to a contradiction. Thus $r_n\to 0$. By choosing S_n sufficiently close to one another we can assure that $\lim D_n=D_\infty$ is a n-disk with boundary S_∞ such that $S_\infty\cap F$ is dense in $S_\infty\cap X$ as required. □

Observe that Lemma 3.3 applies in the case when $X = N$. Now we prove the Main Theorem; for convenience we restate it here.

Main Theorem. *Suppose that $f : M \to N$ is a light and almost 1-to-1 map from an n-manifold M into a connected n-manifold N. Then*

$$f|_{M\setminus\partial M} : M \setminus \partial M \to N$$

is an embedding. In particular, if M is a closed manifold, then f is a homeomorphism.

Proof. Assume first that M is a closed n-manifold. We begin by proving that f is weakly confluent (and, hence, onto). If $n = 1$ then M is a circle and it is easy to see that f is one-to-one. Hence, let us assume that $n \geq 2$.

The following claim immediately implies that f is an onto map. Ultimately it will enable us to prove that f is weakly confluent.

Claim A. *For any $(R_f, f(M))$-basic sphere S^{n-1} in N there exists a continuum C' in M such that $f(C') = S^{n-1}$.*

Proof of Claim A. Suppose S^{n-1} is a $(R_f, f(M))$-basic sphere. Then $S = f^{-1}(S^{n-1})$ separates M. Hence there exists $0 \neq g_{n-1} \in \check{H}^{n-1}(S)$, where we use Čech cohomology. By continuity of Čech cohomology, there exists a minimal closed subset T^{n-1} of S such that g_{n-1} is not homologous to zero on T^{n-1}. Since $n \geq 2$, T^{n-1} is a connected $(n-1)$-Cantor manifold. Hence, since f is light, $f|_{T^{n-1}} : T^{n-1} \to S^{n-1}$ is quasi-interior and almost one-to-one.

We claim that $f(T^{n-1}) = S^{n-1}$. If $n = 2$, this follows immediately from Theorem 2.4 since $\check{H}^1(T^1) \neq 0$. Hence assume $n \geq 3$ and by way of contradiction suppose $f(T^{n-1})$ is a proper non-degenerate subset of S^{n-1}. By Lemma 3.3, there exists a $(R_f, f(T^{n-1}))$-basic $(n-2)$-sphere $S^{n-2} \subset S^{n-1}$ such that $S^{n-2}\setminus f(T^{n-1}) \neq \emptyset$. Then $T = f^{-1}(S^{n-2}) \cap T^{n-1}$ separates T^{n-1}.

Let $T^{n-1} \setminus T = U \cup V$ where U and V are non-empty, disjoint and open sets in T^{n-1}. Let $A = U \cup T$ and $B = V \cup T$, then $A \cup B = T^{n-1}$ and $A \cap B = T$. Since T^{n-1} is minimal, g_{n-1} is homologous to zero on each of A and B. By Mayer-Vietoris,

$$\check{H}^{n-2}(T) \to \check{H}^{n-1}(T^{n-1}) \to \check{H}^{n-1}(A) \oplus \check{H}^{n-1}(B)$$

is exact. Hence there exists $g_{n-2} \in \check{H}^{n-2}(T)$ which maps to g_{n-1}. So $0 \neq g_{n-2}$. By continuity of Čech cohomology there exists a minimal closed set T^{n-2} in T such that g_{n-2} is not homologous to zero on T^{n-2}. As above, T^{n-2} is a connected $(n-2)$-Cantor manifold and $f|_{T^{n-2}} : T^{n-2} \to S^{n-2}$ is almost one-to-one.

This describes the induction step. Inductively, we can construct a sequence of spheres S^{n-i} and connected $(n-i)$-Cantor manifolds T^{n-i} for $i = 1, \dots, n-1$ such that the following hold:

(1) $f(T^{n-i})$ is a proper subset of S^{n-i};
(2) $f|_{T^{n-i}} : T^{n-i} \to S^{n-i}$ is almost one-to-one;
(3) there is $0 \neq g_{n-i}$ in $\check{H}^{n-i}(T^{n-i})$ such that g_{n-i} is 0 when restricted to any proper closed subset of T^{n-i};
(4) S^{n-i-1} is a $(R_f, f(T^{n-i}))$-basic $(n-i-1)$-sphere in S^{n-i}.
(5) $T^{n-i-1} \subset T^{n-i} \cap f^{-1}(S^{n-i-1})$.

Thus for $i = n-1$ the properties (1)-(3) are translated into the following:

(1) $f(T^1)$ is a proper subset of S^1;
(2) $f|_{T^1} : T^1 \to S^1$ is almost one-to-one;
(3) $\check{H}^1(T^1) \neq 0$.

By (1) the set $f(T^1)$ is an interval. By Theorem 2.4 $f|_{T^1}$ is an embedding and so T^1 is an interval, a contradiction with (3). This completes the proof of Claim A. In particular, $f : M \to N$ is an onto map.

Next let K be an arbitrary subcontinuum of N. Observe that K can be approximated by (R_f, N)-basic spheres C_i. By Claim A, for every i there exists a subcontinuum D_i of M such that $f(D_i) = C_i$. We may assume that $\lim D_i = D_\infty$ is a continuum. Then $f(D_\infty) = \lim C_i = K$. This shows that f is weakly confluent. Thus f is onto, by Lemma 3.1 f is quasi-interior, and now Lemma 3.2 it is a homeomorphism.

Our next step is to deal with the remaining case of the Main Theorem. We begin by considering the case of the closed disk. Namely, suppose that $f : D \to N$ is a light and almost 1-to-1 mapping from the closed n-disk into a n-manifold N. We want to show that then f restricted to $D \setminus \partial D$ is an embedding.

The case $n = 1$ is trivial, so let $n \geq 2$. Denote the set R_f by F. By Lemma 2.2 $f^{-1}(F)$ is dense in D and, hence, $f^{-1}(f(\partial D))$ is nowhere dense in D. Then $I = D \setminus f^{-1}(f(\partial D))$ is dense and open in D.

Let J be a component of I. Then $f|_J : J \to f(J)$ is perfect since $f(J) = f(\overline{J}) \setminus f(\partial D)$ is locally compact. Moreover, $f(J)$ is open in $f(\overline{J})$. By the proof of claim (1), $f(J)$ is open in N. (In Claim A of the proof of claim (1) choose basic $(R_f, f(\overline{J}))$-sphere to lie in $N \setminus \partial N \cup f(\partial D)$). As in the proof of claim (1) f restricted to J is weakly confluent. By Theorem 3.2 f restricted to J is an embedding. Since f is almost one-to-one it follows that f restricted to I is an embedding.

Let $x \in f^{-1}(f(\partial D)) \setminus \partial D$. Then since $f^{-1}(f(x))$ is 0-dimensional and closed there exists a closed disk D' in D such that D' contains all of D except for an arbitrary small neighborhood of $f^{-1}(f(x)) \cap \partial D$ and the boundary of D' misses $f^{-1}(f(x))$. Let $I' = D' \setminus f^{-1}(f(\partial D'))$. Then $x \in I'$ and f restricted to I' is an embedding as above. It follows that f restricted to $D \setminus \partial D$ is an embedding. Since every pair of points in $M \setminus \partial M$ are contained in the interior of a closed D in M, the rest of the Main Theorem easily follows. □

Let us make a few remarks concerning the Main Theorem. The assumption that both topological spaces are manifolds is essential here: the map which pinches a circle S at one point and thus maps it onto a space B homeomorphic to a figure 8 is almost 1-to-1, but not even weakly confluent. Also, the standard maps from the 2-disk onto the 2-torus or onto the 2-cylinder show the difficulties of extending the homeomorphism on $M \setminus \partial M$ over the boundary. Another example is an embedding of the open 2-disk into itself which bypasses a radial cut; if extended to the closed disk, it will identify two adjacent arcs on the boundary into that radial cut. All this shows difficulties in strengthening the Main Theorem.

We finish the paper by showing an application of our results. First though we need a general result about light maps of continua.

Lemma 3.4. *Suppose that $f : X \to Y$ is a light mapping from a continuum X onto a continuum Y.*

Then either:

(1) *there exists an open set $U \subset X$ and a dense G_δ subset D' of $f(U) \subset Y$ such that for all $y \in D'$,*
$$|f^{-1}(y) \cap U| = 1, \text{ or}$$
(2) *there exists a dense G_δ-subset D of Y such that for each $y \in D$, $f^{-1}(y)$ is homeomorphic to the Cantor set.*

Proof. Let $g : X \to 2^Y$ be the function defined by $g(y) = f^{-1}(y)$. Then g is upper semi continuous. Hence there is a dense G_δ set $E \subset Y$ such that g is continuous at each point of E. So for each $y \in E$ and each sequence $y_n \to y$, $\lim f^{-1}(y_n) = f^{-1}(y)$. Let $\mathcal{B} = \{B_n\}$ be a countable basis in X, set $F_n = \{y \in E | \, |f^{-1}(y) \cap B_n| = 1\}$ and $E_n = E \setminus F_n$.

We claim that E_n is a G_δ-subset of E. To see this note that $E_n = (P_n \cap E) \cup (R_n \cap E)$ where $P_n = \{y \in Y \, |g(y) \cap B_n = \emptyset\}$ and $R_n = \{y \in Y | \, |g(y) \cap B_n| \geq 2\}$. It follows easily from the continuity of g at each point of E that R_n is an open subset of Y and that P_n is a G_δ-subset of Y. Hence indeed E_n is a G_δ- subset of Y. Consider the set $D = \cap E_n$. Points of D are such that their preimages intersect any open set either in the empty set or in sets of cardinality greater than 2. Thus, preimages of points of D have no isolated points, and since f is light all such preimages are Cantor sets.

If each E_n is dense, then D is a dense G_δ-set and (2) holds. Otherwise there exists an n and an open set $V \subset F_n$. So, if $U = f^{-1}(V) \cap B_n$, and $D' = E \cap f(U)$, then D' is a dense G_δ-set in $f(U)$ and for each $y \in D'$, $f^{-1}(y) \cap U = 1$ and (1) holds. Observe that the set D above may be empty in general (even for an irreducible, nowhere locally one-to-one and light map from the circle onto a locally connected continuum). □

To apply our results we need the following definition: a continuum X is called a *local tree* if for each $x \in X$ there exists a neighborhood U of x such that the closure of U is a finite tree. Phil Boyland asked a question as to whether point preimages of a dense G_δ-set under a nowhere 1-to-1, light map of manifolds or local trees are homeomorphic to the Cantor set. The following theorem answers this question; its proof uses notation introduced in the proof of Lemma 3.4.

Theorem 3.5. *Suppose that X and Y are continua and $f : X \to Y$ is a light and nowhere 1-to-1 map and one of the following holds:*

(1) *Y is a local tree;*
(2) *X and Y are n-manifolds.*

Then there exists a dense G_δ-subset D of Y such that for each $y \in D$, $f^{-1}(y)$ is homeomorphic to the Cantor set.

Proof. Suppose first that Y is a local tree. It remains to be shown that the set D constructed in the proof of Lemma 3.4 is dense in Y. Hence assume that there exists an n such that E_n is not dense. Then there exists an open set O, such that for each $y \in O \cap E$, $|g(y) \cap B_n| = 1$. Recall that a free arc $[a, b] \subset X$ is an arc such that (a, b) is an open subset of X. Clearly every open set in a local tree contains a free arc J. Since E is dense in Y we may assume that O itself is a free arc; we may assume that O does not contain its endpoints.

Let $x \in B_n$ and $f(x) \in E \cap O$. Choose a very small non-degenerate continuum K such that $x \in K \subset B_n$. Then $f(K)$ is a non-degenerate subarc of O. By Theorem 2.4 $f|_K$ is an embedding. Using interval notation we may assume that $K = [a, b]$ and $f(K) = [f(a), f(b)]$. Let us show that (a, b) is open in B_n. To this end we show that $(a, b) = f^{-1}(f(a), f(b)) \cap B_n$. Indeed, otherwise there is a point $z \in B_n \setminus K$ with $f(z) \in (f(a), f(b))$. Choose a small continuum $K' \subset B_n \setminus K$ so that $f(K') \subset (f(a), f(b))$. Since f is light, $f(K')$ is a non-degenerate subarc of $(f(a), f(b))$. Clearly, all points of $f(K')$ have at least two preimages in B_n, a contradiction with the above made assumption that for each $y \in O \cap E$, $|g(y) \cap B_n| = 1$. Hence (a, b) is open which in turn contradicts the assumption that f is nowhere 1-to-1.

Suppose next that X is a compact n-manifold. Then the possibility (1) from Theorem 3.4 is ruled out by the Main Theorem, and the assumption that f is nowhere 1-to-1, hence (2) must hold. $\square$

Note that the assumptions stated above are needed. For example, it is easy to construct a dendrite $D \subset S^2$ such that the branchpoints and endpoints are dense in D and all branchpoints are of valence exactly 3. Consider the Riemann map f from the unit disk onto $S^2 \setminus D$. Since D is a dendrite, f can be extended to the map $\hat{f}$ of the closed unit disk to S^2. The restriction $\hat{f}|_{S^1} : S^1 \to D$ of $\hat{f}$ onto the unit circle is almost 1-to-1, nowhere 1-to-1 and at most 3-to-1. Similar examples are frequent in the study of polynomial Julia sets.

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