ON CONFLUENTLY GRAPH-LIKE COMPACTA

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ABSTRACT. For any class $\mathcal{K}$ of compacta and any compactum X we say that: (a) X is a *confluently $\mathcal{K}$-representable*, if X is homeomorphic to the inverse limit of an inverse sequence of members of $\mathcal{K}$ with confluent bonding mappings, and (b) X is *confluently $\mathcal{K}$-like*, provided that X admits, for every $\varepsilon > 0$, a confluent ε-mapping onto a member of $\mathcal{K}$. The symbol $\mathbb{LC}$ stands for the class of all locally connected compacta. It is proved in this paper that for each compactum X and each family $\mathcal{K}$ of graphs, X is confluently $\mathcal{K}$-representable if and only if X is confluently $\mathcal{K}$-like. We also show that for any compactum the properties of: (1) being confluently graph-representable, and (2) being 1-dimensional and confluently $\mathbb{LC}$-like, are equivalent. Consequently, all locally connected curves are confluently graph-representable. We also conclude that all confluently arc-like continua are homeomorphic to inverse limits of arcs with open bonding mappings, and, all confluently tree-like continua are absolute retracts for hereditarily unicoherent continua.

1. INTRODUCTION

Confluent mappings for compacta, defined by J. J. Charatonik in [6] (very similar classes of mappings were earlier studied in [20]), turned out to be very important. They form a much narrower family than the one of all continuous mappings. However, this family is essentially larger than any of the following classes of: covering, branch-covering, open, or monotone mappings. Yet confluent mappings share some important properties with those listed above. They preserve end points, atriodicity, tree-likeness [13], the property of Kelley and many other topological properties of compacta. They even have (approximate) path lifting properties [7]. The class of confluent mappings is, in a sense, more "regular" than some other well known classes of mappings. For instance, while the inverse limit of open mappings is not necessarily open, the inverse limit (or even the weakly induced limit in the sense of Mioduszewski [14]) of confluent mappings is always confluent (cf. [9, 17, 10]).

The main purpose of this paper is to study *confluently graph-like* compacta and to generalize known results for continuous functions to the class of confluent mappings. A compactum X is said to be *confluently graph-like* provided that for every $\varepsilon > 0$ there exists a confluent ε-mapping from X onto a graph. There are many important examples of confluently graph-like compacta. The Menger curve and the Sierpiński universal plane curve

are such spaces. More generally, in this paper we prove that all locally connected, 1-dimensional compacta are confluently graph-like (see Corollary 3.15). The solenoids are confluently graph-like (precisely, they form the class of all confluently circle-like continua, see [3]). Inverse limits of arcs with open bonding mappings (called "Knaster type continua") are, obviously, confluently graph-like. Actually, this last class is the class of all confluently arc-like continua (see Corollary 3.4 below). "Case continua" [1] and their generalizations [18], [15] are also confluently graph-like. In a recent paper [4] it was proved that every inverse limit of trees with confluent bonding mappings is an absolute retract for hereditarily unicoherent continua, and thus it has many other strong properties including the fixed point property (see [7], [2], [5] and [4]). In this paper we show that every confluently tree-like continuum is such a retract (see Corollary 3.7).

Confluently graph-like compacta share many properties with the members of a larger class of all confluently $\mathbb{LC}$-like compacta, i.e., compacta admitting, for every $\varepsilon > 0$, a confluent ε-mapping onto a locally connected compactum ($\mathbb{LC}$ stands for the class of all locally connected compacta). For instance such compacta have the arc property of Kelley [3] (cf. Remark 1.1). This last property, satisfied by all locally connected compacta and all absolute retracts for hereditarily unicoherent continua, is interesting by its own right. Confluent mappings on such compacta are known to have some approximate lifting properties [7] (cf. Remark 1.1).

Remark 1.1. Actually, the main results of papers [3] and [7] are formulated for continua, i.e. connected compacta. However, those mentioned above remain true also for non-connected compacta. The proofs are the same.

Let $\mathcal{K}$ be a class of compacta and X be a compactum. We say that X is *confluently $\mathcal{K}$-like* provided that X admits, for every $\varepsilon > 0$, a confluent ε-mapping onto a member of $\mathcal{K}$. If X is homeomorphic to the inverse limit of an inverse sequence of members of $\mathcal{K}$ with surjective confluent bonding mappings, then X is called a *confluently $\mathcal{K}$-representable* compactum.

Let $\mathcal{K}$ be a class of compacta. It is an easy observation that if a compactum X is confluently $\mathcal{K}$-representable, then X is confluently $\mathcal{K}$-like. In this paper we ask for what classes $\mathcal{K}$ the converse is true. The following question is crucial to this paper.

Question 1.1. *Let $\mathcal{K}$ be any of the classes of: polyhedra, ANR's, locally connected compacta, and suppose X is a confluently $\mathcal{K}$-like compactum. Does it follow that X is confluently $\mathcal{K}$-representable?*

In this paper we answer this question in the affirmative for each 1-dimensional compactum X, and this is the main result of this paper. Within the class of continuous functions a similar result was obtained by Mardešić and Segal [12] in 1963. Using this result we obtain a number of conclusions, some of them mentioned above.

Answering Question 1.1 for compacta of dimension greater than 1 is the most important problem of this paper that remains open.

All results described above are presented in Section 3. Section 2 contains some results concerning confluent mappings from locally connected continua

onto an arc. Namely, we prove a "confluent" Urysohn's lemma (Theorem 2.12) and confluent extension and retraction theorems (Theorem 2.14 and Corollary 2.15).

Spaces are assumed to be metric and mappings to be continuous in this paper. A (compact) polyhedron of dimension at most 1 is called a *graph*. If $f : X \to Y$ is a mapping, ε is a positive number and $\mathrm{diam}(f^{-1}(y)) < \varepsilon$ for each $y \in Y$, then f is called an ε-*mapping*. A mapping $f : X \to Y$ is said to be *confluent* provided that for every subcontinuum K of Y and every component L of $f^{-1}(K)$ we have $f(L) = K$. All confluent mapping considered in this paper are surjective and their domain and range spaces are compact. A continuum X is called *arc-like* (*circle-like*, *tree-like*, *graph-like*), if, for every $\varepsilon > 0$, there exists an ε-mapping $f_\varepsilon : X \to Y_\varepsilon$, where Y_ε is an arc (a circle, a tree, a graph; respectively). If, additionally, the mappings f_ε are confluent, then we say that X is *confluently arc-like* (*confluently circle-like*, *confluently tree-like*, *confluently graph-like*; respectively). If X is homeomorphic to the inverse limit $\lim_\leftarrow(Y_n, f_n^{n+1})$, where the mappings : $f_n^{n+1} : Y_{n+1} \to Y_n$ are confluent and surjective, and spaces Y_n are arcs (circles, trees, graphs), then X is called a *confluently arc-representable* (*confluently circle-representable*, *confluently tree-representable*, *confluently graph-representable*; respectively). A continuum X is said to be *hereditarily unicoherent* provided that the intersection of every two subcontinua of X is connected. If a hereditarily unicoherent continuum X, whenever embedded into another hereditarily unicoherent continuum Y, is a retract of Y, then we say that X is an *absolute retract for hereditarily unicoherent continua*. A compactum X is said to have the *arc property of Kelley* provided that for every continuum $K \subset X$, every $p \in K$ and every sequence p_n converging to p in X there exists a sequence of arcwise connected continua K_n such that $p_n \in K_n$ and K_n converges to K in the sense of the Hausdorff distance.

2. "CONFLUENT" URYSOHN'S LEMMA, CONFLUENT EXTENSIONS AND RETRACTIONS FOR LOCALLY CONNECTED CONTINUA

In this section we study confluent mappings of locally connected continua onto the unit interval $[0, 1]$. We will show a stronger, "confluent" variant of the Urysohn lemma for locally connected continua (see Theorem 2.12). This theorem will be applied in the next section. Using similar technique we also obtain extension and retraction theorems for confluent mappings (Theorem 2.14, Corollary 2.15). We add these two last results for completeness of this study. Some results of a similar type concerning classes of mappings related to confluent ones can be found in [21].

To obtain these results we need some preparation. Let X be a continuum. For any surjective mapping $g : X \to [0, 1]$ and any $a \in (0, 1)$ let $F(g, a)$ be the union of all components K of $X \setminus g^{-1}(a)$ such that:

$$\text{either} \quad 0 < \inf(g(K)) < a, \quad \text{or} \quad a < \sup(g(K)) < 1.$$

Lemma 2.1. *For any surjective mapping* $g : X \to [0, 1]$*, where* X *is a continuum, the following conditions are equivalent:*

(1) *The mapping g is confluent.*
(2) *For any $a \in (0,1)$, any component C of $g^{-1}(a)$ and any $\varepsilon > 0$ there exists a continuum $L \subset N_\varepsilon(C)$ such that $C \subset L$ and $a \in Int(g(L))$.*
(3) *The set $F(g,a)$ is empty for each $a \in (0,1)$.*

Proof. Let $g = f \circ m$ be the canonical representation of g as the composition of a monotone mapping m and a light mapping f. Assume that g is confluent, then it easy to see that f is also confluent. By Lelek and Read [11], f is open. We show next that condition (2) is equivalent to the openness of f. Suppose f is open and that $a \in (0,1)$ and C is a component is of $g^{-1}(a)$. Let $J_n = [a - 1/n, a + 1/n] \subset (0,1)$ and let H_n be the component of $f^{-1}(J_n)$ containing $m(C)$. Then $f(H_n) = J_n$. Let $L_n = m^{-1}(H_n)$, then $\cap L_n = C$ and $g(L_n) = J_n$. Hence, for n sufficiently large, $L_n \subset N_\varepsilon(C)$ and (2) holds. Conversely, if (2) holds, then for each $z \in m(X)$ and each open set $U \subset m(X)$ containing z, $f(z) \in Int(f(U))$ and f is open. Thus (1) is equivalent to (2).

Suppose (3) does not hold, i.e. there exists an $a \in (0,1)$ with $F(g,a) \neq \emptyset$. Thus there is a component K of $X \setminus g^{-1}(a)$ such that either $0 < \inf(g(K)) < a$, or $a < \sup(g(K)) < 1$. Suppose $0 < \inf(g(K)) < a$ (the other case is similar). Let $b \in K$ be such that $c = g(b) = \inf(g(K))$. Let C be the component of $g^{-1}(c)$ that contains b. Then for any continuum L that contains C and is sufficiently close to C in the sense of the Hausdorff distance, $g(L) \subset K$. Thus $g(L) \subset [c, 1]$ and $c \notin Int(g(L))$ for such L. Thus (2) does not hold. Hence (2) implies (3).

Suppose (2) does not hold, i.e. there are $a \in (0,1)$, a component C of $g^{-1}(a)$ and an $\varepsilon > 0$ such that for every continuum $L \subset N_\varepsilon(C)$ with $C \subset L$ we have $a \notin Int(g(L))$. Since $g(L)$ for $L \neq C$ is an interval, then a is either left hand end point or right hand end-point of $g(L)$ for all such L. Assume the former case (the latter one is similar). Then for sufficiently small $\delta > 0$, letting $a' = a + \delta$ the set $g^{-1}(a')$ separates X between C and any point $p \in X$ with $g(p) < a$. Hence C is a subset of a component K of $X \setminus g^{-1}(a')$ such that $0 < \min g(K) < a'$ and thus $F(g, a') \neq \emptyset$. So (3) does not hold. This completes the proof of implication from (3) to (2) and of the entire lemma. $\square$

Suppose that $g : X \to [0,1]$ is a surjective mapping and X is a locally connected continuum. For any $a \in (0,1)$ and any $x \in F(g,a)$ we let $g^{(a)}(x) = a$, and, for $x \in X \setminus F(g,a)$ let $g^{(a)}(x) = g(x)$. By the local connectedness of X the set $N_\varepsilon(g^{-1}(a))$ contains, for every $\varepsilon > 0$, almost all components of $X \setminus g^{-1}(a)$. In particular, this set contains almost all components K of $F(g,a)$. This observation leads to the conclusion that $g^{(a)} : X \to [0,1]$ is a continuous mapping. Observe that $F(g,a) = \{x \in X : g^{(a)}(x) \neq g(x)\}$. Define $I(g) = \{a \in (0,1) : g^{(a)} = g\} = \{a \in (0,1) : F(g,a) = \emptyset\}$.

In Propositions 2.2-2.11 we assume that X is a locally connected continuum and $g : X \to [0,1]$ is a surjective mapping. By Lemma 2.1 we observe that such a mapping g is confluent if and only if $I(g) = (0,1)$.

Proposition 2.2. *If g is not confluent, then there exists an $\varepsilon > 0$ and an open interval $(p,q) \subset [0,1]$ such that for every surjective mapping $h : X \to [0,1]$ with $d_{sup}(h,g) < \varepsilon$ we have $(p,q) \cap I(h) = \emptyset$.*

Proof. Since g is not confluent, then there exists an number $a \in (0,1)$ such that $g^{(a)} \neq g$, or equivalently, $F(g,a) \neq \emptyset$. Let K be a component of $F(g,a)$. Then either $0 < \min g(K) < a$, or $a < \max g(K) < 1$. Assume $0 < \min g(K) < a$ (the proof in the other case is similar). Fix a point $x_0 \in K$ such that $g(x_0) = \min g(K)$ and two numbers $p, q \in (0,1)$ satisfying $g(x_0) < p < q < a$. Put $\varepsilon = \frac{1}{2} \min \{|g(x_0) - 0|, |p - g(x_0)|, |a - q|\}$, and let $h : X \to [0,1]$ be a surjective mapping such that $d_{sup}(h,g) < \varepsilon$.

Fix a number $r \in (p,q)$. We will complete the proof by showing that $r \notin I(h)$. Indeed, since h is ε-near to g we have $h(x) > 0$ for each $x \in K$, $h(x_0) < p$, and $h(x) > q$ for each $x \in g^{-1}(a)$. Therefore the set $h^{-1}(r)$ separates $\mathrm{Cl}(K)$ between $h(x_0)$ and $\mathrm{Bd}(K)$. Thus the component K_1 of $X \setminus h^{-1}(r)$ that contains x_0 satisfies $0 < \min h(K_1) < r$. Hence $r \notin I(h)$. $\square$

By the definition we obtain the next proposition.

Proposition 2.3. *For each $a \in (0,1)$ we have $g^{(a)}(g^{-1}(0)) = \{0\}$ and $g^{(a)}(g^{-1}(1)) = \{1\}$.*

Proposition 2.4. *For each $a \in (0,1)$ we have $\{a\} \cup I(g) \subset I(g^{(a)})$.*

Proof. Evidently, we have $(g^{(a)})^{(a)} = g^{(a)}$. Thus $a \in I(g^{(a)})$.

Suppose $b \notin I(g^{(a)})$. Thus $b \in (0,1) \setminus \{a\}$. Then there exists a nonempty component K of $F(g^{(a)}, b)$. For any $x \in X$ if $g^{(a)}(x) \neq g(x)$, then $g^{(a)}(x) = a$. Using this implication we conclude:

(1) Since neither 0 nor 1 belong to $g^{(a)}(K)$, then neither 0 nor 1 belong to $g(K)$.
(2) Since $g^{(a)}(\mathrm{Bd}(K)) = \{b\} \neq \{a\}$, we have $g(\mathrm{Bd}(K)) = \{b\}$.
(3) Since $g^{(a)}(K)$ contains some numbers different from b, then $g(K)$ contains some numbers different from b.

From (1), (2) and (3) it follows that K contains a nonempty component K_1 of $X \setminus g^{-1}(b)$ such that $0 \notin g(K_1)$ and $1 \notin g(K_1)$. Therefore $K_1 \subset F(g,b) \neq \emptyset$ and thus $b \notin I(g)$. The proposition is proved. $\square$

Proposition 2.5. *Let a, p, q be numbers such that $0 < p < a < q < 1$ and $p, q \in I(g)$. Then $g(F(g,a)) \subset [p,q]$ and $g^{(a)}(F(g,a)) = \{a\} \subset (p,q)$.*

Proof. The second part of the conclusion follows by the definition. To see the first one suppose, on the contrary, that a component K of $F(g,a)$ contains a point x such that either $g(x) < p$ or $g(x) > q$. Assume $g(x) < p$ (the other case is similar). The the component of $X \setminus g^{-1}(p)$ that contains x is a subset of $F(g,p)$. On the other hand, $p \in g(I)$ and thus $F(g,p) = \emptyset$. A contradiction. $\square$

Proposition 2.6. *Let $a_1, a_2, ...$ be a sequence of numbers in $(0,1)$ such that $\mathrm{Cl}(\{a_1, a_2, ...\}) = [0,1]$. Then the sequence of mappings $f_n : X \to [0,1]$ defined by $f_0 = g$ and $f_{n+1} = f_n^{(a_{n+1})}$ uniformly converges to a confluent mapping $f : X \to [0,1]$.*

Proof. Applying Proposition 2.4 inductively we see that for any $m \geq n$ we have $a_1, ..., a_n \in I(f_m)$. Let $x \in X$. If $a_i < f_n(x) < a_j$ for some $i, j \in \{1, ..., n\}$, then according to Proposition 2.5 we have $a_i < f_m(x) < a_j$

for all $m > n$. By the density of $\{a_n\}$ in $[0,1]$ the sequence f_n must converge uniformly.

Let f be the limit of this sequence. Since $a_1, ..., a_n \in I(f_m)$ for every $m > n$ and the set $\{a_1, a_2, ...\}$ is dense in $[0,1]$, then there are no numbers p, q as in Proposition 2.2 for $g = f$. Hence f is confluent. $\square$

In Propositions 2.7-2.11 we fix a sequence $\{a_n\} \subset (0,1)$ such that $\mathrm{Cl}(\{a_1, a_2, ...\}) = [0,1]$, and, we assume that mappings f_n and f are as in Proposition 2.6. For any $p \in X$ let $C_n(p)$ be the component of $f_n^{-1}(f_n(p))$ that contains p. Define $C(p)$ as the component of $f^{-1}(p)$ that contains p, and, $X_0 = \{x \in X : f(x) = f_n(x) \text{ for each } n \in \{0,1,...\}\}$. Observe that X_0 is closed in X. The proof of the following proposition is straightforward. The details are left to the reader.

Proposition 2.7. *For every $p \in X$ we have:*
  (1) $C_n(p) \subset C_{n+1}(p)$;
  (2) $\bigcup\{C_n(p) : n \in \{0,1,...\}\} \subset C(p)\}$; *and*
  (3) *If $p \in X \setminus X_0$, then $\bigcup\{C_n(p) : n \in \{0,1,...\}\} \setminus X_0$ is open in X and $Bd((\bigcup\{C_n(p) : n \in \{0,1,...\}\}) \setminus X_0) \subset C(p) \cap X_0$.*

Proposition 2.8. *For every $x \in X$ each component C of the set $C(x) \setminus X_0$ is open in X. In particular, we have $Bd(C) \subset X_0 \cap C(x)$ for such C, and thus $C(x) \cap X_0 \neq \emptyset$ for each $x \in X$.*

Proof. Let $z \in C(x) \setminus X_0 = C(z) \setminus X_0$. Then $z \in \mathrm{Int}(\bigcup\{C_n(z) : n \in \{0,1,...\}\} \setminus X_0) \subset C(z) = C(x)$ by Proposition 2.7, parts (3) and (2), and thus $C(x) \setminus X_0$ is open. By the local connectedness of X any component C of $C(x) \setminus X_0$ is open too. $\square$

Proposition 2.9. *Let $t \in [0,1]$ and suppose $f_0^{-1}(t)$ is connected. Then $f^{-1}(t)$ is connected.*

Proof. Let $p \in f_0^{-1}(t)$. If $a_1 \neq t$, then $C_0(p) = C_1(p) = f_0^{-1}(t)$. If $a_1 = t$, then $C_1(p)$ may properly contain $f_0^{-1}(t)$ but it must remain connected. Similarly, we can argue that $f_0^{-1}(t) \subset C_n(p)$ for each n. Thus the set $f^{-1}(t) \cap X_0 = f_0^{-1}(t)$ is connected. According to Proposition 2.8, for each $x \in f^{-1}(t) \setminus X_0$ we have $\emptyset \neq C(x) \cap X_0 \subset f^{-1}(f) \cap X_0$, which completes the proof. $\square$

To prove the next proposition we will apply the following lemma.

Lemma 2.10. *Let Y be a closed subset of a locally connected compactum Z. If Y is the union of two locally connected sets A and B such that A is open in Z, then Y is locally connected.*

Proof. First, note that $Bd(A) \subset B$. Let a sequence $\{x_n\} \subset Y$ converge to a point $x_0 \in Y$. We prove that for almost all n there are connected sets $K_n \subset Y$ such that $x_0, x_n \in K_n$ and $\lim \mathrm{diam}(K_n) = 0$, which will complete the proof. Since A is open in Z, and Z is locally connected, then such K_n's exist for $x_0 \in A$. Suppose $x_0 \in Y \setminus A = B \setminus A$. Since Z is a locally connected compactum, then it is locally arc connected. Thus, for almost all n, there are arcs $A_n \subset Z$ such that $x_0, x_n \in A_n$ and $\lim \mathrm{diam}(A_n) = 0$. Let p_n be the first point in A_n (in the ordering from x_n to x_0) such that $p_n \notin A$, and

A'_n be the arc in A_n from x_n to p_n. Then $p_n \in \mathrm{Bd}(A) \subset B$. Since B is locally connected, there are connected sets $L_n \subset B$ such that $p_n, x_0 \in L_n$ and $\lim \mathrm{diam}(L_n) = 0$. Note that the sets $K_n = A'_n \cup L_n$ are as desired. $\square$

Proposition 2.11. *Let $t \in [0,1]$ and suppose $f_0^{-1}(t)$ is locally connected. Then $f^{-1}(t)$ is locally connected.*

Proof. In view of Proposition 2.8 the set $A_0 = X_0 \cap f^{-1}(t)$ is nonempty. Applying Proposition 2.7 part (1), we see that A_0 is the union of some components of $f_0^{-1}(t)$, and thus A_0 is locally connected. Using again Proposition 2.8 we see that the set $f^{-1}(t) \setminus X_0 = \bigcup \{C(x) \setminus X_0 : x \in f^{-1}(t)\}$ is open in X. Thus the compactum $f^{-1}(t)$ is the union of a locally connected set A_0 and an open subset $f^{-1}(t) \setminus X_0$ of X. Hence $f^{-1}(t)$ is locally connected by Lemma 2.10. $\square$

Theorem 2.12. *Let X be a locally connected continuum. Then for every two disjoint closed subsets A and B of X there exists a confluent map $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

   Moreover, the map f can be chosen so that in addition:

   (1) *$f^{-1}(0)$ and $f^{-1}(0)$ are locally connected; and*
   (2) *if the sets A and B are connected, then $f^{-1}(0)$ and $f^{-1}(1)$ are connected too.*

Proof. Let $f_0 : X \to [0,1]$ be a mapping guaranteed by the Urysohn lemma such that $f_0^{-1}(0) = A$ and $f_0^{-1}(1) = B$. Let $a_1, a_2, \dots$ be a sequence of numbers in $(0,1)$ such that $\mathrm{Cl}(\{a_1, a_2, \dots\}) = [0,1]$. Inductively define $f_{n+1} = f_n^{(a_{n+1})}$. By Proposition 2.6 the sequence f_n uniformly converges to a confluent mapping $f : X \to [0,1]$. Applying Proposition 2.3 inductively we see that $f_n(A) = \{0\}$ and $f_n(B) = \{1\}$ for each n. Hence $f(A) = \{0\}$ and $f(B) = \{1\}$.

   Now we prove part (1). Since X is a locally connected continuum, every point of X has arbitrarily small closed, connected and locally connected neighborhoods. Thus we can slightly enlarge sets A and B to some corresponding locally connected, compact, disjoint sets A' and B', and apply the previous argument for A' and B' in place of A and B, respectively [16, Proposition 8.7]. Then part (1) follows by Proposition 2.11.

   Part (2) of the theorem follows by Proposition 2.9. $\square$

Remark 2.13. The converse of Theorem 2.12 is false. For example, the harmonic fan F is not locally connected while it is possible to construct for each pair of disjoint closed subsets A and B a confluent map $f : F \to [0,1]$ to an arc such that $f(A) = 0$ and $f(B) = 1$.

Theorem 2.14. *Let X be a closed subset of a locally connected continuum Y and $f : X \to [0,1]$ be a confluent mapping. Then there exists a confluent extension $f^* : Y \to [0,1]$ of the mapping f.*

   Moreover, if the sets $f^{-1}(0)$ and $f^{-1}(1)$ are (locally) connected, then we can choose f^* so that, additionally, the sets $(f^*)^{-1}(0)$ and $(f^*)^{-1}(1)$ are (locally) connected too.

Proof. Since $[0,1]$ is an absolute retract there exists some extension $f_0 : Y \to [0,1]$ of the mapping f. Let $a_1, a_2, \dots$ be a sequence of numbers in $(0,1)$

such that $Cl(\{a_1, a_2, ...\}) = [0, 1]$. Inductively define $f_{n+1} = f_n^{(a_{n+1})}$. By Proposition 2.6 the sequence f_n uniformly converges to a confluent mapping $f^* : X \to [0, 1]$. To show that $f^*|X = f$ it suffices to prove that $f_n|X = f$ for each n. Indeed, by the definition of f_0 we have $f_0|X = f$. Suppose $f_n|X = f$ for some n and let $x \in X$. If $f_n(x) = a_{n+1}$, then $f_{n+1}(x) = a_{n+1} = f(x)$. Suppose x belongs to a component K of $Y \setminus f_n^{-1}(a_{n+1})$. Let C be the component of X that contains x. Then the mapping $g = f|C : C \to [0, 1]$ is confluent. Since f_n is an extension of f (and thus of g), the set K must contain the component K' of $C \setminus g^{-1}(a_{n+1})$ that contains x. The mapping g is confluent, and thus $F(g, a_{n+1}) = \emptyset$ (see Lemma 2.1). Therefore either $0 \in g(K')$ or $1 \in g(K')$. Consequently, either $0 \in f_n(K)$ or $1 \in f_n(K)$. This implies that $K \cap F(f_n, a_{n+1}) = \emptyset$. So $f_{n+1}|K = f_n|K$. In particular $f_{n+1}(x) = f_n(x) = f(x)$. Hence $f_{n+1}|X = f$.

We have proved that $f_n|X = f$ for each n, and thus $f^*|X = f$. To see the last part of the theorem observe that the extension f_0 of f can, additionally, satisfy the condition $f_0^{-1}(0) = f^{-1}(0)$ and $f_0^{-1}(1) = f^{-1}(1)$. Then the conclusion follows by Propositions 2.9 and 2.11. The proof is complete.   □

Corollary 2.15. *For any arc $[a, b]$ in a locally connected continuum X there exists a confluent retraction $r : X \to [a, b]$ such that the sets $r^{-1}(a)$ and $r^{-1}(b)$ are locally connected continua.*

Remark 2.16. No nondegenerate continuum different from an arc can replace the segment $[0, 1]$ in Theorem 2.14 or the arc $[a, b]$ in Corollary 2.15. To see this it suffices to show that such a continuum must have each pair of different points joined by a free arc. The details are left to the reader.

3. Confluently graph-like compacta

In this section we prove the main results of the paper. First we prove that for any class $\mathcal{K}$ of graphs and any compactum the properties of being confluently $\mathcal{K}$-representable and the one of being confluently $\mathcal{K}$-like are equivalent (Theorem 3.2). Then we show (Corollary 3.14) that for any compactum the following three conditions are equivalent: (1) being confluently graph-representable, (2) being confluently graph-like, and (3) being 1-dimensional and confluently $\mathbb{LC}$-like. In particular, this theorem implies that each 1-dimensional locally connected compactum is confluently graph-representable (Corollary 3.15).

Theorem 3.1. *Let $f : X \to F$ be a confluent mapping from a compactum X onto a graph F. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that for every confluent δ-mapping $g : X \to G$ from X onto a graph G, there exists an onto confluent mapping $h : G \to F$ satisfying $d_{sup}(h \circ g, f) < \varepsilon$.*

Proof. Let $e_1, ..., e_n$ be mutually different points in F such that all components of $F \setminus \{e_1, ..., e_n\}$ are open arcs of diameter less than $\varepsilon/2$ and each two points e_i, e_j are the two boundary points of at most one, if any, component of $F \setminus \{e_1, ..., e_n\}$. Note that, in particular, all end points, all ramification

points and all isolated points of F are included in $\{e_1, ..., e_n\}$. For any $i \in \{1, ..., n\}$ define

$$L_i = \{e_i\} \cup \bigcup \{L : L \text{ is a component of } F \setminus \{e_1, ..., e_n\} \text{ with } e_i \in \mathrm{Bd}(L)\}.$$

Notice that $\mathrm{diam}(L_i) < \varepsilon$ for each i. Let $\sigma > 0$ be such that $N_\sigma(e_i) \subset L_i$ for each $i \in \{1, ..., n\}$. Put $E_i = f^{-1}(e_i)$ and $\xi = \min\{d_X(E_i, E_j) : i \neq j\}$. Fix a number $\delta > 0$ such that $\delta < \xi/2$ and, for any pair $x, y \in X$, we have $d_F(f(x), f(y)) < \sigma$ whenever $d_X(x, y) < \delta$. Let $g : X \to G$ be a confluent δ-mapping onto a graph G. Define $D_i = g(E_i)$ for $i \in \{1, ..., n\}$ and note the sets $D_1, ..., D_n$ are mutually disjoint by the definition of δ.

Claim 1. For every component K of $G \setminus (D_1 \cup ... \cup D_n)$ if $\mathrm{Bd}(K) \cap D_i \neq \emptyset$, then $f(g^{-1}(K)) \subset L_i$.

Indeed, suppose there exists a point $x \in D_i \cap \mathrm{Bd}(K)$ and let M be a component of $g^{-1}(K)$. Since g is confluent, $g(M) = K$ and also $d_X(M, g^{-1}(x)) = 0$. Moreover, $g^{-1}(x) \cap E_i \neq \emptyset$ and g is a δ-mapping. Therefore $d_X(M, E_i) < \delta$. Thus $d_F(f(M), e_i) < \sigma$ by the choice of δ. Since the set $f(M)$ is a connected subset of $F \setminus \{e_1, ..., e_n\}$, the last inequality implies, by the choice of σ, that $f(M) \subset L_i$. Therefore $f(M) \subset L_i$ for every component M of $g^{-1}(K)$, and thus the claim is proved.

Claim 2. For every component K of $G \setminus (D_1 \cup ... \cup D_n)$ the set $\mathrm{Bd}(K)$ intersects at most two different sets D_i, D_j. Moreover, if $\mathrm{Bd}(K) \cap D_i \neq \emptyset \neq \mathrm{Bd}(K) \cap D_j$ and $i \neq j$, then e_i, e_j are the two different boundary points of a component L of $F \setminus \{e_1, ..., e_n\}$ and $f(g^{-1}(K)) \subset L$.

Indeed, this is a consequence of Claim 1 and of the fact that any three different sets L_i, L_j, L_k have empty intersection. Since any two different sets L_i, L_j are either disjoint, or have a component L of $F \setminus \{e_1, ..., e_n\}$ as their intersection, the last part of the claim follows.

We are ready to define the desired mapping $h : G \to F$. For any $x \in D_i$ put $h(x) = e_i$. To define h on G let K be a component of $G \setminus (D_1 \cup ... \cup D_n)$.

First, suppose D_i, for some fixed $i \in \{1, ..., n\}$, be such that $\mathrm{Bd}(K) \subset D_i$. Then we put $h(x) = e_i$ for each $x \in K$. The only other case is that $\mathrm{Bd}(K) \subset D_i \cup D_j$ and $\mathrm{Bd}(K) \cap D_i \neq \emptyset \neq \mathrm{Bd}(K) \cap D_j$ for two different, fixed sets D_i, D_j such there is a component L of $F \setminus \{e_1, ..., e_n\}$ with e_1, e_2 as their boundary points (see Claim 2). In this case we apply Theorem 2.12 for the graph $\mathrm{Cl}(K)$ with two disjoint nonempty subsets $\mathrm{Bd}(K) \cap D_i$, $\mathrm{Bd}(K) \cap D_j$ obtaining a confluent mapping h from $\mathrm{Cl}(K)$ to $L \cup \{e_1, e_2\}$ such that $h(D_i) = \{e_i\}$ and $h(D_j) = \{e_j\}$. We have defined $h(x)$ for each $x \in G$.

Notice that $h : G \to F$ is well-defined, continuous and onto.

Claim 3. $d_{sup}(h \circ g, f) < \varepsilon$.

Indeed, Let $x \in X$. If $x \in g^{-1}(D_i)$ for some i, then $h \circ g(x) = e_i$ and $d_X(x, E_i) < \delta$. Thus $d_F(f(x), e_i) < \sigma$ by the choice of δ. Therefore $f(x) \in L_i$ by the choice of σ, and thus $d_F(f(x), e_i) = d_F(f(x), h \circ g(x)) < \varepsilon$.

Now assume that $x \in X \setminus (g^{-1}(D_1) \cup ... \cup g^{-1}(D_n))$. Then $g(x)$ belongs to a component K of $G \setminus (D_1 \cup ... \cup D_n)$. This component must satisfy

$\mathrm{Bd}(K) \cap D_i \neq \emptyset$ for some i. By the definition of h we have $h(\mathrm{Cl}(K)) \subset \mathrm{Cl}(L_i)$ and $f(x) \in L_i$ (compare Claim 1). Since $\mathrm{diam}(\mathrm{Cl}(L_i)) < \varepsilon$ we again have $d_F(f(x), h \circ g(x)) < \varepsilon$, which completes the proof of the claim.

It remains to show that h is confluent. Suppose, on the contrary, that there exists a continuum P in F and a component Q of $h^{-1}(P)$ such that $h(Q)$ is a proper subset of P. Let p be a boundary point of $h(Q)$ in P and fix a point q in $Q \cap h^{-1}(p)$. Choose an arc A in P such that

  (i) p is an end point of A;
  (ii) $A \cap h(Q) = \{p\}$;
  (iii) if $p \in F \setminus \{e_1, ..., e_n\}$, then $A \subset F \setminus \{e_1, ..., e_n\}$; and
  (iv) if $p = e_i$ for some i, then $A \subset L_i$.

Then the component C of $h^{-1}(A)$ that contains q satisfies $h(C) = \{p\}$. Suppose first that $C \cap (D_1 \cup \ldots \cup D_n) = \emptyset$. Then C is contained in a component K of $G \setminus (D_1 \cup \ldots \cup D_n)$. Put $K' = \mathrm{Cl}(K)$ and let $\{a_n\}$ be a sequence of points in A converging to p, $A_n = [p, a_n] \subset A$ and let B_n the component of $h^{-1}(A_n)$ containing C. Then $B_n \subset Q$ and there exists an n such that $B_n \subset K$. Since $h|_{K'}$ is confluent, $h(B_n) = A_n$. This contradicts the definition of p and A .

Hence we may assume that there exists an i such that $C \cap D_i \neq \emptyset$ and thus $h(C) = e_i$. Note that A is contained in the closure L' of a unique component L of $F \setminus \{e_1, \ldots, e_n\}$ with end points e_i and e_j. Choose $r \in C \cap D_i$ and $u \in g^{-1}(r) \cap E_i$. Let B be the component of $f^{-1}(L')$ which contains the point u. Since f is confluent, $f(B) = L'$ and, hence, $g(B)$ is a subgraph of G which meets D_i, D_j and $r \in g(B) \cap C$.

Let $\mathcal{K}_{i,j}$ denote the union of D_i, D_j and all components K of $G \setminus (D_1 \cup \ldots \cup D_n)$ such that $\partial K \subset D_i \cup D_j$.

Claim 4. The set $g(B)$ is contained in $\mathcal{K}_{i,j}$.

Indeed, if a component K of $G \setminus (D_1 \cup ... \cup D_n)$ meets $g(B)$ at a point x, then $f(g^{-1}(K)) \cap L \neq \emptyset$. By Claims 1 and 2 the set $f(g^{-1}(K))$ must be contained in some set L_k. But $L_k \cap L \neq \emptyset$ only in the case $k \in \{i, j\}$. Therefore $f(g^{-1}(K)) \subset L_i \cup L_j$. We also have $\mathrm{Bd}(K) \cap D_m = \emptyset$ for each m different from i and j because otherwise $f(g^{-1}(K)) \subset L_m$ (see Claim 1) and $L_m \cap L = \emptyset$, which is impossible.

Let $\mathcal{K}_i$ denote the union of D_i and all components K of $G \setminus (D_1 \cup \ldots \cup D_n)$ such that $\partial K \subset D_i$. Note that by the definition of h, $h(\mathcal{K}_i) = e_i$. Recall that $r \in C \cap D_i$. Let $\mathcal{K}_i(r)$ be the component of $\mathcal{K}_i$ which contains r. Then $\mathcal{K}_i(r) \subset Q$. Since $g(B)$ meets D_j it follows from Claim 4 that there exists a component K of $G \setminus (D_1 \cup \ldots \cup D_n)$ such that $z \in \mathrm{Cl}(K) \cap \mathcal{K}_i(r) \neq \emptyset \neq \partial K \cap D_j$. Hence $h(\mathrm{Cl}(K)) = \mathrm{Cl}(L)$ and $h|_{\mathrm{Cl}(K)}$ is confluent. Let D be the component of $h^{-1}(A)$ containing the point z, then $h(D) = A$ and, hence, $A \subset h(Q)$. This contradicts the choice of p and A and completes the proof of the Theorem. $\square$

Theorem 3.2. *Let $\mathcal{K}$ be any class of graphs and X be a compactum. Then X is confluently $\mathcal{K}$-representable if and only if X is confluently $\mathcal{K}$-like.*

Proof. Suppose that $\mathcal{K}$ is a class of graphs such that for each $\varepsilon > 0$ there exists a confluent ε-map from the continuum X onto a member of $\mathcal{K}$. We

may assume by Theorem 3.1 that there exists a sequence $\varepsilon_n \to 0$, a countable family of graphs G_n, confluent, onto mappings $f_n : X \to G_n$ and $g_n^{n+1} : G_{n+1} \to G_n$ such that

(1) $X \cup (\cup_n G_n)$ is a subset of the Hilbert cube Q with metric d,

(2) $d(x, f_n(x)) < \varepsilon_n$ for each $x \in X$ and each n,

(3) for each $k \leq m \leq n$ and each $x \in X$, $d(g_k^m(f_m(x)), g_k^n(f_n(x))) < e_k$.

By [14], there exists a continuous and onto mapping $f : X \to \lim_{\leftarrow}(G_n, g_n^{n+1})$ defined by $f(x) = (y_k)$, where each $y_k = \lim_{n \to \infty} g_k^n(f_n(x))$. It suffices to show that f is one-to-one. Let $\pi_n : \lim_{\leftarrow}(X_n, g_n^{n+1}) \to X_n$ denote the natural projection. Suppose that $x, y \in X$ such that $d(x, y) > \eta > 0$. Choose n such that $\sum_n^\infty \varepsilon_i < \eta/6$. Then $d(\pi_n(f(x)), \pi_n(f(y))) > d(x, y) - 4\sum_n^\infty \varepsilon_i > 0$. This completes the proof of the Theorem. $\square$

Using a different approach the following result about confluently circle-like continua was also proved in [3].

Corollary 3.3. *A continuum X is confluently circle-like if and only if X is a solenoid.*

The next result answers a question from [3].

Corollary 3.4. *A continuum X is confluently arc-like if and only if X is a Knaster type continuum, i.e. X is homeomorphic to an inverse limit of arcs with open bonding mappings.*

Let $\mathbb{LC}$ stand for the class of all locally connected compacta.

Corollary 3.5. *An atriodic continuum X is confluently $\mathbb{LC}$-like if and only if X is either a solenoid, or a Knaster type continuum.*

Corollary 3.6. *Each confluently tree-like continuum is confluently tree-representable.*

In a recent paper [4] it was proved that confluently tree-representable continua are absolute retracts for hereditarily unicoherent continua. The question whether the same is true for confluently tree-like ones was an important inspiration to the research presented here. The next result answers this question in the affirmative.

Corollary 3.7. *Each confluently tree-like continuum is an absolute retract for hereditarily unicoherent continua.*

It is known that if a dendroid X is an absolute retract for hereditarily unicoherent continua, then X is a dendroid with the property of Kelley (in particular, such a dendroid must be smooth). In view of Corollary 3.7, the following question is of some interest (compare [4]).

Question 3.1. *Is every dendroid with the property of Kelley a confluently tree-like continuum?*

Let G be a graph and S be a simplicial complex structure on G. The 0-dimensional elements of such a structure will be denoted by $\{e_1\}, ..., \{\varepsilon_n\}$ and points $e_1, ..., e_n$ will also be called *vertices* of S. Denote by $\mathcal{S}(G)$ the family of all simplicial complex structures S such that each pair of two

different vertices e_i, e_j of S can be boundary points of at most one, if any, 1-dimensional element of S. For any $S \in \mathcal{S}(G)$ and vertex e_i of S, let

$$L_i(S) = \{e_i\} \cup \bigcup \{\mathrm{Int}(F) : F \in S, \dim F = 1 \text{ and } e_i \in \mathrm{Bd}(F)\},$$

and, if $F \in S$ with $\dim F = 1$,

$$L_F(S) = L_i(S) \cup L_j(S), \text{ where } e_i, e_j \text{ are the boundary points of } F.$$

If $F = \{v\} = \{e_i\}$, i.e., $v = e_i$ is a vertex of S, we also write $L_i(S) = L_v(S) = L_F(S)$. Let $f : X \to G$ be a mapping, where X is compact. We say that a family $\mathcal{P}$ of sets *refines* the family $\mathcal{Q}$ if for each $P \in \mathcal{P}$ there exists $Q \in \mathcal{Q}$ such that $P \subset Q$. If $\mathcal{Q}$ is a family of sets in X and $f : X \to Y$ is a mapping such that the family $\{f^{-1}(y) : y \in Y\}$ refines $\mathcal{Q}$, we say that the mapping f *refines* $\mathcal{Q}$. The following observation follows by the compactness of the domain of the function f.

Observation 3.8. *If $\mathcal{U}$ is a family of open sets covering a compactum X and $f : X \to G$ is a mapping to a graph G such that f refines $\mathcal{U}$, then there exists $S \in \mathcal{S}(G)$ such that the family $\{f^{-1}(L_F(S)) : F \in S\}$ refines $\mathcal{U}$.*

Lemma 3.9. *Let $\mathcal{U}$ be an open cover of a locally connected compactum X and suppose X admits a mapping, that refines $\mathcal{U}$, onto a graph H. Then there exists a graph G, a structure $S \in \mathcal{S}(G)$ and a surjective mapping $g : X \to G$ such that the family $\{g^{-1}(L_F(S)) : F \in S\}$ refines $\mathcal{U}$ and the set $g^{-1}(F)$ is connected for each $F \in S$.*

Proof. We begin with the following claim.

Claim 1. There exists a graph H, a mapping $h : X \to H$ and a structure $S \in \mathcal{S}(H)$ such that the family $\{h^{-1}(L_F(S)) : F \in S\}$ refines $\mathcal{U}$ and $h^{-1}(v)$ has finitely many components only for each vertex v of S.

Indeed, let $h_1 : X \to H$ be a mapping onto a graph H that refines $\mathcal{U}$. By Observation 3.8 there exists a structure $S \in \mathcal{S}(H)$ such that the family $\{h_1^{-1}(L_F(S)) : F \in S\}$ refines $\mathcal{U}$. Let ξ be the minimum of the numbers $d_X(h_1^{-1}(e_i), h_1^{-1}(X \setminus L_i(S)))$ for all vertices e_i of S. Since X is a locally connected compactum and $h_1^{-1}(\{e_1, ..., e_n\})$ is compact, there exists a finite collection of continua $K_1, ..., K_m$ in X such that $\mathrm{diam}(K_j) < \xi$, $h_1^{-1}(\{e_1, ..., e_n\}) \cap K_j \neq \emptyset$ for each $j \in \{1, ..., m\}$, and $h_1^{-1}(\{e_1, ..., e_n\}) \subset K_1 \cup ... \cup K_m$. The condition $\mathrm{diam}(K_j) < \xi$ implies that for each $j \in \{1, ..., m\}$ there exists exactly one $i \in \{1, ..., n\}$ such that $K_j \cap h_1^{-1}(e_i) \neq \emptyset$, and, we have $K_j \subset h_1^{-1}(L_i(S))$ for such i. Define $M_i = \bigcup \{K_j : K_j \cap h_1^{-1}(e_i) \neq \emptyset\}$ for $i \in \{1, ..., n\}$. For any $F \in S$ with $\dim F = 1$ and with end points e_i, e_j let $P_{i,j} = M_i \cup M_j \cup h_1^{-1}(F)$ and observe that $P_{i,j}$ is compact. Then by the Urysohn lemma there exists a mapping $h_{i,j} : P_{i,j} \to F$ such that $h_{i,j}^{-1}(e_i) = M_i$ and $h_{i,j}^{-1}(e_j) = M_j$. For each $F \in S$ with $\dim F = 1$ and $\mathrm{Bd}(F) = \{e_i, e_j\}$, and for each $x \in M_i \cup M_j \cup h_1^{-1}(F)$ we define $h(x) = h_{i,j}(x)$. Observe that $h : X \to H$ is a well defined continuous function. Moreover, the set $h^{-1}(e_i) = M_i$ is the finite union of continua, and, $h^{-1}(L_F(S)) \subset h_1^{-1}(L_F(S))$ for 1-dimensional sets $F \in S$. Hence the family $\{h^{-1}(L_F(S)) : F \in S\}$ refines $\mathcal{U}$, which completes the proof of the claim.

For any graph H, any structure $S \in \mathcal{S}(H)$ and any mapping $h : X \to H$ such that the family $\{h^{-1}(L_F(S)) : F \in S\}$ refines $\mathcal{U}$ let $k(h, S)$ be the difference between the number of components of $h^{-1}(\{e_1, ..., e_n\})$ and the number of nonempty sets $h^{-1}(\{e_i\})$. Fix a graph G_1, a structure $S_1 \in \mathcal{S}(G_1)$ with vertices $e_1, ..., e_n$, and a mapping $g_1 : X \to G_1$ such that the family $\{g_1^{-1}(L_F(S_1)) : F \in S_1\}$ refines $\mathcal{U}$ and the number $k(g_1, S_1)$ is minimal. By Claim 1, $k(g_1, S_1)$ is finite.

Claim 2. For every $i \in \{1, ..., n\}$ each two different components of $g_1^{-1}(e_i)$ are subsets of two different components of $g_1^{-1}(L_i(S_1))$.

Suppose, on the contrary, that two different components of $g_1^{-1}(e_i)$ are contained in the same component of $g_1^{-1}(L_i(S_1))$. Since components of open subsets of locally connected compacta are arcwise connected and $g_1^{-1}(L_i(S_1))$ is open in X, there exists an arc ab in $g_1^{-1}(L_i(S_1))$ joining two different components A and B of $g_1^{-1}(e_i)$ such that $ab \cap g_1^{-1}(e_i) = \{a, b\}$, $a \in A$, and $b \in B$. Let $K_1, ..., K_m$ be all components of $g_1^{-1}(\{e_1, ..., e_n\})$ with $K_{m-1} = A$ and $K_m = B$. Define $K'_{m-1} = A \cup ab \cup B$. Using the continua $K_1, ... K_{m-2}, K'_{m-1}$ we modify the mapping g_1 to a mapping $g'_1 : X \to G_1$ (in the same way as the mapping h_1 was modified to the mapping h in the proof of Claim 1) so that the family $\{(g'_1)^{-1}(L_F(S_1)) : F \in S_1\}$ refines $\mathcal{U}$ and $K_1, ... K_{m-2}, K'_{m-1}$ are all components of $(g'_1)^{-1}(\{e_1, ..., n\})$. Thus we have $k(g'_1, S_1) < k(g_1, S_1)$. So $k(g_1, S_1)$ is not minimal, a contradiction.

Now we will prove that $k(g_1, S_1) = 0$. Suppose, on the contrary, that $k(g_1, S_1) > 0$. Thus the set $g_1^{-1}(e_i)$ is not connected for some i. Without loss of generality we can assume that $g_1^{-1}(e_n)$ is not connected. Let D be a component of $g_1^{-1}(e_n)$ and W be the component of $g_1^{-1}(L_n(S_1))$ that contains D. Using Claim 2 we see that other components of $g_1^{-1}(e_n)$ are contained in components of $g_1^{-1}(L_n(S_1))$ different from W.

Let G'_1 be a homeomorphic copy of G_1 disjoint with G_1, and $f : G_1 \to G'_1$ be a homeomorphism. For each $x \in X \setminus W$ we identify the pair x and $f(x)$ in the disjoint union $G_1 \cup G'_1$. Let $q : G_1 \cup G'_1 \to q(G_1 \cup G'_1)$ be the quotient map of this identification. Put $G_2 = q(G_1 \cup G'_1)$ and $e_{n+1} = (f \circ q)(e_n)$. We will identify the graph G_1 with $q(G_1)$ by the homeomorphism $q|G_1$. So $G_1 \subset G_2$. Let $S_2 \in \mathcal{S}(G_2)$ be the simplicial complex structure on G_2 introduced by vertices $e_1, ..., e_n, e_{n+1}$. Define $g_2(x) = g_1(x)$ for $x \in X \setminus W$, and $g_2(x) = (g_1 \circ q \circ f)(x)$ for $x \in W$. Observe that $g_2 : X \to G_2$ is a well defined, continuous mapping. Moreover, $g_2^{-1}(L_i(S_2)) \subset g_1^{-1}(L_i(S_1))$ for $i \in \{1, ..., n\}$ and $g_2^{-1}(L_{n+1}(S_2)) \subset g_1^{-1}(L_n(S_1))$. This implies that the family $\{g_2^{-1}(L_F(S_2)) : F \in S_2\}$ refines $\mathcal{U}$. Note that the sets $g_1^{-1}(\{e_1, ..., e_n\})$ and $g_2^{-1}(\{e_1, ..., e_n, e_{n+1}\})$ are identical while the number of nonempty sets $g_2^{-1}(e_i)$ (for $i \in \{e_1, ..., e_n, e_{n+1}\}$) is greater, by 1, than the number of nonempty sets $g_1^{-1}(e_i)$ (for $i \in \{1, ..., n\}$). So $k(g_2, S_2) < k(g_1, S_1)$, and thus $k(g_1, S_1)$ is not minimal, a contradiction.

We have proved that $k(g_1, S_1) = 0$. Thus $g_1^{-1}(e_i)$ is connected for each $i \in \{1, ..., n\}$. Now we are ready to define the required mapping g. Let $x \in g_1^{-1}(\{e_1, ..., e_n\})$. Then we put $g(x) = g_1(x)$. Let P be a component of $G_1 \setminus \{e_1, ..., e_n\}$ with the boundary points e_i, e_j, and let x belong to a

component Q of $g_1^{-1}(P)$. If $\mathrm{Bd}(Q) \cap g_1^{-1}(e_i) \neq \emptyset \neq \mathrm{Bd}(Q) \cap g_1^{-1}(e_j)$, we put $g(x) = g_1(x)$. If $\mathrm{Bd}(Q)$ meets $g_1^{-1}(e_i)$ only, we put $g(x) = e_i$, and, if $\mathrm{Bd}(Q)$ meets $g_1^{-1}(e_j)$ only, we put $g(x) = e_j$. Since X is locally connected, then g is a well defined continuous mapping into G_1. Define $G = g(X)$ and let $S \in \mathcal{S}(G)$ be the simplicity complex structure on G introduced by the set of vertices $E = \{e_1, ..., e_n\} \cap G$. Then for each $e_i \in E$ we have $g^{-1}(L_i(S)) \subset g_1^{-1}(L_i(S_1))$, and thus the family $\{g^{-1}(L_F(S)) : F \in S\}$ refines $\mathcal{U}$. By the construction the set $g^{-1}(e_i)$ is connected for every $e_i \in E$. If $F \in S$, $\dim F = 1$ and e_i, e_j are the boundary points of F, then every component Q of $g^{-1}(F \setminus \{e_i, e_j\})$ satisfies $g(Q) = F \setminus \{e_i, e_j\}$. Employing this last observation and the fact that the sets $g^{-1}(e_i)$, $g^{-1}(e_j)$ are connected, we conclude that $g^{-1}(F)$ is connected too. The proof is complete.   □

Theorem 3.10. *Let $\mathcal{U}$ be an open cover of a locally connected continuum X and suppose X admits a mapping, that refines $\mathcal{U}$, onto a graph. Then there exists a graph G, a structure $S \in \mathcal{S}(G)$ and a confluent mapping $g : X \to G$ such that the family $\{g^{-1}(L_F(S)) : F \in S\}$ refines $\mathcal{U}$ and the set $g^{-1}(F)$ is a locally connected, connected continuum for each $F \in S$.*

Proof. First, let $g' : X \to G$ be a mapping guaranteed by Lemma 3.9 with a structure $S \in \mathcal{S}(G)$ and vertices $\{e_1, ..., e_n\}$. Then the sets $D_i = (g')^{-1}(e_i)$ for $i \in \{1, ..., n\}$ are continua. Since X is locally connected, then, similarly as in the proof of Theorem 2.12, part 1, we can slightly enlarge continua D_i to some locally connected, mutually disjoint continua V_i containing D_i, correspondingly. Let $F \in S$, $\dim F = 1$ and e_i, e_j be the end points of F. Then the continuum $D_F = (g')^{-1}(F) \cup V_i \cup V_j$ is the union of the open subset $(g')^{-1}(F \setminus \{e_i, e_j\})$ of X and the locally connected one $V_i \cup V_j$. So D_F is locally connected in view of Lemma 2.10. Using Theorem 2.12, parts 1 and 2 including, there exists a confluent mapping $g_F : D_F \to F$ such that $V_i \subset g_F^{-1}(e_i)$, $V_j \subset g_F^{-1}(e_j)$ and the sets $g_F^{-1}(e_i)$, $g_F^{-1}(e_i)$ are locally connected continua in X. Define $g : X \to G$ as the combination of mappings g_F for all 1-dimensional $F \in S$, i.e. $g(x) = g_F(x)$ for $x \in F$, and observe that g is a well defined, continuous, surjective mapping. Note that the set $g^{-1}(e_i) = \bigcup \{g_F^{-1}(e_i) : F \in S, \dim F = 1, e_i \in F\}$ is a finite union of locally connected continua with nonempty intersection. Thus $g^{-1}(e_i)$ is a locally connected continuum for each i. Moreover, for each $F \in S$ with $\dim F = 1$ and end points e_i, e_j the continuum $g^{-1}(F) = D_F \cup g^{-1}(e_i) \cup g^{-1}(e_j)$ is a finite union of locally connected continua. Thus $g^{-1}(F)$ is a locally connected continuum. For each e_i we have $g^{-1}(L_i(S)) \subset (g')^{-1}(L_i(S))$. Therefore the family $\{g^{-1}(L_F(S)) : F \in S\}$ refines $\mathcal{U}$. Hence g is a mapping as desired.   □

Corollary 3.11. *A continuum X is a locally connected curve if and only if it admits, for every $\varepsilon > 0$, a confluent mapping $g : X \to G$ onto a graph G with a structure $S \in \mathcal{S}(G)$ such that for each $F \in S$ the set $g^{-1}(F)$ is a locally connected continuum of diameter less than ε.*

Let X be a compactum, $\mathcal{W}$ and $\mathcal{U} = (U_1, ..., U_n)$ be open covers of X. Define $\mathrm{ord}(\mathcal{U})$ as the maximal number of U_i's with nonvoid intersection. Consider the following conditions:

 (i) $\mathrm{ord}(\mathcal{U}) = 2$.
 (ii) $\mathrm{Cl}(U_i) \cap \mathrm{Cl}(U_j) \neq \emptyset$ if and only if $U_i \cap U_j \neq \emptyset$.

(iii) The cover $(\mathrm{Cl}(U_1), ..., \mathrm{Cl}(U_n))$ refines the cover $\mathcal{W}$.

(iv) $\mathrm{Bd}(U_i) \cap \mathrm{Bd}(U_j) = \emptyset$ for all $i, j \in \{1, ..., n\}$ with $i \neq j$.

It is known that for any 1-dimensional compactum X and any open cover $\mathcal{W}$ of X there exists a cover $\mathcal{U} = (U_1, ..., U_n)$ that satisfies conditions (i)-(iv) [8]. The following lemma is easy to see.

Lemma 3.12. *Let X be a compactum and $\mathcal{U} = (U_1, ..., U_n)$ be an open cover of X that satisfies conditions (i), (ii) and (iv). Then for sufficiently small positive number δ and for any δ-mapping $f : X \to Y$ onto some compactum Y the family $\mathcal{V} = (V_1, ..., V_n)$, where $V_i = \{y \in Y : f^{-1}(y) \subset U_i\}$ for $i \in \{1, ..., n\}$, is an open cover of Y that satisfies (i) and (ii).*

Let X be a compactum with an open cover $\mathcal{U} = (U_1, ..., U_n)$ that satisfies conditions (i), (ii) and, G be a graph with a structure $S \in \mathcal{S}(G)$ having vertices $e_1, ..., e_n$ such that there is 1-dimensional element $L_{i,j} \in S$ with end points e_i, e_j if and only if $U_i \cap U_j \neq \emptyset$. Let $D_{i,j} = (U_i \cup U_j) \setminus \bigcup\{U_k : j \neq k \neq i, k \in \{1, ..., n\}\}$ be defined for all pairs i, j with $U_i \cap U_j \neq \emptyset$. For every such a pair i, j, according to the Urysohn lemma, there exists a mapping $g_{i,j} : D_{i,j} \to L_{i,j}$ such that $g_{i,j}^{-1}(e_i) = U_i \setminus \bigcup\{U_k : k \neq i, k \in \{1, ..., n\}\}$, $g_{i,j}^{-1}(e_j) = U_j \setminus \bigcup\{U_k : k \neq j, k \in \{1, ..., n\}\}$. If some set U_l does not intersect other elements of $\mathcal{U}$ we let $g_l(U_l) = \{e_l\}$. Define $g : X \to G$ as the combination of all mappings $g_{i,j}$ and g_l. Then g is a well defined continuous mapping. Observe that g refines $\mathcal{U}$. We call the pair (G, S) the *nerve* of the cover $\mathcal{U}$ and g a *natural map of X into the nerve of (G, S)* (compare [8]).

Theorem 3.13. *Every confluently $\mathbb{LC}$-like compactum X with $\dim X \leq 1$ is confluently graph-like.*

Proof. Fix an $\varepsilon > 0$ and let $\mathcal{U} = (U_1, ..., U_n)$ be an open cover of X such that $\mathrm{diam}(U_i) < \varepsilon$ for $i \in \{1, ..., n\}$ and the conditions (i), (ii) and (iv) are satisfied. Let δ be a positive number guaranteed by Lemma 3.12 for the cover $\mathcal{U}$. By the assumption there exists a confluent δ-mapping $f : X \to Y$ onto a locally connected compactum Y. Then the family $\mathcal{V} = (V_1, ..., V_n)$, where $V_i = \{y \in Y : f^{-1}(y) \subset U_i\}$, is an open cover of Y such that (i) and (ii) are satisfied for $\mathcal{V}$. Let (G_1, S_1) be the nerve of the cover $\mathcal{V}$ and $g_1 : Y \to G_1$ be a natural map into the nerve of (G_1, S_1). Then g_1 refines $\mathcal{V}$. By Theorem 3.10 there exists a confluent mapping $g : Y \to G$ onto a graph G with a structure $S \in \mathcal{S}(G)$ such that the family $\{g^{-1}(L_F(S)) : F \in S\}$ refines $\mathcal{V}$. In particular, g refines $\mathcal{V}$. Define $h = g \circ f : X \to G$ and note that h is a confluent mapping. Since the family $\{f^{-1}(V_1), ..., f^{-1}(V_n)\}$ refines $\mathcal{U}$, we observe that the composition mapping h refines $\mathcal{U}$, i.e. for each $z \in G$ we have $h^{-1}(z) \subset U_i$ for some $i \in \{1, ..., n\}$. We also have $\mathrm{diam}(U_i) < \varepsilon$. Hence $h : X \to G$ is a confluent ε-mapping onto a graph G. The proof is complete. $\square$

The following corollary is a consequence of Theorems 3.2 and 3.13, and this is one of the main results of the paper.

Corollary 3.14. *For each continuum X the following three conditions are equivalent:*

(1) *X is confluently graph-representable;*

(2) *X is confluently graph-like;*
(3) *X is confluently* $\mathbb{LC}$*-like and* $\dim X \leq 1$.

Corollary 3.15. *Each locally connected curve is confluently graph-like, and consequently, it is confluently graph-representable.*

We end the paper with the two following questions. The first one is related to the study of absolute retracts for hereditarily unicoherent continua and for tree-like continua (compare questions of [4]).

Question 3.2. *Let X be an absolute retract for the class of hereditarily unicoherent continua (of tree-like continua). Must X be confluently tree-like?*

The second question relates this study to the one of homogeneous curves. All known homogeneous curves that contain arcs are confluently graph-like, so the following question naturally appears (compare a discussion of classifying homogeneous curves in [19]).

Question 3.3. *Let X be a homogeneous curve that contains an arc. Must X be confluently graph-like?*

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