EXTENDING ISOTOPIES OF PLANAR CONTINUA

LEX G. OVERSTEEGEN AND E. D. TYMCHATYN

Abstract. In this paper we solve the following problem in the affirmative: Let Z be a continuum in the plane $\mathbb{C}$ and suppose that $h : Z \times [0, 1] \to \mathbb{C}$ is an isotopy starting at the identity. Can h be extended to an isotopy of the plane? We will provide a new characterization of an accessible point in a planar continuum Z and use it to show that an accessible point is preserved during the isotopy. We show next that the isotopy can be extended over hyperbolic crosscuts. The proof makes use of the notion of a metric external ray, which mimics the notion of a conformal external ray, but is easier to control during an isotopy.

1. Introduction

Denote the complex plane by $\mathbb{C}$, the origin by O, the open unit disk by $\mathbb{D}$ and the complex sphere by $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. Suppose that $h : Z \times [0, 1] \to \mathbb{C}$ is an isotopy of a continuum $Z \subset \mathbb{C}$ such that if we denote $h^t = h|_{Z\times\{t\}}$, then $h^0 = id_Z$. We consider the old problem whether the isotopy h can be extended to an isotopy of the plane.[1] Under the much more restrictive assumption of a holomorphic motion (where the parameter t belongs to the open unit disk and h is holomorphic in t) the λ-Lemma shows that h^t can be extended to a quasi-conformal homeomorphism of the entire plane (in this case the assumption that h is continuous can be relaxed, while the continuity of the extension still follows, see [MSS83, Lyu83, ST86] and [Slo91] for further details). Although the λ-Lemma also holds for arbitrary (in particular not connected) sets Z, easy examples show that an isotopy of a convergent sequence cannot necessarily be extended over the plane (see [Fab05, p. 991]). It follows from Rado's Theorem [Wen91, Theorem 4.2] that the isotopy h^t can be extended to an isotopy of $\mathbb{C}$ if Z is a simple

Date: July 11, 2007.

1991 *Mathematics Subject Classification.* Primary 57N37; Secondary 57N05.

Key words and phrases. Isotopy, extension, planar continuum.

The first named author was supported in part by NSF-DMS-0405774, and the second named author by NSERC OGP0005616.

[1] We are indebted to Professor R. D. Edwards who communicated this problem to us.

closed curve (see [Bea27, Bea28] for related results and [Eps66] for a generalization). Analytic techniques, in particular, boundary values of conformal maps have been powerful tools for studying plane continua. However, they appear insufficient to answer the general question.

One of the main complications addressed in this paper is that Carathéodory kernel convergence is insufficient to allow us control of the behavior of conformal external rays under an isotopy. To this end we introduce metric external rays which depend only on distance and, hence, behave well under an isotopy. The existence of metric external rays was alluded to in [Bel67] and more fully developed in [Ili70, Bel76]. For our purpose it will be easier to define them as the equidistant set between two disjoint and closed sets in the covering space of $\mathbb{C}\setminus\{O\}$ by the exponential map. Equidistant sets and metric external rays were studied in detail by G. Brouwer in [Bro05]. We will use metric external rays to show that the isotopy can be extended over conformal external rays. Even though the entire proof could have been carried out using metric external rays, it suffices for this paper to rely for the final extension over $\mathbb{C}^*$ on the existing analytic theory.

We will always denote by Z a proper subcontinuum in the sphere $\mathbb{C}^*$ (or equivalently in the plane $\mathbb{C}$), by $h : Z\times[0,1]\to\mathbb{C}$ an isotopy such that $h^0 = id_Z$ and by U a component of $\mathbb{C}^*\setminus Z$ (or equivalently $\mathbb{C}\setminus Z$). Given a fixed component U of $\mathbb{C}^*\setminus Z$ we may assume, without loss of generality, that U contains the point at infinity (or is the unbounded component of Z) and $\infty\in\mathbb{C}^*\setminus h^t(Z)$ for all $t\in[0,1]$. Denote by U^t the component of $\mathbb{C}^*\setminus h^t(Z)$ containing the point at infinity (or the unbounded component of $\mathbb{C}\setminus h^t(Z)$), then $U^t\cup\{\infty\}$ is simply connected. We always denote by $\varphi^t:\mathbb{D}\to U^t\cup\{\infty\}$ the conformal map such that $\varphi^t(O)=\infty$ and $(\varphi^t)'(O)>0$. Then the maps φ^t are unique and, by Carathéodory kernel convergence, uniformly convergent in t on compact subsets of $\mathbb{D}$. By slightly abusing the language we will identify points in the boundary S^1 of the disk $\mathbb{D}$ with their arguments and call them *angles*.

We say that $x\in Z$ is *accessible* from U if there exists an angle $\theta\in[0,2\pi)$ such that the *(conformal) external ray* $R_\theta=\varphi(\{re^{i\theta}\mid r<1\})$ lands on x (i.e., $\overline{R_\theta}\setminus R_\theta=\{x\}$). It is well-known that a point $x\in Z$ is accessible from U if and only if there exists a continuum $Y\subset\overline{U}$ such that $Y\cap Z=\{x\}$. Moreover, in this case $\overline{\varphi^{-1}(Y\setminus\{x\})}\cap S^1=\{\theta\}$ is a single point and R_θ lands on x in Z [Mil00]. It is clearly necessary that the corresponding point $x^t=h^t(x)$ remains accessible in $h^t(Z)$ from U^t. However, Carathéodory kernel convergence is insufficient to show this and one of the first steps of the proof is to show that this is indeed the case. If we assume in addition that x is not a cut point of Z, then there exists for each t a unique angle θ^t such that the external ray $R^t_{\theta^t}$ of Z^t lands on x^t.

The next step of the proof is to show that this correspondence of angles is continuous in t and there exists an isotopy $\alpha^t : S^1 \to S^1$ of the unit circle such that if R^0_θ lands on x^0 in Z^0, then $R^t_{\alpha^t(\theta)}$ lands on x^t in Z^t for each t. Extending α^t to an isotopy $f^t : \mathbb{D} \to \mathbb{D}$, defined by $f^t(re^{i\theta}) = re^{i\alpha^t(\theta)}$ does not, however, provide a proper extension over $\overline{U^0}$ since simple examples show that in general the isotopy $H : U^0 \times [0,1] \to \mathbb{C}^*$ defined by $H(w,t) = \varphi^t \circ f^t \circ (\varphi^0)^{-1}(w)$ does not have a continuous extension over $\mathrm{Bd}(U^0)$. In the final step of the proof we use the Kulkarni-Pinkall [KP94] lamination of U^0 in the sphere to define the proper extension.

We denote by exp the covering map $\exp : \mathbb{C} \to \mathbb{C} \setminus \{O\}$ defined by $\exp(z) = e^z$. Given a set $X \subset \mathbb{C}$ we denote by $\widehat{\mathbf{X}} = \exp^{-1}(X \setminus \{O\})$ and we use bold face letters for subsets of $\widehat{\mathbf{X}}$. However, for points $x \in \mathbb{C} \setminus \{O\}$ we denote by $\mathbf{x}$ a point in the set $\exp^{-1}(x)$. We also denote by $\pi_j : \mathbb{C} \to \mathbb{R}$, $j = 1, 2$, the projections onto the x-axis and y-axis, respectively. The open ball with center x and radius r is denoted by $B(x,r)$ and its boundary by $S(x,r)$. For a set $A \subset \mathbb{C}$ we denote by $B(A,\varepsilon) = \bigcup\{B(a,\varepsilon) \mid a \in A\}$. By a *ray* R we mean a subset of $\mathbb{C}$ homeomorphic to the real line $\mathbb{R}$ so that $|\overline{R} \setminus R| \leq 1$ and $\overline{R}$ is not a simple closed curve. If $\overline{R} \setminus R = \emptyset$, then we say that R is a *closed ray*.

We will use the following notation throughout: for any set $A \subset Z$ we denote by A^t the set $h^t(A)$. We are initially only interested in extending the isotopy over the unbounded component U of $\mathbb{C} \setminus Z$. Recall that U^t is the component of $\mathbb{C}^* \setminus h^t(Z)$ containing ∞ and denote by X^t the continuum $\mathbb{C}^* \setminus U^t$. Then X^t is a non-separating plane continuum. We may identify any particular point $z \in \mathrm{Bd}(U)$ with the origin O, assume that it is fixed under the isotopy and that $X^t \subset B(O,1)$ for all $t \in [0,1]$. We will denote the Euclidean metric on $\mathbb{C}$ by d and the spherical metric on $\mathbb{C}^*$ by ρ. Finally, given two points $x, y \in \mathbb{C}$, we denote by xy the straight line segment joining them.

2. Preliminaries

Crucial to our study is the notion of an equidistant set between two disjoint closed sets in $\mathbb{C}$. Since the reference [Bro05] is not easily accessible and our results require a slightly different setting, we will sketch proofs for some of these results in this section. We start with the following definition from [Bro05]. Suppose that A and B are two disjoint closed subsets of the plane. For $z \in \mathbb{C} \setminus [A \cup B]$, let $r(z) =$

$d(z, A \cup B)$. Then we say that A and B are *non-interlaced* if for each $z \in \mathbb{C} \setminus [A \cup B]$, $A \cap S(z, r(z))$ and $B \cap S(z, r(z))$ are contained in two disjoint closed and connected subsets of $S(z, r(z))$ (one may be empty). Let $E(A, B) = \{z \in \mathbb{C} \mid d(z, A) = d(z, B)\}$ be the equidistant set between A and B.

Let A and B be two disjoint, closed and non-interlaced sets. By Gaston Brouwer [Bro05][Theorem 3.4.4], $E(A, B)$ is a 1-manifold. Moreover, if A and B are connected, then $E(A, B)$ is connected and, hence, it is either a closed ray in the plane or a simple closed curve. In particular if A and B are also both unbounded, then $E(A, B)$ is a closed ray which separates $\mathbb{C}$ into two disjoint open and connected sets one containing A and the other containing B. We will slightly generalize this case by replacing the condition that A and B are connected by the weaker condition that A lies above B (see Definition 2.1 and Theorem 2.8).

Since $O \in X^t \subset B(O, 1)$ for all t, $\max\{\pi_1(\widehat{\mathbf{X}}^t)\} < 0$. It follows from this and the fact that X is a continuum that for any component $\mathbf{C}$ of $\widehat{\mathbf{X}}$, $\pi_1(\mathbf{C}) = (-\infty, m_{\mathbf{C}}]$ with $m_{\mathbf{C}} < 0$. Moreover, since X is non-separating, $\widehat{\mathbf{X}}$ is also non-separating and, hence, each component $\mathbf{C}$ of $\widehat{\mathbf{X}}$ is also non-separating. To see this note that $\widehat{\mathbf{X}}$ has a unique complementary domain W such that $\pi_1^{-1}([0, \infty)) \subset W$. If V is any complementary domain of $\widehat{\mathbf{X}}$, then V must contain a point $\mathbf{v} \in V \setminus \widehat{\mathbf{X}}$ and $\exp(\mathbf{v}) = v \in \mathbb{C} \setminus X$. Hence there exists a ray $R \subset \mathbb{C} \setminus X$ joining v to infinity. Then the lift $\mathbf{R}$ of R with initial point $\mathbf{v}$ is a ray in $\mathbb{C} \setminus \widehat{\mathbf{X}}$ which intersects W and $V = W$ as required. These facts will allow us to define what it means for one component of $\widehat{\mathbf{X}}$ to lie above another component.

Definition 2.1. Let $\mathbf{C}$ and $\mathbf{D}$ be two distinct components of $\widehat{\mathbf{X}}$. We say that $\mathbf{C}$ *lies above* $\mathbf{D}$ if there is a path $s : [0, 1] \to \pi_1^{-1}((-\infty, 0]) \setminus \mathbf{C}$ such that the initial point $s(0)$ is in $\mathbf{D}$, $s(1) = O$ and if $R = s([0, 1]) \cup [0, \infty) \times \{0\}$, then $\mathbf{C}$ lies in the unique unbounded component of $\mathbb{C} \setminus [\mathbf{D} \cup R]$ which contains the point $1 + 2\pi i$.

Moreover, if $\widehat{\mathbf{X}} = \mathbf{A} \cup \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are disjoint closed sets, such that every component of $\mathbf{A}$ lies above every component of $\mathbf{B}$, then we say that $\mathbf{A}$ *lies above* $\mathbf{B}$.

Lemma 2.2. *The notion of* $\mathbf{C}$ *lying above* $\mathbf{D}$ *is well-defined and for any two components* $\mathbf{C}$ *and* $\mathbf{D}$ *of* $\widehat{\mathbf{X}}$*, one must lie above the other.*

Proof. Let s_1 and s_2 be two paths satisfying the conditions in definition 2.1. Put $R_1 = s_1([0, 1]) \cup [0, \infty) \times \{0\}$, $R_2 = s_2([0, 1]) \cup [0, \infty) \times \{0\}$ and suppose that $\mathbf{C}$ lies in the unbounded component of $\mathbf{D} \cup R_1$ which contains the point $1+2\pi i$. Suppose first that R_1 and R_2 have the same initial point $s_1(0) = s_2(0)$. Since $\pi_1^{-1}((-\infty, 0]) \setminus \mathbf{C}$ is simply connected, there exists a homotopy $j : [0,1] \times [0,1] \to \pi_1^{-1}((-\infty, 0]) \setminus \mathbf{C}$, with endpoints fixed, between s_1 and s_2. Since j^t misses the connected set $\mathbf{C}$ for each t, it follows that $\mathbf{C}$ lies in the component of $\mathbb{C} \setminus [R_2 \cup \mathbf{D}]$ containing $1 + 2\pi i$.

Next suppose that R_1 and R_2 have initial points $\mathbf{z}_1$ and $\mathbf{z}_2$, respectively. Let $\mathcal{U} = \{B(y, \varepsilon) \mid y \in \mathbf{D} \text{ and } \varepsilon = (1/3)\, d(y, [C \cup \pi_1^{-1}([0, \infty))])\}$. Then $\mathcal{U}$ is an open cover of $\mathbf{D}$. Since $\mathbf{D}$ is connected, there exists a chain $\{B_1, \ldots, B_n\}$ of balls in $\mathcal{U}$ such that $\mathbf{z}_1 \in B_1$, $\mathbf{z}_2 \in B_n$ and $B_j \cap B_{j+1} \neq \emptyset$ for $j = 1, \ldots, n-1$. Let J be a piecewise linear arc in $\cup B_j$ from $\mathbf{z}_1$ to $\mathbf{z}_2$. Then there exists a path s_3 such that $s_3([0,1])) = J \cup s_2([0,1])$ is a path with initial point $\mathbf{z}_1$ and terminal point O. Put $R_3 = s_3([0,1]) \cup [0, \infty) \times \{0\}$. Then $\mathbf{C}$ lies in the unbounded component of $\mathbb{C} \setminus [\mathbf{D} \cup R_3]$ which contains the point $1+2\pi i$. Hence, $\mathbf{C}$ lies in the unbounded component of $\mathbb{C} \setminus [\mathbf{D} \cup R_2]$ which contains the point $1 + 2\pi i$.

Suppose that $\mathbf{C}$ and $\mathbf{D}$ are any two components of $\widehat{\mathbf{X}}$. Then $U_{\mathbf{C}} = \mathbb{C} \setminus \mathbf{C}$ and $U_{\mathbf{D}} = \mathbb{C} \setminus \mathbf{D}$ are open and connected sets homeomorphic to $\mathbb{C}$. Hence there exist two arcs $J_{\mathbf{C}} \subset U_{\mathbf{D}} \cap \pi_1^{-1}((-\infty, 0])$ and $J_{\mathbf{D}} \subset U_{\mathbf{C}} \cap \pi_1^{-1}((-\infty, 0])$ joining points $\mathbf{c} \in \mathbf{C}$ and $\mathbf{d} \in \mathbf{D}$ to O, respectively. In addition we may assume that $J_{\mathbf{C}} \cap J_{\mathbf{D}} = \{O\}$. If $\mathbf{D}$ is not contained in the component of $\mathbb{C} \setminus J_{\mathbf{C}} \cup [0, \infty) \times \{0\}$ containing $1+2\pi i$, then $\mathbf{C}$ is contained in the component of $\mathbb{C} \setminus J_{\mathbf{D}} \cup [0, \infty) \times \{0\}$ containing $1+2\pi i$ and $\mathbf{C}$ lies above $\mathbf{D}$. □

Our goal is to show that the condition that $\mathbf{A}$ lies above $\mathbf{B}$ is preserved under the lift of the isotopy h^t.

Lemma 2.3. *Suppose $h^t : Bd(X) \to \mathbb{C}$ is an isotopy such that $h^0 = id_{Bd(X)}$, $O \in Bd(X)$ and $h^t(O) = O$ for all t. Then there exist an isotopy $\mathbf{h}^t : Bd(\widehat{\mathbf{X}}) \to \mathbb{C}$ which lifts h^t such that $\mathbf{h}^0 = id_{Bd(\widehat{\mathbf{X}})}$.*

Proof. For each $x \in \mathrm{Bd}(X) \setminus \{O\}$ and each $\mathbf{x} \in \exp^{-1}(x)$ the path $h|_{\{x\} \times [0,1]}$ has a unique lift to a path $\mathbf{h}_{\mathbf{x}} : [0,1] \to \mathbb{C}$ with initial point $\mathbf{x}$. Define $\mathbf{h}^t(\mathbf{x}) = \mathbf{h}_{\mathbf{x}}(t)$. By uniqueness of lifts, $\mathbf{h}^t$ is one-to-one. It now follows easily that $\mathbf{h}^t$ is an isotopy of $\mathrm{Bd}(\widehat{\mathbf{X}})$ lifting h^t with $\mathbf{h}^0 = id_{\mathrm{Bd}(\widehat{\mathbf{X}})}$. □

The following easy Lemma follows immediately from the fact that $h^t(O) = O$ for all t and that h^t is uniformly continuous.

Lemma 2.4. *Suppose that $h^t(O) = O$ for all t and let $\mathbf{h}^t : Bd(\widehat{\mathbf{X}}) \to \mathbb{C}$ be the isotopy which is the lift of h^t to $Bd(\widehat{\mathbf{X}}) = \exp^{-1}(Bd(X) \setminus \{O\})$*

such that $\mathbf{h}^0 = id_{Bd(\widehat{\mathbf{X}})}$*. Denote* $\mathbf{h}^t(\mathbf{x})$ *by* $\mathbf{x}^t$*. For all* $\varepsilon > 0$ *there exists* $\delta \in \mathbb{R}$ *such that if there exists* $t_0 \in [0,1]$ *such that* $\mathbf{x}^{t_0} \in \widehat{\mathbf{X}}^{t_0}$ *and* $\pi_1(\mathbf{x}^{t_0}) \leq \delta$*, then* $\pi_1(\mathbf{x}^t) < \varepsilon$ *for all* $t \in [0,1]$*. In otherwords, if there exists* t_0 *such that* $\pi_1(\mathbf{x}^{t_0}) \geq \varepsilon$*, then* $\pi_1(\mathbf{x}^t) > \delta$ *for all* $t \in [0,1]$*.*

Given the existence of the lifted isotopy $\mathbf{h}^t$ we will use similar notation as for h^t: for any set $\mathbf{A} \subset \mathrm{Bd}(\widehat{\mathbf{X}})$ we denote by $\mathbf{A}^t$ the set $\mathbf{h}^t(\mathbf{A})$. Recall that U^t is the unbounded component of $\mathbb{C} \setminus h^t(\mathrm{Bd}(X))$, $X^t = \mathbb{C} \setminus U^t$ and $\widehat{\mathbf{X}}^t = \exp^{-1}(X^t \setminus \{O\})$. Also, if $\mathbf{C}^0$ is a component of $\widehat{\mathbf{X}}^0$ choose a point $\mathbf{x}^0 \in \mathrm{Bd}(\widehat{\mathbf{X}}^0) \cap \mathbf{C}$. Then we denote by $\mathbf{C}^t$ the component of $\widehat{\mathbf{X}}^t$ containing the point $\mathbf{x}^t = \mathbf{h}^t(\mathbf{x})$. Next we show that the notion of the component $\mathbf{C}$ being above $\mathbf{D}$ in $\widehat{\mathbf{X}}$ is preserved throughout the isotopy $\mathbf{h}$.

Lemma 2.5. *Let* $\mathbf{C} = \mathbf{C}^0$ *and* $\mathbf{D} = \mathbf{D}^0$ *be components of* $\widehat{\mathbf{X}}^0$ *such that* $\mathbf{C}$ *lies above* $\mathbf{D}$*. Then* $\mathbf{C}^t$ *lies above* $\mathbf{D}^t$ *for each* $t \in [0,1]$*.*

Proof. It suffices to show that there exists $0 < t_0$ such that for all $t \leq t_0$ $\mathbf{C}^t$ lies above $\mathbf{D}^t$. Let $R = s([0,1]) \cup [0,\infty) \times \{0\}$ be a piecewise linear ray landing on $\mathbf{d}^0 \in \mathbf{D}^0$ which satisfies the conditions of Definition 2.1 and such that $R \cap \mathbf{C}^0 = \emptyset$ and $R \cap \mathbf{D}^0 = \{\mathbf{d}^0\}$. Then $R \cup \mathbf{D}^0$ has exactly two complementary domains and each is homeomorphic to $\mathbb{C}$. Hence there exists an arc $A \subset \mathbb{C} \setminus [\mathbf{D}^0 \cup R]$ joining a point $\mathbf{c}^0 \in \mathbf{C}^0$ to the point $1 + 2\pi i$. Choose $a < 0$ such that $A \cup R \subset \pi^{-1}([a,\infty))$. Choose $\varepsilon < (1/3)\, d(A \cup [\pi_1^{-1}([2a,\infty)) \cap \mathbf{C}^0], R \cup [\pi_1^{-1}([2a,\infty)) \cap \mathbf{D}^0])$. Let $0 < t_0$ such that for each $\mathbf{x} \in \mathrm{Bd}(\widehat{\mathbf{X}}) \cap \pi_1^{-1}([2a,\infty))$, $|\mathbf{h}^t(\mathbf{x}) - \mathbf{h}^0(\mathbf{x})| < \varepsilon/2$ and $\pi_1(\mathbf{h}^t(\mathbf{x})) < a$ for all $\mathbf{x} \in \pi_1^{-1}((-\infty, 2a]) \cap \mathrm{Bd}(\widehat{\mathbf{X}})$ for all t. Then for all $t \leq t_0$, $\mathbf{C}^t \cup \mathbf{c}^0\mathbf{c}^t \cup A$ is connected, closed and disjoint from $\mathbf{D}^t \cup \mathbf{d}^0\mathbf{d}^t \cup R$. The first set contains an arc A^* from $\mathbf{c}^t$ to the point $1 + 2\pi i$ and the latter set contains a half ray R^* satisfying the conditions in 2.1 from the point $\mathbf{d}^t \in \mathbf{D}^t$ to ∞. Since $1 + 2\pi i$ is above R^* and $A^* \cap [\mathbf{D}^t \cup R^*] = \emptyset$, $\mathbf{C}^t$ is above $\mathbf{D}^t$ for all $t \in [0, t_0]$. □

Lemma 2.6. *Suppose that* $\widehat{\mathbf{X}}^0 = \mathbf{A}^0 \cup \mathbf{B}^0$ *are disjoint closed sets such that* $\mathbf{A}^0$ *lies above* $\mathbf{B}^0$*. Then for each* t*,* $\widehat{\mathbf{X}}^t = \mathbf{A}^t \cup \mathbf{B}^t$ *and* $\mathbf{A}^t$ *and* $\mathbf{B}^t$ *are disjoint, closed and non-interlaced sets.*

Proof. By Lemma 2.5, every component of $\mathbf{A}^t$ lies above every component of $\mathbf{B}^t$ for all t. Since $\mathbf{h}^t$ is an isotopy it only remains to show that $\mathbf{A}^t$ and $\mathbf{B}^t$ are non-interlaced. To see this fix t, let $\mathbf{w} \in E(\mathbf{A}^t, \mathbf{B}^t)$ and let $K \subset \mathbb{C} \setminus \mathbf{A}^t \cup \mathbf{B}^t$ be the minimal open ball with center $\mathbf{w}$ whose boundary S meets both $\mathbf{A}^t$ and $\mathbf{B}^t$. Suppose that there exist $\mathbf{x}, \mathbf{x}' \in S \cap \mathbf{A}^t$ and $\mathbf{y}, \mathbf{y}' \in S \cap \mathbf{B}^t$ such that $\{\mathbf{y}, \mathbf{y}'\}$ separates $\mathbf{x}$ and $\mathbf{x}'$

in S. For $\mathbf{z} \in \widehat{\mathbf{X}}^t$, let $\mathbf{C_z}$ denote the component of $\widehat{\mathbf{X}}^t$ which contains the point $\mathbf{z}$. Suppose, without loss of generality, that $\mathbf{C_y}$ lies above $\mathbf{C_{y'}}$. We may suppose that $\mathbf{C_y} \cup \mathbf{C_{y'}} \cup \mathbf{yy'}$ separates $\mathbf{x}$ from $1 + 2\pi i$ in $\mathbb{C}$. Since $\mathbf{C_y} \cup \mathbf{C_{y'}}$ does not separate $\mathbb{C}$ by unicoherence, we can choose an arc D in $\mathbb{C} \setminus [\mathbf{C_y} \cup \mathbf{C_{y'}}]$ irreducible from O to $\mathbf{yy'}$ such that $\pi_1(D) \subset (-\infty, 0]$. Let $\{d\} = D \cap \mathbf{yy'}$. Then $\mathbf{C_y} \cup \mathbf{y}d \cup D \cup [0,\infty) \times \{0\}$ separates $\mathbf{y'}$, and hence also $\mathbf{x}$, from $1 + 2\pi i$ and $\mathbf{C_x}$ is below $\mathbf{C_y}$. This contradicts Lemma 2.5 and completes the proof. □

Lemma 2.7. *Suppose $\widehat{\mathbf{X}} = \mathbf{A} \cup \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are disjoint closed subsets of $\mathbb{C}$ such that $\mathbf{A}$ lies above $\mathbf{B}$. Let E be a component of $E(\mathbf{A},\mathbf{B})$. Then E is a closed ray. If $e \in E$ and $r = d(e, \mathbf{A} \cup \mathbf{B})$, then there exist disjoint irreducible arcs or points $J_\mathbf{A}$ and $J_\mathbf{B}$ in $S(e,r)$ such that $\mathbf{A} \cap S(e,r) \subset J_\mathbf{A}$ and $\mathbf{B} \cap S(e,r) \subset J_\mathbf{B}$, and E separates $J_\mathbf{A}$ from $J_\mathbf{B}$ in $\mathbb{C}$.*

Proof. By Lemma 2.6, $\mathbf{A}$ and $\mathbf{B}$ are non-interlaced. By [Bro05][Theorem 3.4.4], E is a 1-manifold. Let E be a component of $E(\mathbf{A},\mathbf{B})$, $e \in E$ and $d(e, \mathbf{A} \cup \mathbf{B}) = r$. Since $\mathbf{A}$ and $\mathbf{B}$ are non-interlaced, there exist disjoint irreducible arcs or points $J_\mathbf{A}$ and $J_\mathbf{B}$ in $S(e,r)$ such that $\mathbf{A} \cap S(e,r) \subset J_\mathbf{A}$ and $\mathbf{B} \cap S(e,r) \subset J_\mathbf{B}$. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be the endpoints of $J_\mathbf{A}$. For $\mathbf{z} \in \widehat{\mathbf{X}}$, let $\mathbf{C_z}$ be the component of $\mathbf{z}$ in $\widehat{\mathbf{X}}$. Let V be the component of $\mathbb{C} \setminus [\mathbf{C}_{\mathbf{a}_1} \cup J_\mathbf{A} \cup \mathbf{C}_{\mathbf{a}_2}]$ containing e and let $W = \mathbb{C} \setminus V$. It follows from the proof of Lemma 2.6 that $W \cap \mathbf{B} = \emptyset$.

Choose $\mathbf{z} \in J_\mathbf{A} \setminus \mathbf{A}$ and $\mathbf{w} \in \mathbf{B}$. If $\mathbf{zw} \cap \mathbf{A} \neq \emptyset$, then $d(\mathbf{z}, \mathbf{A}) < d(\mathbf{z}, \mathbf{w})$. If $\mathbf{zw} \cap \mathbf{A} = \emptyset$, then it follows easily that $d(\mathbf{z}, \mathbf{w}) > \min\{d(\mathbf{z}, \mathbf{a}_1), d(\mathbf{z}, \mathbf{a}_2)\}$. Hence for all $\mathbf{w} \in \mathbf{B}$, $d(\mathbf{z}, \mathbf{A}) < d(\mathbf{z}, \mathbf{w})$ and $E(\mathbf{A},\mathbf{B}) \cap J_\mathbf{A} = \emptyset$. Choose $\mathbf{b} \in J_\mathbf{B} \cap \mathbf{B}$. Note that $E(\mathbf{A},\mathbf{B}) \cap \mathbf{a}_1 e \setminus \{e\} = \emptyset = E(\mathbf{A},\mathbf{B}) \cap e\mathbf{b} \setminus \{e\}$. Now $E(\mathbf{A},\mathbf{B})$ separates $\mathbf{a}_1$ and $\mathbf{b}$. By unicoherence of $\mathbb{C}$ a component of $E(\mathbf{A},\mathbf{B})$ separates $\mathbf{a}_1$ and $\mathbf{b}$. Since this component must contain e, E separates $\mathbf{a}_1$ and $\mathbf{b}$ in $\mathbb{C}$. Hence E separates $\mathbf{C}_{\mathbf{a}_1} \cup J_\mathbf{A}$ and $\mathbf{C_b} \cup J_\mathbf{B}$ which both are unbounded sets. Hence, E is an unbounded closed 1-manifold, i.e., E is a closed ray. □

Theorem 2.8. *Suppose that $\widehat{\mathbf{X}} = \mathbf{A} \cup \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are disjoint, closed, non-empty sets such that $\mathbf{A}$ lies above $\mathbf{B}$. Then $E(\mathbf{A},\mathbf{B})$ is a closed ray such that $\pi_1(E(\mathbf{A},\mathbf{B})) = (-\infty,\infty)$ and for $x > 0$, $|\pi_1^{-1}(x) \cap E(\mathbf{A},\mathbf{B})| = 1$.*

Proof. By Lemma 2.7, each component of $E(\mathbf{A},\mathbf{B})$ is a closed ray which stretches to $-\infty$. For $\mathbf{z} \in \widehat{\mathbf{X}}$, let $\mathbf{C_z}$ be the component of $\mathbf{z}$ in $\widehat{\mathbf{X}}$.

Let $\mathbf{a} \in \widehat{\mathbf{X}}$ such that $\pi_1(\mathbf{a}) = \max(\pi_1(\widehat{\mathbf{X}})) < 0$. Without loss of generality, $\mathbf{a} \in \mathbf{A}$. Let $R = \mathbf{a}O \cup [0,\infty) \times \{0\}$, then $R \setminus \{\mathbf{a}\}$ is a ray disjoint from $\widehat{\mathbf{X}}$ which lands on $\mathbf{a}$. Note that $\mathbb{C} \setminus [R \cup \mathbf{C_a}] = W \cup V$, where W and V are disjoint, connected, open and non-empty sets. Without loss of generality, $1 + 2\pi i \in W$. Then every component of $\widehat{\mathbf{X}} \cap W$ is above $\mathbf{C_a} \subset \mathbf{A}$. Hence $\mathbf{B} \subset V$. Since $\mathbf{B} \neq \emptyset$, there exists $\mathbf{b} \in \mathbf{B}$ such that $\pi_1(\mathbf{b}) = \pi_1(\mathbf{a})$. By compactness of $X \cap S(O, e^{\pi_1(\mathbf{a})})$, we may assume that

$$\pi_2(\mathbf{a}) = \min(\pi_2(\mathbf{A} \cap \pi_1^{-1}(\pi_1(\mathbf{a})))) \text{ and}$$
$$\pi_2(\mathbf{b}) = \max(\pi_2(\mathbf{B} \cap \pi_1^{-1}(\pi_1(\mathbf{a})))).$$

Then $0 < \pi_2(\mathbf{a}) - \pi_2(\mathbf{b}) \leq 2\pi$ and we may assume that $0 < \pi_2(\mathbf{a}) \leq \pi$. For $z \in [\pi_1(\mathbf{a}), \infty) \times [\pi_2(\mathbf{a}), \infty)$, $d(z, \mathbf{A}) < d(z, \mathbf{B})$ and for $z \in [\pi_1(\mathbf{a}), \infty) \times (-\infty, \pi_2(\mathbf{b})]$, $d(z, \mathbf{B}) < d(z, \mathbf{A})$. By unicoherence there exists a component E of $E(\mathbf{A}, \mathbf{B})$ which separates $\mathbf{a}$ and $\mathbf{b}$. Then E separates $\mathbf{C_a} \cup [\pi_1(\mathbf{a}), \infty) \times \pi_2(\mathbf{a})$ from $\mathbf{C}_b \cup [\pi_1(\mathbf{a}), \infty) \times \pi_2(\mathbf{b})$. So $\pi_1(E) = (-\infty, \infty)$ and $E(\mathbf{A}, \mathbf{B}) \cap [\pi_1(\mathbf{a}), \infty) \times \mathbb{R}] \subset [\pi_1(\mathbf{a}), \infty) \times [\pi_2(\mathbf{b}), \pi_2(\mathbf{a})]$. In particular, $E(\mathbf{A}, \mathbf{B}) \cap \pi_1^{-1}(x)$ is compact for $x > 0$.

It remains to be shown that $E(\mathbf{A}, \mathbf{B})$ is connected. We prove first that E separates $\mathbf{A}$ and $\mathbf{B}$. Suppose that $\mathbb{C} \setminus E = W' \cup V'$, where W' and V' are disjoint, non-empty, open and connected sets, and $1 + 2\pi i \in W'$. Just suppose there exist $\mathbf{y} \in \mathbf{A} \cap V'$. Since neither of the disjoint closed sets E nor $\mathbf{B}$ separates $\mathbf{y}$ from $1 - 2\pi i$, neither does their union. Let $D \subset \mathbb{C} \setminus [E \cup B]$ be an arc from $\mathbf{y}$ to $1 - 2\pi i \in V'$. Choose $e \in E$ such that if $r = d(e, \widehat{\mathbf{X}})$, then $\pi_1(e) + r < \min\{\pi_1(D)\}$. Let $\mathbf{w} \in S(e, r) \cap \mathbf{B}$. Then $\mathbf{C_w} \cup \mathbf{w}e \cup E'$, where E' is the component of $E \setminus \{e\}$ which projects under π_1 over $[\pi_1(y), \infty)$, does not separate $\mathbf{y}$ from $1 - 2\pi i$. It now follows easily that $\mathbf{C_y}$ lies below $\mathbf{C_w}$, a contradiction.

Hence, we can conclude that $\mathbf{A} \subset W'$, $\mathbf{B} \subset V'$ and $E(\mathbf{A}, \mathbf{B}) \cap [W' \cup V'] = \emptyset$. It follows that for all $z \in W'$, $d(z, \mathbf{A}) < d(z, \mathbf{B})$ and for all $z \in V'$, $d(z, \mathbf{B}) < d(z, \mathbf{A})$, and $E(\mathbf{A}, \mathbf{B}) = E$.

Suppose that $e_1 \neq e_2 \in E$ and $d(e_i, \mathbf{A} \cup \mathbf{B}) = r_i$. Then for all $\mathbf{z}_i \in S(e_i, r_i) \cap [\mathbf{A} \cup \mathbf{B}]$, $[e_1\mathbf{z}_1 \setminus \{\mathbf{z}_1\}] \cap [e_2\mathbf{z}_2 \setminus \{\mathbf{z}_2\}] = \emptyset$. Since $\mathbf{A}$ lies above $\mathbf{B}$, for $e \in E$ with $\pi_1(e) > 0$ and $r = d(e, \mathbf{A} \cup \mathbf{B})$, for all $\mathbf{z} \in \mathbf{A} \cap S(e, r)$ and $\mathbf{w} \in \mathbf{B} \cap S(e, r)$, $\pi_2(\mathbf{z}) > \pi_2(\mathbf{w})$. It now follows easily that for such e, $\pi_1^{-1}(\pi_1(e)) \cap E(\mathbf{A}, \mathbf{B}) = \{e\}$.

□

3. Characterizing accessibility

In this section we provide a characterization of accessibility for points in X and show that accessibility is preserved under the isotopy h. In this section we will always assume that $O \in \mathrm{Bd}(X)$ is fixed under the isotopy h. Easy examples (e.g., a half ray spiralling around the closed interval $[-1,1]\times\{0\}$) show that accessibility of O in X is not equivalent to $\widehat{\mathbf{X}}$ being not connected. Nevertheless the spirit of this idea is correct:

Lemma 3.1. *Suppose that $O \in Bd(X)$. Then the following are equivalent:*

(1) *O is accessible,*
(2) *$\widehat{\mathbf{X}} = \mathbf{A}\cup\mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are non-empty, disjoint and closed such that:*

$$\mathbf{A} \textit{ lies above } \mathbf{B}$$

and for all $x \in \mathbb{R}$ there exists $y_1 < y_2$ in $\mathbb{R}$ such that

$$\pi_1^{-1}([x,\infty)) \cap \pi_2^{-1}([y_2,\infty)) \cap \mathbf{B} = \emptyset \textit{ and} \tag{2a}$$

$$\pi_1^{-1}([x,\infty)) \cap \pi_2^{-1}((-\infty,y_1]) \cap \mathbf{A} = \emptyset. \tag{2b}$$

Proof. Suppose first that O is accessible, let R be a conformal external ray landing on O and let $\mathbf{J}$ be a component of $\exp^{-1}(R)$. Then $\mathbf{J}$ is a closed ray in $\mathbb{C}\setminus\widehat{\mathbf{X}}$ such that $\pi_1(\mathbf{J}) = (-\infty,\infty)$ and for every vertical line ℓ, $\mathbf{J}\cap\ell$ is compact. Note that $\mathbb{C}\setminus\mathbf{J} = U \cup V$, where U and V are disjoint open and connected sets. We may assume that for some vertical line ℓ, $\pi_2(\ell\cap U)$ has no upper bound and, since $\widehat{\mathbf{X}}$ is invariant under vertical translations by 2π, that $\{1+2\pi i\}\subset U$. Put $\mathbf{A} = \widehat{\mathbf{X}}\cap U$ and $\mathbf{B} = \widehat{\mathbf{X}}\cap V$ then condition (2) holds. The fact every component of $\mathbf{A}$ lies above every component of $\mathbf{B}$ follows from the fact that U and V are open and connected, and U is "above" V. To see that (2a) and (2b) hold note that close to infinity R behaves like a radial line segment in the plane and, hence, $\mathbf{J}$ behaves like a horizontal line segment near $+\infty$ so $\pi_2(\mathbf{J}\cap\pi_1^{-1}([a,\infty)))$ is bounded for each $a\in\mathbb{R}$.

Suppose next that conditions (2), (2a) and (2b) hold. By Theorem 2.8, $E = E(\mathbf{A},\mathbf{B})$ is a closed ray which runs from $-\infty$ to ∞ and separates $\mathbf{A}$ from $\mathbf{B}$.

Claim: For every compact arc $C \subset x$-axis, $\pi_1^{-1}(C)\cap E$ is compact.

Proof of Claim. Let $C\subset x$-axis be a compact interval. Suppose without loss of generality that $\pi_1^{-1}(C)\cap E$ contains points e_i with $\lim\pi_2(e_i) = +\infty$. Note that there exists $K>0$ such that for each $z\in\pi_1^{-1}(C)$, $d(z,\widehat{\mathbf{X}})\le K$. Hence for each i there exists $\mathbf{b}_i\in\mathbf{B}$ such that $d(e_i,\mathbf{b}_i)\le K$ and $\lim\pi_2(\mathbf{b}_i) = +\infty$. This contradicts (2a) and completes the proof of the claim.

Now let $MR = \overline{\exp(E)}$. Then MR is a closed and connected set in $\mathbb{C}$ and it suffices to show that $MR\cap X = \{O\}$. Since $\pi_1(E) = (-\infty,\infty)$, $O\in MR$. Suppose that $x\in X\setminus\{O\}$ is also a limit point of MR. Choose $z_i\in\exp(E)$ such that $\lim z_i = x$ and $\mathbf{z}_i\in E$ such that $\exp(\mathbf{z}_i) = z_i$. Then $d(\mathbf{z}_i,\exp^{-1}(x))\to 0$. Since E is closed and disjoint from $\widehat{\mathbf{X}}$, the sequence $\mathbf{z}_i$ cannot be convergent and so $\lim |\pi_2(\mathbf{z}_i)| = \infty$. By choosing a compact arc C in R which contains $\pi_1(\exp^{-1}(x))$ in its interior, we see that $E \cap \pi_1^{-1}(C)$ is not compact. This contradiction completes the proof. □

The characterization Lemma 3.1 allows us to show that an accessible point remains accessible throughout the isotopy.

Theorem 3.2. *If x is accessible from $U = \mathbb{C} \setminus X$, then $h^t(x)$ is accessible from U^t, where U^t is the unbounded component of $\mathbb{C} \setminus h^t(Bd(X))$, for each $t \in [0,1]$.*

Proof. Suppose that x^0 is an accessible point of X^0. We may assume that $x^0 = O$, $h^t(O) = O$ and $X^t \subset B(O,1)$ for all t.

By Lemma 3.1 $\widehat{\mathbf{X}}^0 = \mathbf{A}^0 \cup \mathbf{B}^0$ such that conditions (2), (2a) and (2b) of Lemma 3.1 hold. By Lemma 2.3 we can lift the isotopy h^t to an isotopy $\mathbf{h}^t : \mathrm{Bd}(\widehat{\mathbf{X}}^0) \to \mathbb{C}$ such that $\mathbf{h}^0 = id_{\mathrm{Bd}(\widehat{\mathbf{X}}^0)}$. By Lemma 2.5, $\mathbf{A}^t$ lies above $\mathbf{B}^t$ for all t. It remains to be shown that conditions (2a) and (2b) are satisfied for all t. By symmetry it suffices to show that (2a) holds. Suppose $x \in \mathbb{R}$. By Lemma 2.4 there exists $x' \in \mathbb{R}$ such that for all t, $\max(\pi_1(\mathbf{h}^t(\pi_1^{-1}((-\infty, x']) \cap \widehat{\mathbf{X}}^0)) < x$. By (2a) for $t = 0$, there exists y_2 such that $\pi_1^{-1}([x', \infty)) \cap \pi_2^{-1}([y_2, \infty)) \cap B^0 = \emptyset$. Choose y_3 such that for all t, $\max(\pi_2(\mathbf{h}^t(\pi_1^{-1}([x', \infty)) \cap \pi_2^{-1}((-\infty, y_2])) \cap \widehat{\mathbf{X}}))) < y_3$. Then $\pi_1^{-1}([x, \infty)) \cap \pi_2^{-1}([y_3, \infty)) \cap B^t = \emptyset$ and (2a) holds for all t. Hence by Lemma 3.1, O is accessible for all t. □

4. Continuity of external angles

Given a non-separating continuum X, an isotopy $h^t : \mathrm{Bd}(X) \to \mathbb{C}$ such that $h^0 = id_{\mathrm{Bd}(X)}$, let U^t be the unbounded component of $\mathbb{C} \setminus h^t(\mathrm{Bd}(X))$. We construct an isotopy $\alpha^t : S^1 \to S^1$ such that if the conformal ray $R_\theta \subset U^0$ lands on x, then $R_{\alpha^t(\theta)} \subset U^t$ lands on x^t in X^t for each t. This is accomplished in two steps. We first construct for each t a continuously (in the sense of Hausdorff metric) varying arc L^t landing on x^t. This arc is contained in the image under the exponential map of the equidistant set in Section 2.

Lemma 4.1. *Let O be an accessible point of X. Then there exists for each t an arc L^t such that $X^t \cap L^t = \{O^t\}$ is an endpoint of L^t and the function $\beta : [0,1] \to C(\mathbb{C})$ defined by $\beta(t) = L^t$ is a continuous function to the space $C(\mathbb{C})$ of compact subsets of $\mathbb{C}$ with the Hausdorff metric.*

Proof. We assume as usual that $h^t(O) = O$ and $X^t \subset B(O,1)$ for all t. Since every half ray in the plane is tame, we may assume that the positive x-axis is contained in $\mathbb{C} \setminus X^0$. Then $\widehat{\mathbf{X}} \subset \mathbb{C} \setminus \pi_2^{-1}(\{0\})$. Let $\mathbf{A}^0 = \widehat{\mathbf{X}} \cap \pi_2^{-1}((0,\infty))$ and $\mathbf{B}^0 = \widehat{\mathbf{X}} \cap \pi_2^{-1}((-\infty,0))$. Then $\widehat{\mathbf{X}}$ is the union of these two disjoint closed sets and $\mathbf{A}^0$ lies above $\mathbf{B}^0$. Since $\widehat{\mathbf{X}}$ is invariant under vertical translation by 2π, it follows that $E(\mathbf{A}^0, \mathbf{B}^0)$ is contained in $\pi_2^{-1}([-2\pi, 2\pi])$. By Lemma 2.5, $\mathbf{A}^t$ lies above $\mathbf{B}^t$ for each t. By Theorem 2.8, $E(\mathbf{A}^t, \mathbf{B}^t)$ is a ray which separates $\mathbf{A}^t$ and $\mathbf{B}^t$ in $\mathbb{C}$ and $\pi_1(E(\mathbf{A}^t, \mathbf{B}^t)) = (-\infty, \infty)$.

Let $t_i \to t_0 \in [0,1]$. Then $\mathbf{A}^{t_i} \to \mathbf{A}^{t_0}$ and $\mathbf{B}^{t_i} \to \mathbf{B}^{t_0}$ on compact sets (i.e., K compact in $\mathbf{A}^0$ implies $K^{t_i} \to K^{t_0}$). It is easy to check that if $e_i \in E(\mathbf{A}^{t_i}, \mathbf{B}^{t_i})$ and $e_i \to e$, then $e \in E(\mathbf{A}^{t_0}, \mathbf{B}^{t_0})$.

By Theorem 2.8 $|E(\mathbf{A}^t, \mathbf{B}^t) \cap \pi_1^{-1}(1)| = 1$. For each t, let $M^t = E(\mathbf{A}^t, \mathbf{B}^t) \cap \pi_1^{-1}((-\infty, 1])$. Then M^t is connected and $\lim M^{t_i} = M^{t_0}$. Hence $L^t = \overline{\exp(M^t)}$ is the required arc. □

By a *crosscut* C of a non-separating continuum $X \subset \mathbb{C}$ we mean an open arc $C \subset \mathbb{C} \setminus X$ whose closure is a closed arc with distinct endpoints a and b which are in X. In this case we say that the crosscut C *joins* the points a and b of X. By the *shadow* of C, denoted by $Sh(C)$, we mean the closure of the bounded complementary domain of $\mathbb{C} \setminus [X \cup C]$.

Theorem 4.2. *Suppose that $O \in Bd(X)$, $h^t : Bd(X) \to \mathbb{C}$ is an isotopy such that $h^0 = id_{Bd(X)}$, $h^t(O) = O$ and $diam(X^t) < 1$ for all t. Let U^t be the component of $\mathbb{C}^* \setminus h^t(Bd(X))$ containing ∞, let $\varphi^t : \mathbb{D} \to U^t$ be the normalized Riemann map and let L^t be an arc with one endpoint at O such that L^t varies continuously with t in the Hausdorff metric and $L^t \cap X^t = \{O\}$ for all t. Then the function $\alpha : [0,1] \to S^1$ defined by $\alpha(t) = S^1 \cap \overline{(\varphi^t)^{-1}(L^t)}$ is continuous and the external ray $\varphi^t(\{re^{2\pi i \alpha(t)} \mid r < 1\}) = R^t_{\alpha(t)}$ lands on O in X^t for each t.*

We shall refer to the function $\alpha : [0,1] \to S^1$ in Theorem 4.2 as the *continuous angle function.*

Proof. By [Mil00], α as defined in the statement of the Lemma is a function. It remains to be shown that α is continuous. We will first present an outline of the proof. Fix $\varepsilon > 0$. Let $a(t_0)$ and $b(t_0)$ be endpoints of a crosscut $C(t_0)$ of X^{t_0} in $B(O, 1/2)$ such that L^{t_0} lands in the shadow of the crosscut $C(t_0)$ and $\mathrm{diam}((\varphi^{t_0})^{-1}(C(t_0))) < 2\varepsilon/3$. Choose $\beta < 1/5\ \min(d(a(t_0), b(t_0)), d(L^{t_0}, \{a^{t_0}, b^{t_0}\}), d(O, C(t_0)))$. We shall choose K, a large compact subarc of $C(t_0)$, such that $B(L^{t_0}, \beta) \cap C(t_0) \subset K$ and such that $L^t \subset B(L^{t_0}, \beta)$ and $K \cap X^t = \emptyset$ whenever t is close to t_0. We shall define K^t, a crosscut of X^t, which contains a large sub-arc of K together with two small arcs $J(a,t)$ and $J(b,t)$ which join points close to the endpoints of K to X^t such that $(\varphi^t)^{-1}(K^t)$ is a small crosscut of $\mathbb{D}$ whose shadow contains $\alpha(t_0)$ and $\alpha(t)$ for t sufficiently close to t_0.

Let $C(t_0)$, $a(t_0)$, $b(t_0)$ and β be defined as above and let $\hat{a}(0)$ and $\hat{b}(0)$ be the endpoints of $(\varphi^{t_0})^{-1}(C(t_0))$. We may assume that $\alpha(t_0)$ is contained in the interval $(\hat{a}(0), \hat{b}(0))$ in the boundary circle S^1 contained in the shadow of $(\varphi^{t_0})^{-1}(C(t_0))$. Then $|\hat{b}(0) - \hat{a}(0)| < 2\varepsilon/3$. Choose $\delta_1 > 0$ such that $\delta_1 < (1/5)\min(|\hat{b}(0) - \alpha(t_0)|, |\alpha(t_0) - \hat{a}(0)|, \varepsilon/4, \beta)$. Let $\rho < \sqrt{\rho} < \min\{\delta_1, 1\}$ such that $\frac{2\pi}{\sqrt{\ln(1/\rho)}} < \delta_1$ and

$$(\varphi^{t_0})^{-1}(B(a(t_0), 2\sqrt{\rho}) \cap C(t_0)) \subset B(\hat{a}^{t_0}, \delta_1), \tag{4.1}$$

$$(\varphi^{t_0})^{-1}(B(b(t_0), 2\sqrt{\rho}) \cap C(t_0)) \subset B(\hat{b}^{t_0}, \delta_1), \tag{4.2}$$

and there is just one component K of $C(t_0)\setminus[B(a(t_0), \rho/2) \cup B(b(t_0), \rho/2)]$ which meets both $S(a(t_0), \rho/2)$ and $S(b(t_0), \rho/2)$.

Next choose $\delta_2 > 0$ such that for all $|t - t_0| < \delta_2$:

(1) $L^t \subset B(L^{t_0}, \rho/2)$,
(2) $d(h^t, h^{t_0}) < \rho/2$,
(3) $X^t \cap K = \emptyset$ and
(4) $d((\varphi^t)^{-1}|_K, (\varphi^{t_0})^{-1}|_K) < \delta_1$.

By [Pom92, Prop. 2.2] there exist $\rho \le r, s \le \sqrt{\rho}$ such that if $J(a, t)$ is a component of $S(a(t_0), r) \setminus X^t$ which meets K and and $J(b, t)$ is a component of $S(b(t_0), s) \setminus X^t$ which meets K, then

$$\operatorname{diam}(\varphi^t)^{-1}(J(z, t)) \le \frac{2\pi}{\sqrt{\ln(1/\rho)}} < \delta_1 \text{ for } z \in \{a, b\}. \tag{4.3}$$

Then $K \cup J(a, t) \cup J(b, t)$ contains a crosscut $C(t)$ of X^t and L^t lands in the shadow of $C(t)$. Since the endpoints of $C(t)$ are joined by a subarc of $J(z, t) \subset B(z, \sqrt{\rho})$ to K it follows from (4.1), (4.2), (4.3) and (4) that the endpoints of $(\varphi^t)^{-1}(C(t))$ are within $3\delta_1 < (3/20)\varepsilon$ of the endpoints $\hat{a}^{t_0}$ and $\hat{b}^{t_0}$ of $(\varphi^{t_0})^{-1}(C(t_0))$. Then for $|t - t_0| < \delta_2$, $\alpha(t_0)$ is in the shadow of $(\varphi^t)^{-1}(C(t))$, $\varphi^t(L^t)$ lands in this shadow and the distance between the endpoints of $(\varphi^t)^{-1}(C(t))$ is less than $6\delta_1 + 2\varepsilon/3 < (6/20 + 2/3)\varepsilon < \varepsilon$ as desired. □

Theorem 4.3. *Suppose h^t is an isotopy of the boundary of a non-separating continuum $X \subset \mathbb{C}$ such that $h^0 = id_{Bd(X)}$. Let U^t be the component of $\mathbb{C}^* \setminus h^t(Bd(X))$ containing ∞ and let $\varphi^t : \mathbb{D} \to U^t$ denote the normalized Riemann map. Then there exists an isotopy $\alpha^t : S^1 \to S^1$ such that $\alpha^0 = id_{S^1}$ and if R^0_θ lands on $x^0 \in X^0$, then $R^t_{\alpha^t(\theta)}$ lands on x^t for each t.*

Proof. Suppose that R^0_θ lands on $x^0 \in X^0$. By Theorem 4.2, there exists a continuous function $\alpha_\theta : [0,1] \to S^1$ such that $\alpha_\theta(0) = \theta$ and $R^t_{\alpha_\theta(t)}$ lands on x^t for each t. Let $\mathcal{A}$ be the set of angles in S^1 such that for each $\theta \in \mathcal{A}$, R^0_θ lands on a point $x(\theta) \in X^0$. Define $\alpha^t : \mathcal{A} \to S^1$ by $\alpha^t(\theta) = \alpha_\theta(t)$. Then α^t is a circular order preserving isotopy of $\mathcal{A}$ such that $\alpha^0 = id_{\mathcal{A}}$. Since $\mathcal{A}$ is dense in S^1, α^t can be extended to an isotopy of S^1. □

We will refer to the isotopy α^t as the *continuous angle isotopy.*

5. Extension over hyperbolic crosscuts

Suppose U is a component of $\mathbb{C} \setminus Z$ and $h^t : Z \to \mathbb{C}$ is an isotopy such that $h^0 = id_Z$. Then there exists a path $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) \in U$ and $\gamma(t) \in \mathbb{C} \setminus Z^t$ for all t. Then we denote by U^t the component of $\mathbb{C} \setminus Z^t$ which contains the point $\gamma(t)$. Note that U^t is independent of the choice of the path γ. We have shown in the previous section that if there exists a crosscut in U^0 joining the points a^0 and b^0, then for each t there exists a crosscut in U^t joining the points a^t and b^t. We will show next that we can choose for each t a natural crosscut C^t joining these points such that the isotopy h can be extended over $X^0 \cup C^0$. For this purpose we will use hyperbolic geodesics defined by the Poincaré metric on $\mathbb{D}$.

Suppose that a^0 and b^0 are the landing points of the external rays $R^0_{\theta(a)}$ and $R^0_{\theta(b)}$ in $\mathbb{C} \setminus X^0$. By Theorem 4.2, there exist continuous angle functions $\alpha : [0,1] \to S^1$ and $\beta : [0,1] \to S^1$ such that for each t, $R^t_{\alpha(t)}$ and $R^t_{\beta(t)}$ land on a^t and b^t, respectively. Let G^t be the hyperbolic geodesic joining the points $\alpha(t)$ and $\beta(t)$ in $\mathbb{D}$ (i.e., G^t is the intersection of the round circle through the points $\alpha(t)$ and $\beta(t)$ with $\mathbb{D}$ which crosses S^1 perpendicularly at both of these points). Let $C^t = \varphi^t(G^t)$. We will call C^t the *hyperbolic crosscut of X^t joining the points a^t and b^t*. In the final part of this section we will consider Z as a subset of the sphere and show that the isotopy $h : X^0 \times [0,1] \to \mathbb{C}^*$ can be extended to an isotopy $H : X^0 \cup C^0 \times [0,1] \to \mathbb{C}^*$ such that $H^t(C^0) = C^t$, where C^t is the hyperbolic crosscut of X^t joining a^t to b^t. We will make use of the following well-known Theorem [Pom92, Theorem 4.20][2].

Theorem 5.1 (Gehring-Hayman Theorem)**.** *There exists a universal constant K such that for for any conformal map $\varphi : \mathbb{D} \to \mathbb{C}$, if $z_1, z_2 \in \overline{\mathbb{D}}$, γ is an arc in $\mathbb{D}$ from z_1 to z_2, and S is the hyperbolic geodesic from z_1 to z_2, then $diam(\varphi(S)) \leq K\, diam(\varphi(\gamma))$.*

[2]We are indebted to Paul Fabel for this reference.

Recall that X is a non-separating plane continuum. Each angle $\theta \in S^1$ corresponds to a *prime end* of $\mathbb{C}^* \setminus X$. By a fundamental chain C_j of crosscuts we mean a sequence of crosscuts of X such that $\lim \operatorname{diam}(C_j) = 0$, $C_i \subset Sh(C_j)$ for $i > j$ and the arcs $\{\overline{C_i}\}$ are all pairwise disjoint. A naturally defined equivalence class of fundamental chains is called a prime end of $\mathbb{C}^* \setminus X$ (see [Mil00] for further details).

Lemma 5.2. *Let h be an isotopy of $Bd(X)$, $O \in Bd(X)$ and $h^t(O) = O$ for all t. Suppose that R^0_θ is a conformal external ray of X^0 landing on O. Then the isotopy h can be extended to an isotopy $H : [X \cup R^0_\theta] \times [0,1] \to \mathbb{C}$ such that $H^t(R^0_\theta)$ is an external ray of X^t landing on O.*

Proof. By Lemma 4.2, there exists a continuous angle function $\alpha : [0,1] \to S^1$ such that $\alpha(0) = \theta$ and the (conformal) external ray $R^t_{\alpha(t)}$ lands on O for each t. Extend the isotopy h over $R^0_{\alpha(0)}$ by

$$H(z,t) = \varphi^t \circ \rho^t \circ (\varphi^0)^{-1}(z) \tag{5.1}$$

for $z \in R^0_{\alpha(0)}$, where ρ^t is the rotation of $\mathbb{D}$ by the angle $\alpha(t) - \alpha(0)$. By Carathéodory kernel convergence, H is an isotopy of every compact subset of $R^0_{\alpha(0)}$. Hence it suffices to show that if $z_i \to O$ in $R^0_{\alpha(0)}$ and $t_i \to t_\infty$, then $H(z_i, t_i) \to O = H(O, t_\infty)$.

To see this fix $\varepsilon > 0$. It suffices to show that there exists an open disk B containing O with simple closed curve boundary S and $\delta > 0$ such that for all t with $|t - t_\infty| < \delta$, if z^t is the first point of $R^t_{\alpha(t)}$ (from ∞) on S and if $CR^t_{z^t}$ is the component of $R^t_{\alpha(t)} \setminus z^t$ from z^t to O, then $CR^t_{z^t} \subset B(O, \varepsilon)$.

Let K be the universal constant from Theorem 5.1. By Lemma 4.1 there exists a continuously varying arc $L^t \subset \mathbb{C} \setminus X^t$ landing on O in X^t for each t such that $\overline{(\varphi^0)^{-1}(L^0)} \cap S^1 = \{\theta\}$. Choose a fundamental chain of crosscuts $C^{t_\infty}_n$ of X^{t_∞} for the prime-end $\alpha(t_\infty)$. Then both L^{t_∞} and $R^{t_\infty}_{\alpha(t_\infty)}$ cross $C^{t_\infty}_n$ essentially (that is $X \cup C^{t_\infty}_n$ separates the endpoints of L^{t_∞} and also the ends of the ray $R^{t_\infty}_{\alpha(t_\infty)}$). Hence we can choose n sufficiently large and a simple closed curve S containing O in its bounded complementary domain B such that $C^{t_\infty}_n \subset S$, $[L^{t_\infty} \cup R^{t_\infty}_{\alpha(t_\infty)}] \cap [S \setminus C^{t_\infty}_n] = \emptyset$ and $\operatorname{diam}(S) < \varepsilon/K$. From now on fix this n and let a and b be the endpoints of $C^{t_\infty}_n$.

For t close to t_∞, let w^t be the first point (from O) of L^t on S. Let C^t be the component of $S \setminus X^t$ containing the point w^{t_∞}. Choose $\rho < (1/3)d(\{a,b\}, [L^{t_\infty} \cup R^{t_\infty}_{\alpha(t_\infty)}])$ and let C^t_- be the component of $C^t \setminus [B(a,\rho) \cup B(b,\rho)]$ which contains w^{t_∞}. Choose $\delta > 0$ such that if $|t - t_\infty| < \delta$, then

(1) $w^{t_\infty} \in \mathbb{C} \setminus [X^t \cup B(a,\rho) \cup B(b,\rho)]$,
(2) $C^t_- = C^{t_\infty}_-$,
(3) $L^t \subset B(L^{t_\infty}, \rho)$,
(4) if z^t is the first point of $R^t_{\alpha(t)}$ (from infinity) on S, then $z^t \in C^t_-$.

The first and second conditions follow from the continuity of h and the third from the continuity of L^t. The last condition follows from Carathéodory kernel convergence: recall that $d(R^{t_\infty}_{\alpha(t_\infty)}, S \setminus C^{t_\infty}) = \eta > 0$. Let $v \in R^{t_\infty}_{\alpha(t_\infty)} \cap B$ such that the component of $R^t_{\alpha^t(\theta)} \setminus \{v\}$ from v to O is contained in B and let $(\varphi^{t_\infty})^{-1}(v) = r_0 \exp(\alpha(t_\infty))$. By Carathéodory kernel convergence, $I^t = \varphi^t(\{r \exp(\alpha(t)) \mid 0 \le r \le r_0\})$ converges to the segment from v to ∞ in $R^{t_\infty}_{\alpha(t_\infty)}$. Hence $d(I^t, S \setminus C^{t_\infty}) > \eta/2$ and $d(I^t, v) < (1/2)d(v,S)$ for t close to t_∞, and (4) hold for δ sufficiently small.

By (2), (3) and (4), the sub-arc A^t of C^t joining the points w^t and z^t, is contained in $\mathbb{C} \setminus X^t$. Hence the union of the arcs A^t and $[w^t, O] \subset L^t$ is an arc in $[\mathbb{C} \setminus X^t] \cup \{O\}$, joining z^t to O, of diameter less than ε/K. By Theorem 5.1, the terminal segment $CR^t_{z^t} \subset B(O, \varepsilon)$ as required.

□

In the remaining part of the paper we will consider Z as a subset of the unit sphere $\mathbb{C}^* \subset \mathbb{R}^3$ with *spherical metric* ρ. Hence the distance between two points $z, w \in \mathbb{C}^*$ is the length of the shortest arc in the great circle which is the intersection of $\mathbb{C}^*$ and the plane through z, w and the origin in $\mathbb{R}^3$.

Since every hyperbolic crosscut is conformally equivalent to a diameter of $\mathbb{D}$ it follows that we can extend the isotopy h^t over any hyperbolic crosscut $C^0 \subset U^0$ joining two points a^0 and b^0 in Z^0 to an isotopy $H : Z^0 \cup C^0 \to \mathbb{C}^*$ (since in this case the point at infinity is not fixed, the range of the isotopy must be the sphere). Note that if C_i is a convergent sequence of hyperbolic crosscuts whose limit contains a non-degenerate subcontinuum $Y \subset Z$, then this extension of the isotopy over $\cup C_i$ is not necessarily continuous at Y. However, we can extend over a suitable compact set of hyperbolic crosscuts in U as follows. At this point it will be convenient to change to the Cayley-Klein model[3] of the hyperbolic disk. There exists a homeomorphism $g : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, which is the identity on the boundary S^1 of $\mathbb{D}$, such that g preserves radial line segments and for any two points $\theta_1, \theta_2 \in S^1$ the hyperbolic geodesic G joining θ_1 to θ_2 is mapped to the straight line segment $\theta_1\theta_2$, which is a chord of the unit disk with endpoints θ_1 and θ_2.

[3]We are indebted to Nandor Simanyi for suggesting the Cayley-Klein model.

Suppose that $\mathcal{H}$ is a collection of disjoint hyperbolic crosscuts in U such that the set $\bigcup_{c\in\mathcal{H}}\overline{C}$ is compact, then we call $\mathcal{H}$ a *compact set of disjoint hyperbolic crosscuts in* U. The compactness implies that there exists $\varepsilon > 0$ such that for each $C \in \mathcal{H}$, $\operatorname{diam}(C) \geq \varepsilon$. Let a_C and b_C denote the endpoints of $C \in \mathcal{H}$, let α_C and β_C be the corresponding endpoints of $(\varphi^0)^{-1}(C)$ and let $\mathcal{A}^0$ denote the union of all the angles $\{\alpha_C, \beta_C\}$ for $C \in \mathcal{H}$. Let α^t be the continuous angle isotopy and let $\mathcal{A}^t = \alpha^t(\mathcal{A}^0)$. Then for each $t \in [0,1]$, the collection of chords $\alpha^t(\alpha_C)\alpha^t(\beta_C)$ is a compact lamination in the unit disk in the sense of Thurston [Thu85]. We will denote the family of all such chords by $\mathcal{L}^t$. We will say that $\mathcal{L}^0$ is the *pullback of the lamination* $\mathcal{H}$ *to the unit disk.* Let $\mathcal{L}^{t^*} = \bigcup \mathcal{L}^t$. Note that any two distinct chords in $\mathcal{L}^t$ meet at most in a common endpoint and there exists $\delta > 0$ such that for each t and each chord in $\mathcal{L}^t$, $\operatorname{diam}(\alpha^t(\alpha_C)\alpha^t(\beta_C)) > \delta$. Let $L^t : \mathcal{L}^0 \to \mathcal{L}^t$ be the *linear isotopy on* $\mathcal{L}$ which extends α^t and maps each chord $\alpha^0(\alpha_C)\alpha^0(\beta_C)$ linearly onto the chord $\alpha^t(\alpha_C)\alpha^t(\beta_C)$. Then the following theorem follows.

Theorem 5.3. *Suppose that* $\mathcal{H} = \mathcal{H}^0$ *is a compact set of disjoint hyperbolic crosscuts in* U^0. *Then the isotopy* $h : Z^0 \times [0,1] \to \mathbb{C}^*$ *can be extended to an isotopy* $H : [Z \cup \mathcal{H}^*] \times [0,1] \to \mathbb{C}^*$ *such that* $H^t(\mathcal{H}^*) = \mathcal{H}^{t^*} = \varphi^t(g^{-1}(\mathcal{L}^{t^*}))$ *and* H *is defined by:*

$$H^t(z) = \begin{cases} h^t(z), & \text{if } z \in Z^0; \\ \varphi^t \circ g^{-1} \circ L^t \circ g \circ (\varphi^0)^{-1}(z) & \text{if } z \in \mathcal{H}^*, \end{cases}$$

where L^t *is the linear isotopy on the pullback* $\mathcal{L}$ *of* $\mathcal{H}$

We will say that the extended isotopy H defined in Theorem 5.3 is the *linear extended isotopy which preserves the hyperbolic crosscuts* in $\mathcal{H}$.

6. Existence of short crosscuts

It follows from the results of the previous section that if C is the hyperbolic crosscut of Z which joins the points a and b in a complementary domain U of Z in $\mathbb{C}^*$, then we can extend the isotopy to an isotopy H of $Z \cup C$ such that $H^t(C) = C^t$ is the hyperbolic crosscut joining the points a^t and b^t. We need to show that if the crosscut C has small diameter, then the crosscut C^t also has small diameter. If C is contained in the component U of $\mathbb{C}^* \setminus Z$, then we denote by U^t the component of $\mathbb{C}^* \setminus Z^t$ which contains C^t.

Given a hyperbolic crosscut C of a continuum $Z \subset \mathbb{C}^*$, we say that C is a *δ-hyperbolic crosscut* if the diameter of C is less than δ. Note that we see Z as a subset of the sphere $\mathbb{C}^*$ with the spherical metric ρ.

Theorem 6.1. *For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in Z$ can be joined by a δ-hyperbolic crosscut $C \subset U$, where U is a component of $\mathbb{C}^* \setminus Z$, then for each t, x^t and y^t are joined by an ε-hyperbolic crosscut in U^t.*

Proof. Suppose that the Theorem fails for some $\varepsilon > 0$. Then there exist $x_n, y_n \in Z^0 = Z$ and a sequence of $1/n$-hyperbolic crosscuts C_n in complementary domains U_n joining them and $t_n \in [0,1]$ such that the points $x_n^{t_n}$ and $y_n^{t_n}$ are not joined by an ε-hyperbolic crosscut in $U_n^{t_n}$. Then there exists $0 < \varepsilon' < \varepsilon$ and $w_n \in \mathrm{Bd}(U_n)$, accessible from U_n, such that $\rho(w_n^t, O) > \varepsilon'$ for all n and all $t \in [0,1]$. Without loss of generality, the origin $O \in Z$, $\lim C_n = \{O\}$ and $h^t(O) = O$ for all t.

Let K be the universal constant from Theorem 5.1. Choose $0 < \delta < \varepsilon'/3$ such that if $h^t(z) \in \overline{B(O,\delta)}$ for some $t \in [0,1]$, then $h^s(z) \in B(O, \varepsilon/3K)$ for all $s \in [0,1]$. Choose n_0 such that $C_{n_0} \subset B(O,\delta)$ and $\{x_{n_0}^s, y_{n_0}^s\} \subset B(O,\delta)$ for all $s \in [0,1]$. From now on we fix this $n = n_0$ and, hence, we can omit n from the notation. In particular we have a fixed component U of $\mathbb{C}^* \setminus Z$, three points $x, y, w \in \mathrm{Bd}(U)$ with x and y joined by the hyperbolic crosscut $C \subset U \cap B(O,\delta)$, with $x^s, y^s \in B(O,\delta)$ and $w^s \in \mathbb{C} \setminus \overline{B(O,\delta)}$ for all $s \in [0,1]$. By Theorem 5.3 we can extend the isotopy h to an isotopy H of $Z \cup C$ such that $H^t(C) = C^t$ is the hyperbolic crosscut joining x^t to y^t in $U^t \subset \mathbb{C}^* \setminus Z^t$ for each t.

Let D be the closed δ-ball centered at O. For each $t \in [0,1]$, let $P^t = D \cup C^t$. Since $\mathrm{Bd}(P^t)$ is contained in $S(O,\delta) \cup C^t$, which is a finite union of arcs, $\mathrm{Bd}(P^t)$ contains no continuum of convergence and each sub-continuum of $\mathrm{Bd}(P^t)$ is locally connected and arcwise connected [Why42].

Since C^t is an arc, the components $\{T_i\}$ of $C^t \setminus D$ form a null sequence. For each i, $\overline{T_i}$ is an arc and $\overline{T_i} \cap D$ consists of the endpoints of T_i. Each point of $\mathbb{C}^* \setminus P^t$ can be joined to w^t by an arc in $\mathbb{C}^* \setminus D$ which meets P^t in a finite set.

Suppose that V is a component of $\mathbb{C}^* \setminus P^t$. We say that V is an *odd domain* (respectively *even domain*) *of* $\mathbb{C}^* \setminus P^t$ if there is a closed arc $A \subset \mathbb{C}^* \setminus D$ from w^t to a point in V such that $|A \cap C^t|$ is odd (respectively, even) and A is transverse to C^t at each point of $A \cap C^t$. This definition is independent of the choice of the arc and the point in V.

Let $Q^t = P^t \cup \bigcup\{V \mid V$ is an odd domain of $\mathbb{C}^* \setminus P^t\}$. The boundary of each odd domain V of $\mathbb{C}^* \setminus P^t$ is a simple closed curve which meets D and there exists a T_i such that T_i is contained in $\mathrm{Bd}(V)$ and $T_i \cup D$ separates V from w^t in $\mathbb{C} \setminus D$. Also each T_i is contained in the boundary of exactly one odd domain of $\mathbb{C}^* \setminus P^t$. Since the odd domains form a null family, Q^t is a locally connected continuum.

Let t_i converge to $t \in [0,1]$. We prove that $\lim Q^{t_i} = Q^t$. Note that $\lim P^{t_i} = P^t$. Let $z \in \mathbb{C}^* \setminus P^t$. It suffices to prove that $z \in Q^t$ if and only if $z \in Q^{t_i}$ for all sufficiently large i. Let $A \subset \mathbb{C}^* \setminus B(O, \rho(O,z))$ be a piecewise linear arc from z to w^t which witnesses whether or not $z \in Q^t$. Then A meets only finitely many, without loss of generality $T_1, \ldots, T_n$, of the open arcs T_j. Let $H : Z \cup C \to \mathbb{C}^*$ be the extended linear isotopy of Theorem 5.3 such that $H^t(C) = C^t$. Let $\delta < \delta' < \min(\rho(z,O), 2\varepsilon'/3)$ then for all i sufficiently large $\overline{B(O,\delta')} \cup T_j$ separates z from w^t if and only if $\overline{B(O,\delta')} \cup H^{t_i}((H^t)^{-1}(T_j))$ does for each $j = 1, \ldots, n$ and $T_j \cap A \neq \emptyset$ if and only if $H^{t_i}((H^t)^{-1}(T_j)) \cap A \neq \emptyset$ for all j.

Note that $|T_j \cap A|$ is odd if and only if $\overline{B(O,\delta')} \cup T_j$ separates z from w^t. Fix any large i and choose an arc M very close to A which witnesses whether z is in Q^{t_i}. Then $|M \cap H^{t_i}((H^t)^{-1}(T_j))| = |A \cap T_j|$ mod 2 for $j = 1, \ldots, n$ and $M \cap H^{t_i}((H^t)^{-1}(T_j)) = \emptyset$ for all $j > n$. Hence $z \in Q^{t_i}$ if and only if $z \in Q^t$ as desired.

Let $z^t \in Q^t \cap Z^t$. We prove that $\rho(z^t, O) < \varepsilon/3K$. We may assume that $z^t \notin \{x^t, y^t\} \cup D$. Let $s_0 = \inf\{s \in [0,1] \mid z^s \in Q^s \setminus D\}$. Since $Q^0 = D$ and $Z^s \cap C^s = \emptyset$ for all s, $z^{s_0} \in D$. Hence, by the choice of δ, $\rho(O, z^t) < \varepsilon/3K$.

It remains to prove the following:

Claim. $Q^t \cap B(O, \varepsilon/2K)$ contains an arc A such that $A \cap Z^t = \{x^t, y^t\}$.

Proof of Claim. Fix $t \in [0,1]$. Then C^t is an arc such that $C^t \cap Z^t = \{x^t, y^t\}$. After a small perturbation of C^t we may assume that $C^t \cap S(O, \varepsilon/2K)$ is finite and all intersections are transverse. Note that the definition of an odd domain of $\mathbb{C}^* \setminus P^t$ was with respect to $P^t = D \cup C^t$. In what follows we will use the same definition but now with respect to $P^t_M = D \cup M$, where $M \subset U^t \cup \{x^t, y^t\}$ is an arc such that:

(1) $M \cap Z^t = \{x^t, y^t\}$ and x^t and y^t are endpoints of M,
(2) $M \cap S(O, \varepsilon/2K)$ is finite and all intersections are transverse,
(3) for each odd domain V of $\mathbb{C}^* \setminus P^t_M$ and each $z^t \in Z^t \cap V$, $\rho(z^t, O) < \varepsilon/3K$,
(4) $n = |M \cap S(O, \varepsilon/2K)|$ is minimal.

If $n = 0$ we are done. Note that $n = 1$ is impossible since all intersections of $S(O, \varepsilon/2K)$ and M are transverse and both endpoints of M are in $B(O, \varepsilon/2K)$. Hence, assume $n > 1$. Let $Q^t_M = P^t_M \cup \bigcup\{V \mid V$ is an odd domain of $\mathbb{C}^* \setminus P^t_M\}$. Let S_i be all components of $S(O, \varepsilon/2K) \setminus M$. Since each component M_j of $M \setminus D$ is an arc which locally separates the plane, points on one side of M_j are in an even domain and points on the other side are in an odd domain. Hence, each arc S_i is contained in a complementary domain V_i of $\mathbb{C}^* \setminus P^t_M$ and these domains are alternately even and odd moving around the circle $S(O, \varepsilon/2K)$. In particular, n is even. We may order M so that $x^t < y^t$ and we write intervals in M as in $\mathbb{R}$.

Let $\mathcal{M} = \{M_i\}$ be the collection of all components of $M \setminus \overline{B(O, \varepsilon/2K)}$. We can define a partial order $\prec$ on $\mathcal{M}$ by $M_1 \prec M_2$ if M_2 separates M_1 from w^t in $\mathbb{C} \setminus \overline{B(O, \varepsilon/2K)}$. Assume that $M_1 = (a_1, b_1)$ (with $a_1 < b_1$) is a minimal element of $\mathcal{M}$. Then $M_1 \cup \overline{B(O, \varepsilon/2K)}$ bounds a disk V_1 whose closure meets $\overline{B(O, \varepsilon/2K)}$ in an arc $S_1 \subset S(O, \varepsilon/2K)$ and $S_1 \cap M = \{a_1, b_1\}$. Then S_1 is either contained in an an even or an odd domain of $\mathbb{C}^* \setminus P^t_M$.

Subcase 0. Suppose that $Z^t \cap S_1 = \emptyset$ (this must be the case if S_1 is contained in an odd domain). In this case choose $a'_1 < a_1 < b_1 < b'_1$, with a'_1 in $B(O, \varepsilon/2K)$ very close to a_1 and b'_1 in $B(O, \varepsilon/2K)$ very close to b_1, and an arc $S'_1 \subset B(O, \varepsilon/2K)$ very close to S_1 from a'_1 to b'_1 such that $S'_1 \cap Z^t = \emptyset$. Then Z^t is disjoint from the bounded complementary domain B of the simple closed curve $F = S'_1 \cup (a'_1, b'_1)$. Hence there exists a homotopy of the plane which is the identity on $Z^t \cup S'_1 \cup [x^t, a'_1] \cup [b'_1, y^t]$ and shrinks B to S'_1. Let $M' = S'_1 \cup [M \setminus (a_1, b_1)]$. Then $z \in Z^t$ lies in an odd domain of $\mathbb{C}^* \setminus P^t_M$ if and only if z lies in an odd domain of $\mathbb{C}^* \setminus [D \cup M']$. Thus M' satisfies (3). Clearly M' satisfies (1-2) and since $|M' \cap S(O, \varepsilon/2K| < n$ we have a contradiction with the minimality of n.

Hence we may assume that $S_1 \cup V_1$ is contained in an even domain and $Z^t \cap S_1 \neq \emptyset$. Then there exists $M_2 = (a_2, b_2) \in \mathcal{M}$ (with $a_2 < b_2$) such that M_2 is the immediate successor of M_1 in $\mathcal{M}$. Let V_2 be the component of $\mathbb{C}^* \setminus [V_1 \cup \overline{B(O, \varepsilon/2K)} \cup M_2]$ whose closure contains the arc (a_1, b_1). Since M_2 is the immediate successor of M_1 in $\mathcal{M}$, there exists an arc $J \subset [V_2 \setminus M] \cup \{j_1, j_2\}$ with one endpoint of J, $j_1 \in (a_1, b_1)$ and the other endpoint of J, $j_2 \in (a_2, b_2)$. Moreover, since V_1 was even, J is contained in an odd domain and $J \cap Z^t = \emptyset$. We will examine the circular (counter clockwise) order $<_C$ of the four points a_1, b_1, a_2, b_2 around the circle $S(O, \varepsilon/2K)$.

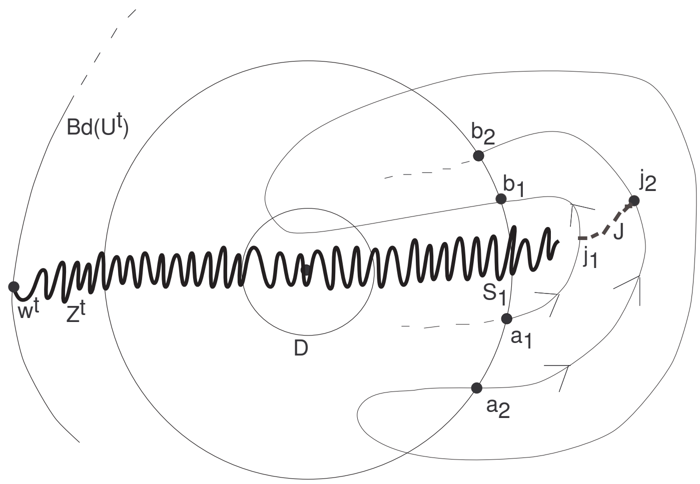

FIGURE 1. Subcase 1 in the proof of Theorem 6.1

Subcase 1. $a_2 <_C a_1 <_C b_1 <_C b_2$. We have either $a_1 < a_2 < b_1 < b_2$ (see figure 1) or $a_2 < b_2 < a_1 < b_1$. Since $w^t \in Z^t \cap \mathrm{Bd}(U^t)$, $Z^t \cap S_1 \neq \emptyset$ and $M \cap Z^t = \{x^t, y^t\}$, in either case (see the gate theorems in [Bec74, page 36]), the simple closed curve $F = J \cup [j_1, j_2]$ separates x^t from y^t. Since $Z^t \cap F = \emptyset$, this contradicts the connectedness of Z^t.

Subcase 2. $b_2 <_C a_1 <_C b_1 <_C a_2$. Then either $a_1 < b_1 < a_2 < b_2$ or $a_2 < b_2 < a_1 < b_1$. Since $M \cap Z^t = \{x^t, y^t\}$, $w^t \in Z^t \cap \mathrm{Bd}(U^t)$ and $Z^t \cap S_1 \neq \emptyset$, x^t and y^t are contained in the unbounded component of $F = J \cup [j_1, j_2]$ and the proof proceeds as in Subcase 0, where F is now $J \cup [j_1, j_2]$.

Other cases are similar. This completes the proof of the Claim.

Hence for each t there exists a crosscut $M(t)$ joining x^t to y^t in U^t such that $\mathrm{diam}(M(t)) < \varepsilon/K$. By Theorem 5.1, the diameter of the hyperbolic crosscut C^t is less than ε for all t. This contradiction completes the proof.

□

7. EXTENDING THE ISOTOPY OVER $\mathbb{C}$

Now that we know how to extend the isotopy over hyperbolic crosscuts, it remains to define the extension over all complementary domains U of Z. Easy examples show that if we choose the hyperbolic crosscuts without care the extension may not be continuous. Fortunately a suitable set of hyperbolic crosscuts exists. Fix a component U of $\mathbb{C}^* \setminus Z$ and let $\mathcal{B}$ be the collection of all maximal open balls $B(z, r) \subset U$ (that is open balls in the spherical metric and such that $|S(z, r) \cap Z| \geq 2$). Let $\mathcal{C}$ be the collection of all centers of such balls and for $c \in \mathcal{C}$ let $r(c)$ be the corresponding radius. Note that for each $c \in \mathcal{C}$, $B(c, r(c))$ is conformally equivalent with the unit disk $\mathbb{D}$ and, hence, can be endowed with the hyperbolic metric. Let $F(c)$ be the convex hull of the set $S(c, r(c)) \cap X$ in $B(c, r(c))$ using the hyperbolic metric on the ball $B(c, r(c))$. The following Theorem is due to Kulkarni and Pinkall:

Theorem 7.1 ([KP94]). *For each $z \in U$ there exists a unique $c \in \mathcal{C}$ such that $z \in F(c)$.*

Note that the collection of chords in the boundaries of all $F(c)$ form a lamination of U in the sense of Thurston [Thu85]. As in [Thu85] we will call the chords in this lamination leaves. Then two such leaves do not cross each other (i.e., if $\ell \neq \ell'$ are leaves, then $\ell \cap \ell' \cap U = \emptyset$) and any convergent sequence of leaves is either a leaf, or a point in Z. In particular, the subcollection of leaves of diameter greater or equal to ε is compact for each $\varepsilon > 0$. This collection of leaves will naturally provide us with the required collection of hyperbolic crosscuts by simply replacing each leaf in the lamination by the hyperbolic crosscut joining its endpoints. The collection $\mathcal{H}$ of such hyperbolic crosscuts will be called the *hyperbolic KP-lamination of* U^0. The union of all the hyperbolic crosscuts in $\mathcal{H}$ will be denoted by $\mathcal{H}^*$. A *gap* G of $\mathcal{H}$ is the closure of a component of $U \setminus \mathcal{H}^*$. By Theorems 5.3 and 6.1 we can extend the isotopy h over $\mathcal{H}^*$. To finish the proof we must extend the isotopy over all gaps.

Theorem 7.2. *Suppose that h^t is an isotopy of a plane continuum Z, which we consider as a subset of the sphere $\mathbb{C}^*$, with $h^0 = id|_Z$. Then there exists an extension to an isotopy $H^t : \mathbb{C}^* \to \mathbb{C}^*$ such that $H^0 = id_{\mathbb{C}^*}$.*

Proof. Let $\{U_n\}$ be all the components of $\mathbb{C}^* \setminus Z$. For each n let $\mathcal{H}_n$ be the hyperbolic KP-lamination of U_n. Since the diameter of maximal balls contained in distinct components U_n converges to 0, it follows from Theorems 6.1 and 5.1 that for any sequence $C_n \in \mathcal{H}_n$, such that $U_n \neq U_m$ when $n \neq m$,

$$\lim \operatorname{diam}(C_n) = 0. \tag{7.1}$$

By Theorems 6.1 and 5.3 we can extend the isotopy h of Z to an isotopy H_n of $Z \cup \mathcal{H}_n^*$ such that H_n preserves the hyperbolic crosscuts in $\mathcal{H}_n^0$ (i.e., $H_n^t(\mathcal{H}_n^0) = \mathcal{H}_n^t$). Each gap G^t in U_n^t is a hyperbolic convex set with barycenter b_G^t using the Cayley-Klein model of U_n^t. For each t define $H_n^t(b_G^0) = b_G^t$ and extend H_n over G by taking the "cone"of H_n over the boundary of G and its barycenter b_G^0 (using the Cayley-Klein model). Note that if G_i is a convergent sequence of gaps in U_n, then either $\lim G_i$ is a point in the boundary of U_n, or a leaf C in $\mathcal{H}_n$. In the latter case the barycenters b_i of G_i converge to the barycenter of C. It follows that the isotopy h can be extended to an isotopy H_n of $Z \cup U_n$ for each n.

Let $H = \cup H_n$. Then $H : \mathbb{C}^* \times [0,1] \to \mathbb{C}^*$ is continuous by (7.1). Hence H is the required extension of h. □

Theorem 7.2 shows that we can extend an isotopy h of a planar continuum Z, starting at the identity, to an isotopy $H : \mathbb{C}^* \times [0,1] \to \mathbb{C}^*$ of the sphere. Let U denote the component of $\mathbb{C}^* \setminus Z$ containing the point at infinity. By composing the isotopy H by an isotopy K of the sphere such that $K^0 = id_{\mathbb{C}^*}$ and $K^t|_{\mathbb{C}^* \setminus U} = id_{\mathbb{C}^* \setminus U}$, and $K^t(H^t(\infty)) = \infty$ for all $t \in [0,1]$ we obtain an isotopy which extends h and fixes the point at infinity. Hence the following theorem follows.

Theorem 7.3. *Suppose that h^t is an isotopy of a plane continuum $Z \subset \mathbb{C}$ with $h^0 = id|_Z$. Then there exists an extension to an isotopy $H^t : \mathbb{C} \to \mathbb{C}$ such that $H^0 = id$.*

References

[Bea27] R. Bear, *Kurventypen auf flächen*, J. Reine Angew. Math. **156** (1927), 231–246.

[Bea28] ———, *Isotopie von kurven auf orientierbaren, geschlossen flächen und ihr zusammenhang mit der topologischen deformation der flächen*, J. Reine Angew. Math. **159** (1928), 101–111.

[Bec74] Anatole Beck, *Continuous flows in the plane*, Die Grundlehren der Math. Wissenschaften, vol. 201, Springer-Verlag, New York-Heidelberg, 1974.

[Bel67] H. Bell, *On fixed point properties of plane continua*, Trans. A. M. S. **128** (1967), 539–548.

[Bel76] ———, *A correction to my paper "Some topological extensions of plane geometry", Rev. Colombiana* **9** *(1975), pp. 125–153*, Rev. Colombiana **10** (1976), 93.

[Bro05] Gaston A. Brouwer, *Green's functions from a metric point of view*, Ph.d. dissertation, University of Alabama at Birmingham, 2005.

[Eps66] D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. **115** (1966), 83–107.

[Fab05] Paul Fabel, *Completing Artin's braid group on infintely many strands*, Journal of Knot Theory and its Ramifications **14** (2005), 979–991.

[Ili70] S. D. Iliadis, *Location of continua on a plane and fixed points*, Vestnik Moskovskogo Univ. Matematika **25** (1970), no. 4, 66–70, Series I.

[KP94] Ravi S. Kulkarni and Ulrich Pinkall, *A canonical metric for Möbius structures and its applications*, Math. Z. **216** (1994), no. 1, 89–129.

[Lyu83] M. Yu. Lyubich, *Some typical properties of the dynamics of rational mappings*, Uspekhi Mat. Nauk. **38** (1983), no. 5 (233), 197–198.

[Mil00] J. Milnor, *Dynamics in one complex variable*, second ed., Vieweg, Wiesbaden, 2000.

[MSS83] R. Mané, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Scient. Éc. Norm. Sup., 4^e serie **16** (1983), 193–217.

[Pom92] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Math. Wissenschaften, vol. 299, Springer-Verlag, New York, 1992.

[Slo91] Z. Slodkowski, *Holomorphic motions and polynomial hulls*, Proc. AMS **111** (1991), 347–355.

[ST86] D. Sullivan and W. Thurston, *Extending holomorphic motions*, Acta Math. **157** (1986), 243–257.

[Thu85] W. P. Thurston, *On the geometry and dynamics of iterated rational maps*, Preprint, 1985.

[Wen91] Guo-Chun Wen, *Conformal mappings and boundary values problems*, American Mathematical Society, 1991, Translations of mathematical Monograms, Volume 106.

[Why42] G. T. Whyburn, *Analytic topology*, vol. 28, AMS Coll. Publications, Providence, RI, 1942.

University of Alabama at Birmingham, Department of Mathematics, Birmingham, AL 35294, USA

E-mail address, Lex Oversteegen: overstee@math.uab.edu

University of Saskatchewan, Department of Mathematics and Statistics, 106 Wiggins road, Saskatoon, Canada, S7N 5E6

E-mail address, Ed Tymchatyn: tymchat@math.usask.ca