ON OPEN MAPS BETWEEN DENDRITES

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ABSTRACT. In this paper we use a result by J. Krasinkiewicz to present a description of the topological behavior of an open map defined between dendrites. It is shown that, for every such map $f: X \to Y$, there exist n subcontinua $X_1, X_2, \ldots, X_n$ of X such that $X = X_1 \cup X_2 \cup \cdots \cup X_n$, each set $X_i \cap X_j$ consists of at most one element which is a critical point of f, and each map $f_{|X_i}: X_i \to Y$ is open, onto and can be lifted, in a natural way, to a product space $Z_i \times Y$ for some compact and zero-dimensional space Z_i. We also study the ω-limit sets $\omega(x)$ of a self-homeomorphism $f: X \to X$ defined on a dendrite X. It is shown that $\omega(x)$ is either a periodic orbit or a Cantor set (and if this is the case, then $f_{|\omega(x)}$ is an adding machine).

1. INTRODUCTION

It is well known that each open map from the interval $[0, 1]$ to itself is an n-fold branched covering map (i.e., there exist $n \in \mathbb{N}$ and n subcontinua $X_1, X_2, \ldots, X_n$ of $[0, 1]$ such that $[0, 1] = X_1 \cup X_2 \cup \cdots \cup X_n$, each set $X_i \cap X_j$ contains at most one element, for $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, and each map $f_i = f_{|X_i}: X_i \to [0, 1]$ is a homeomorphism). Based on this fact, the dynamics of such maps have been extensively investigated (see for example [MT88]). Since every open map of a finite tree, with at least one branch-point, onto itself is a homeomorphism (Theorem 3.1), it is natural to investigate open maps on dendrites. Easy examples show that a straight forward generalization of the above result for the interval is false. In this paper we formulate a correct generalization for the class of dendrites (see Theorem 4.4).

Dendrites appear naturally as the Julia set of a complex polynomial. If, for example, $p: \mathbb{C} \to \mathbb{C}$ is the map defined by $p(z) = z^2 + c$, then for certain values of c, the Julia set J of p is a dendrite and the map $p_{|J}: J \to J$ is a branched covering [Mil00]. In particular, $p_{|J}$ is open. The dynamics of such maps is still not well understood (cf. [BL02] and [Thu85]) and serves as a motivation for this paper.

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The paper is divided in 5 sections. After the introduction, we write in Section 2 some notions and auxiliary results. Then in Section 3 we present some conditions under which an open map defined between dendrites must be a homeomorphism. In this section we also study the ω-limits sets of a self-homeomorphism $f\colon X \to X$ defined on a dendrite X. Later in Section 4 we present a consequence of a theorem by Krasinkiewicz that will allow us to prove the main theorem of the paper (Theorem 4.3). Finally, in Section 5 we collect some other results involving open maps between dendrites.

2. Notions and auxiliary results

All spaces considered in this paper are assumed to be metric. If X is a space, $p \in X$ and $\epsilon > 0$, then $B_X(p,\epsilon)$ denotes the open ball around p of radius ϵ. If $A \subset X$, then the symbols $\mathrm{cl}_X(A)$, $\mathrm{int}_X(A)$ and $\mathrm{bd}_X(A)$ stands for the closure, the interior and the boundary of A in X, respectively. Moreover, the symbol $|A|$ represents the cardinality of A.

A *continuum* is a nonempty, compact and connected metric space. The topological limit, with respect to the Hausdorff metric, of a sequence of closed nonempty sets $(Y_n)_n$ in a metric space is denoted by $\mathrm{Lim}\, Y_n$.

A *dendrite* is a locally connected continuum that contains no simple closed curves. For a dendrite X it is known that any subcontinuum of X is a dendrite [Nad92, Corollary 10.6], every connected subset of X is arcwise connected [Nad92, Proposition 10.9], and the intersection of any two connected subsets of X is connected [Nad92, Theorem 10.10]. Given points p and q in a dendrite X, there is only one arc from p to q in X. We denote such an arc by pq.

A *map* is a continuous function. A map f from a continuum X onto a continuum Y is said to be

- *open* if the image of any open subset of X is an open subset of Y;
- *interior at* $x \in X$ if for every open set U of X such that $x \in U$, we have $f(x) \in \mathrm{int}_Y(f(U))$;
- *confluent* provided that for any subcontinuum Q of Y and any component C of $f^{-1}(Q)$, we have $f(C) = Q$;
- *monotone* if for any $y \in Y$, the set $f^{-1}(y)$ is connected;
- *light* if for any $y \in Y$, the set $f^{-1}(y)$ is zero-dimensional.

It is well known that a map is open if and only if it is interior at each point of its domain. Moreover, any open map is confluent [Nad92, Theorem 13.14]. It is also known that confluent light maps onto a locally connected continuum are open.

For a dendrite X and a point $p \in X$ we denote the *order of* p *at* X by ord_pX. Points of order 1 in X are called *end-points* of X. The set of all such points is denoted by $E(X)$. It is known that $E(X)$ is zero-dimensional. It is easy to see that if C is a connected subset of X, then the set $C \setminus E(X)$

is arcwise connected. Points of order 2 in X are called *ordinary points* of X. The set of all such points is denoted by $O(X)$. It is known that $O(X)$ is dense in X [Nad92, 10.42]. Points of order greater than 2 are called *branch points* of X. The set of all such points is denoted by $B(X)$. It is known that $B(X)$ is countable [Nad92, Theorem 10.23]. Moreover $ord_p X \leq \aleph_0$ for any $p \in X$. Note that $X = E(X) \cup O(X) \cup B(X)$.

For a dendrite X and subcontinua A and B of X such that $A \cap B \neq \emptyset$ we define a map $r \colon A \cup B \to A$ as follows. If $x \in A$ we put $r(x) = x$ and if $x \in (A \cup B) \setminus A$ then $r(x)$ is the unique point of $A \cap C$ where C is any irreducible arc in $A \cup B$ from x to a point of A. It is known that r is a monotone retraction from $A \cup B$ onto A [Nad92, Lemma 10.25]. The map r is called the *first point map from* $A \cup B$ *to* A.

If $f \colon X \to Y$ is a map then a point $p \in X$ is said to be

- a *fixed point of* f if $f(p) = p$;
- a *periodic point of* f if there exists $n \in \mathbb{N}$ such that $f^n(p) = p$;
- a *critical point of* f if for any neighborhood U of p there exist $x_1, x_2 \in U$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

We denote by $\mathrm{Fix}(f)$, $P(f)$ and $\mathfrak{C}$ the set of fixed, periodic and critical points of f, respectively. It is known that if $f \colon X \to X$ is a map and X is a dendrite, then $\mathrm{Fix}(f) \neq \emptyset$ [Why, Corollary 3.21, p. 243].

If X is a space then an arc pq in X is called a *free arc in* X provided that $pq \setminus \{p, q\}$ is open in X. The following theorem collects some results from Section 6 of [CCP94].

Theorem 2.1. *Let $f : X \to Y$ be an open map from a dendrite X onto a continuum Y. Then*

(2.1.1) *Y is a dendrite;*
(2.1.2) *f is light;*
(2.1.3) *$ord_{f(p)} Y \leq ord_p X$ for any $p \in X$;*
(2.1.4) *if $ord_p X = \aleph_0$, then $ord_{f(p)} Y = \aleph_0$;*
(2.1.5) *$f(E(X)) \subset E(Y)$;*
(2.1.6) *$f^{-1}(B(Y)) \subset B(X)$;*
(2.1.7) *the set $f^{-1}(y)$ is finite for any $y \in Y \setminus E(Y)$;*
(2.1.8) *the set $f^{-1}(E(Y)) \setminus E(X)$ is finite;*
(2.1.9) *the image under f of a free arc in X is a free arc in Y;*
(2.1.10) *for each subcontinuum B of Y and for each $p \in f^{-1}(B)$, there is a subcontinuum A of X containing p and such that the map $f_{|A} : A \to B$ is a homeomorphism.*

The following basic result will be used in Section 4.

Theorem 2.2. *Let X be a dendrite and let M be a subset of X such that $E(X) \subset M$ and $M \setminus E(X)$ is closed in X. Let C be a component of $X \setminus M$. Then C is open and closed in $X \setminus M$.*

Proof. Note that $X \setminus E(X)$ is connected and locally connected. Hence $X \setminus M$ is locally connected and the required result follows easily. □

3. Homeomorphisms and ω-limit sets

In this section we provide sufficient conditions which imply that an open map, defined between dendrites, must be a homeomorphism. Later we will study the ω-limit sets of a self-homeomorphism $f\colon X \to X$ defined on a dendrite X. We start with a self-open map defined on a non-trivial tree, in which case no additional assumptions are needed, i.e. such a map must be a homeomorphism.

Theorem 3.1. *Let $f\colon X \to X$ be an open map from a finite tree X onto itself. If $B(X) \neq \emptyset$, then f is a homeomorphism.*

Proof. Put $n = |B(X)|$ and let $B(X) = \{b_1, b_2, \ldots, b_n\}$. For any given $i \in \{1, 2, \ldots, n\}$, let $a_i \in X$ be such that $f(a_i) = b_i$. Put $B = \{a_1, a_2, \ldots, a_n\}$. By (2.1.6), $B \subset B(X)$ and since $|B| = n$, it follows that $B(X) = B$. This shows that $f(B(X)) = B(X)$ and $f^{-1}(B(X)) = B(X)$. Hence the map $f_{|B(X)}: B(X) \to B(X)$ is one-to-one and onto. To finish the proof it suffices to show, by (2.1.5), that $f^{-1}(E(X)) \subset E(X)$.

To see this, suppose there exists $v \in X \setminus E(X)$ such that $w = f(v) \in E(X)$. Since $B(X)$ is finite and $f(B(X)) = B(X)$ it follows that $v \in O(X)$ and there is a connected open subset U of X such that $v \in U \subset O(X)$. Since f is light U can be chosen so that $X \setminus f^{-1}(f(U))$ has at least two components C and D. By (2.1.3) and the inclusion $U \subset O(X)$ we have $f(U) \cap B(X) = \emptyset$ and $f(U) \cap E(X) = \{w\}$. Thus $X \setminus f(U)$ is a subcontinuum of X that contains $B(X)$. Note that $f^{-1}(X \setminus f(U)) = X \setminus f^{-1}(f(U))$, so both C and D are components of $f^{-1}(X \setminus f(U))$. By the confluence of f we have $f(C) = f(D) = X \setminus f(U)$. The latter contradicts the fact that $f_{|B(X)}$ is one-to-one and completes the proof. □

In the following theorem we give some conditions under which a confluent map between dendroids must be a homeomorphism. Recall that a *dendroid* is an arcwise connected continuum such that the intersection of any two of its subcontinua is connected. Note that dendrites are locally connected dendroids. We extend the definition of an end-point in a dendrite as follows. Suppose X is a dendroid. Then a point $e \in X$ is called an *end-point of X* if e is an end-point of every arc in X which contains e. Note that if X is locally connected (and hence if X is a dendrite), this implies that the order of X at e is one. As before we denote the set of all end-points of a dendroid X by $E(X)$.

Theorem 3.2. *Let $f\colon X \to Y$ be a map from a dendroid X onto a dendroid Y. Let us assume that:*

(3.2.1) *f is confluent and light,*
(3.2.2) *$f^{-1}(E(Y)) = E(X)$ and the map $f_{|E(X)}: E(X) \to E(Y)$ is one-to-one.*

Then f is a homeomorphism.

Proof. Let us assume, on the contrary, that there exist $x, y \in X$ with $x \neq y$ and $f(x) = f(y)$. By (3.2.2) $f(x) \notin E(Y)$ and, by (3.2.1), the set $f^{-1}(f(x))$ is zero-dimensional. Then we can assume, without loss of generality, that $xy \cap f^{-1}(f(x)) = \{x, y\}$. Since $f(x) \notin E(Y)$, and $f(z) \neq f(x)$ for all $z \in xy \setminus \{x, y\}$, there is $e \in E(Y)$ such that $ef(x) \cap f(xy) = \{f(x)\}$. Let C_x and C_y be the components of $f^{-1}(ef(x))$ such that $x \in C_x$ and $y \in C_y$. Since f is confluent, we have $f(C_x) = f(C_y) = ef(x)$. Take points $a \in C_x$ and $b \in C_y$ such that $f(a) = f(b) = e$. By (3.2.2) we have $a, b \in E(X)$ and $a = b$. Then the continuum $C_x \cup xy \cup C_y$ contains a simple closed curve, a contradiction. □

The following easy corollary will be used in the proof of Theorem 4.4. Another proof can be obtained using the corollary that appears at the end of page 199 of [Why].

Corollary 3.3. *Let $f\colon X \to Y$ be an open map from a dendrite X onto a dendrite Y. If f has no critical points, then f is a homeomorphism.*

Proof. Let f be as assumed. Since f has no critical points, $f^{-1}(E(Y)) \subset E(X)$, and since f is onto and $f(E(X)) \subset E(Y)$ we have $f^{-1}(E(Y)) = E(X)$. This implies that $f_{|E(X)}$ is one-to-one. To see this consider two distinct points $e_1, e_2 \in E(X)$ such that $f(e_1) = f(e_2)$. Then, since f is light, $f(e_1e_2) = Z$ is a (non-degenerate) continuum. Let $y \in E(Z) \setminus \{f(e_1)\}$ and $x \in e_1e_2 \setminus \{e_1, e_2\}$ such that $f(x) = y$. Then x is a critical point of f, a contradiction. By Theorem 3.2, f is a homeomorphism. □

Now we turn our attention to self-homeomorphisms defined on a dendrite. The next two results involves the set of fixed points of any such map.

Lemma 3.4. *Let X be a dendrite and $g\colon X \to X$ a homeomorphism from X onto itself. Let $a, b \in X$ be such that $a \neq b$ and $g(b) \in X \setminus ab$. Let D be the component of $X \setminus \{b\}$ that contains $g(b)$. Then $\mathrm{Fix}(g) \cap \mathrm{cl}_X(D) \neq \emptyset$.*

Proof. By a standard construction of a maximal Borsuk ray (see [Hag86]), there is a map $\varphi\colon [0, \infty) \to \mathrm{cl}_X(D)$ such that $\varphi(0) = b$, $\varphi(t) \in bg(\varphi(t)) \setminus \{g(\varphi(t))\}$ for every $t \in [0, \infty)$, $\mathrm{cl}_X(\varphi([0, \infty))) \setminus \varphi([0, \infty)) = \{y\}$ and $g(y) = y$. Then $y \in \mathrm{Fix}(g) \cap \mathrm{cl}_X(D)$. □

Lemma 3.5. *Let X be a dendrite and $g\colon X \to X$ a homeomorphism from X onto itself. If $E(X) \cap \mathrm{Fix}(g) \neq \emptyset$, then $|\mathrm{Fix}(g)| \geq 2$.*

Proof. Let $e \in E(X) \cap \mathrm{Fix}(g)$ and assume that $\mathrm{Fix}(g) = \{e\}$. Let $p \in X \setminus \{e\}$. Note that $C = ep \cap eg(p)$ is an arc that contains e as one end-point. Let v be the other end-point of C. Since $g(e) = e$ and g is a homeomorphism, we have $g(ep) = eg(p)$, so $g(v) \in eg(p)$. Thus either $v \in eg(v) \setminus \{g(v)\}$ or $g(v) \in ev \setminus \{v\}$. Let us assume first that $v \in eg(v) \setminus \{g(v)\}$. Let D be the component of $X \setminus \{v\}$ that contains $g(v)$. By Lemma 3.4, $\mathrm{Fix}(g) \cap \mathrm{cl}_X(D) \neq \emptyset$. Let us assume now that $g(v) \in ev \setminus \{v\}$ and let E be the component of $X \setminus \{g(v)\}$ that contains v. By Lemma 3.4, applied to g^{-1}, we have $\mathrm{Fix}(g^{-1}) \cap \mathrm{cl}_X(E) \neq \emptyset$. In any case we found a fixed point of g different than e. □

From now on, in this section, $f\colon X \to X$ represents a homeomorphism from a dendrite X onto itself. Given $x \in X$ the set $\omega(x)$ of points $y \in X$ such that, for any neighborhood U of y and any $N \in \mathbb{N}$, there is $n > N$ such that $f^n(x) \in U$ is called the *ω-limit set of f*. Note that $\omega(x) = \limsup f^n(x)$. In this section we will prove that either $\omega(x)$ is a periodic orbit or a Cantor set. To this aim let us consider the collection $\mathcal{C}$ of all components of $X \setminus \mathrm{Fix}(f)$. Since $\mathrm{Fix}(f)$ is a closed subset of the locally connected continuum X, the elements of $\mathcal{C}$ are open subsets of X. Moreover if $C \in \mathcal{C}$, then $C \cap \mathrm{Fix}(f) = \emptyset$ so $\mathrm{cl}_X(C) \cap \mathrm{Fix}(f) \subset E(\mathrm{cl}_X(C))$. In the following lemma we present more properties of $\mathcal{C}$ and its elements.

Lemma 3.6. *The following properties are satisfied:*

(3.6.1) *$\mathcal{C}$ is countable;*
(3.6.2) *$f(C) \in \mathcal{C}$ for any $C \in \mathcal{C}$;*
(3.6.3) *if $C \in \mathcal{C}$, then $|\mathrm{cl}_X(C) \cap \mathrm{Fix}(f)| \leq 2$;*
(3.6.4) *if $C \in \mathcal{C}$ and $|\mathrm{cl}_X(C) \cap \mathrm{Fix}(f)| = 2$, then $f(C) = C$ and if we write $\mathrm{cl}_X(C) \cap \mathrm{Fix}(f) = \{a, b\}$ then for any $x \in C$ either $\omega(x) = \{a\}$ or $\omega(x) = \{b\}$;*
(3.6.5) *if $C \in \mathcal{C}$, $|\mathrm{cl}_X(C) \cap \mathrm{Fix}(f)| = 1$ and $f^n(C) \neq C$ for all $n \in \mathbb{N}$, then $\omega(x) = \mathrm{cl}_X(C) \cap \mathrm{Fix}(f)$ for any $x \in C$.*

Proof. Let D be a countable dense subset of X and $C_i \in \mathcal{C}$. Since C_i is open it follows that $C_i \cap D \neq \emptyset$ so we can pick a point $d_i \in C_i \cap D$. Note that if C_i and C_j are different elements of $\mathcal{C}$, then $d_i \neq d_j$. Thus since D is countable, the collection $\mathcal{C}$ is countable as well. This shows (3.6.1).

To show (3.6.2) let $C \in \mathcal{C}$. Note that $f(\mathrm{Fix}(f)) = \mathrm{Fix}(f)$. Since f is a homeomorphism $f(C)$ is a component of $f(X \setminus \mathrm{Fix}(f)) = f(X) \setminus f(\mathrm{Fix}(f)) = X \setminus \mathrm{Fix}(f)$, so $f(C) \in \mathcal{C}$.

To show (3.6.3) let $C \in \mathcal{C}$ and assume that $|\mathrm{cl}_X(C) \cap \mathrm{Fix}(f)| \geq 3$. Let a, b and c be three different elements of $\mathrm{cl}_X(C) \cap \mathrm{Fix}(f)$. Consider the arcs ab, bc and ac in $\mathrm{cl}_X(C)$ and note that $ab \cap bc \cap ac = \{t\} \subset C$. Since f is a homeomorphism that fixes a, b and c we have $t \in C \cap \mathrm{Fix}(f)$, a contradiction. This shows (3.6.3).

Now assume that $C \in \mathcal{C}$ is such that $|\mathrm{cl}_X(C) \cap \mathrm{Fix}(f)| = 2$. Put $\mathrm{cl}_X(C) \cap \mathrm{Fix}(f) = \{a, b\}$ and take $x \in C$. Let r be the first point map from X to $ab \subset \mathrm{cl}_X(C)$. It is easy to see that

1) $r(z) = az \cap ab \cap bz$, for any $z \in X$.

In particular $r(x) = ax \cap ab \cap bx$. Since $a, b \in \mathrm{Fix}(f)$, $f(ab) = ab$, $f(ax) = af(x)$, $f(bx) = bf(x)$, and f is a homeomorphism

$$f(r(x)) = f(ax \cap ab \cap bx) = af(x) \cap ab \cap bf(x).$$

Applying 1) to $z = f(x)$ we have $af(x) \cap ab \cap bf(x) = r(f(x))$. Hence $f(r(x)) = r(f(x))$, so $f^n(r(x)) = r(f^n(x))$ for any $n \in \mathbb{N}$. Note that $r(x) \in ab \setminus \{a,b\}$, so $f(r(x)) \neq r(x)$. This implies that the arcs $xr(x)$ and $f(x)r(f(x))$ are disjoint. Now, since $r(x), f(r(x)) \in ab \setminus \{a,b\}$ and $r(x) \neq f(r(x))$ either $f(r(x)) \in br(x)$ or $f(r(x)) \in ar(x)$. Let us assume, without loss of generality, that $f(r(x)) \in r(x)b$. Then $f_{|ab}$ is a homeomorphism whose graph lays above the diagonal (except at points a and b), so $f^n(z) \to b$ for any $z \in ab \setminus \{a,b\}$. In particular $f^n(r(x)) \to b$ and since the arcs in the sequence $(f^n(x)r(f^n(x)))_n$ are mutually disjoint, it follows that $f^n(x) \to b$. Thus $\omega(x) = \{b\}$. To complete the proof of (3.6.4) we have to see that $f(C) = C$. Let us assume that there is $y \in C$ such that $f(y) \notin C$. Then $r(f(y)) \in \{a,b\}$, so $r(y)$ is an element of X such that $f(r(y)) = r(f(y)) \in \{a,b\}$, and this contradicts the fact that f is one-to-one. Thus $f(C) \subset C$. By (3.6.2), $C \subset f(C)$, so $f(C) = C$. The proof of (3.6.4) is complete.

To show (3.6.5) let $C \in \mathcal{C}$ be such that $|\mathrm{cl}_X(C) \cap \mathrm{Fix}(f)| = 1$ and $f^n(C) \neq C$, for all $n \in \mathbb{N}$. Put $\mathrm{cl}_X(C) \cap \mathrm{Fix}(f) = \{a\}$ and let $x \in C$. By (3.6.2) and (3.6.4), $(f^n(C))_n$ is a sequence of mutually disjoint elements of $\mathcal{C}$ such that, for any $n \in \mathbb{N}$, $f^n(\mathrm{cl}_X(C)) \cap \mathrm{Fix}(f) = \{a\}$. Since X is locally connected $f^n(\mathrm{cl}_X(C)) \to \{a\}$, so $\omega(x) = \{a\}$ for any $x \in C$. □

Let $C \in \mathcal{C}$ be such that $|\mathrm{cl}_X(C) \cap \mathrm{Fix}(f)| = 1$ and $f^n(C) = C$ for some $n \in \mathbb{N}$. Put $\mathrm{cl}_X(C) \cap \mathrm{Fix}(f) = \{a\}$. If $n = 1$ then $f_{|\mathrm{cl}_X(C)}$ is a homeomorphism from the dendrite $\mathrm{cl}_X(C)$ onto itself such that $a \in \mathrm{Fix}(f_{|\mathrm{cl}_X(C)}) \cap E(\mathrm{cl}_X(C))$. Then, by Lemma 3.5, $|\mathrm{cl}_X(C) \cap \mathrm{Fix}(f)| = |\mathrm{Fix}(f_{|\mathrm{cl}_X(C)})| \geq 2$. Since this is a contradiction, we have $n > 1$.

We say that an element $C \in \mathcal{C}$ is an *end-periodic component* of $X \setminus \mathrm{Fix}(f)$ (or simply, that C is end-periodic) if $|\mathrm{cl}_X(C) \cap \mathrm{Fix}(f)| = 1$ and $f^n(C) = C$ for some $n > 1$. By (3.6.2), (3.6.4) and (3.6.5) the image under f of an end-periodic component of $X \setminus \mathrm{Fix}(f)$ is an end-periodic component of $X \setminus \mathrm{Fix}(f)$. We say that, for an element $x \in X$, $\omega(x)$ is a *periodic orbit* if there exists $y \in P(f)$ such that $\omega(x) = \{f^n(y) \colon n \in \mathbb{N} \cup \{0\}\}$. We understand that $f^0(y) = y$ for any $y \in X$.

Let us assume that $x \in X$ is such that $\omega(x)$ is not a periodic orbit. Then if $j \in \mathbb{N}$ we have $x \in X \setminus \mathrm{Fix}(f^j)$. Since f^j is a homeomorphism from X onto itself, the family $\mathcal{C}_j$ of components of $X \setminus \mathrm{Fix}(f^j)$ satisfies properties (3.6.1)-(3.6.5) where $\mathcal{C}$ is replaced by $\mathcal{C}_j$ and f by f^j. Let $C(j-1) \in \mathcal{C}_j$ be such that $x \in C(j-1)$. If $C(j-1)$ is not end-periodic then, by (3.6.4) and (3.6.5), $\omega(x) \in \mathrm{Fix}(f^j)$. Since this contradicts the fact that $\omega(x)$ is not a periodic orbit, $C(j-1)$ is end-periodic. Put $\mathrm{cl}_X(C(j-1)) \cap \mathrm{Fix}(f^j) = \{d(j-1)\}$ and note that $d(j-1)$ is an end-point of $\mathrm{cl}_X(C(j-1))$. Moreover, since $C(j-1)$ is end-periodic, there exists $n_{j-1} > 1$ such that $f^{jn_{j-1}}(C(j-1)) = C(j-1)$. We have shown the following result.

Lemma 3.7. *If $x \in X$ is such that $\omega(x)$ is not a periodic orbit then, for any $j \in \mathbb{N}$, we have $x \in C(j-1)$ where $C(j-1)$ is an end-periodic component of $X \setminus \mathrm{Fix}(f^j)$. Moreover if $\mathrm{cl}_X(C(j-1)) \cap \mathrm{Fix}(f^j) = \{d(j-1)\}$, then $d(j-1)$ is an endpoint of $\mathrm{cl}_X(C(j-1))$ and $f^{jn_{j-1}}(C(j-1)) = C(j-1)$ for some integer $n_{j-1} > 1$.*

Let $N = \{n_0, n_1, n_2, \dots\}$ be a sequence of positive integers and let $\mathbb{Z}/n_i$ denote the cyclic group of integers $\mod (n_i)$, with the discrete topology. Then $C_N = \prod_{i=0}^{\infty} \mathbb{Z}/n_i$ is a Cantor set. Define a homeomorphism $h_N : C_N \to C_N$ by $h_N(x_0, x_1, \dots) = (y_0, y_1, \dots)$, where y_i is defined as follows. If $x_0 < n_0 - 1$, then $y_0 = x_0 + 1$ and $y_i = x_i$ for all $i > 0$. If there is $j > 0$ such that $x_i = n_i - 1$ for all $i < j$ and $x_j < n_j - 1$, then $y_i = 0$ for all $i < j$, $y_j = x_j + 1$ and $y_l = x_l$ for all $l > j$. If $x_i = n_i - 1$ for all i, then $y_i = 0$ for all i (one can think of $h_N(x_0, x_1, \dots)$ informally as $(x_0, x_1, \dots) + (1, 0, 0, \dots)$ by adding in each coordinate modulo n_i and carrying). It is not difficult to see that h_N is a minimal homeomorphism. Any homeomorphism $f : C \to C$ on a Cantor set C for which there exists a sequence of positive integers $N = \{n_0, n_1, \dots\}$ and a homeomorphism $\varphi : C \to C_N$ such that $f = \varphi^{-1} \circ h_N \circ \varphi$ will be called an *adding machine* (or a *generalized odometer*) [BKP97, D86]. Similarly, given a finite sequence $N(k) = \{n_0, \dots, n_k\}$ of positive integers, we can define a periodic homeomorphism $h_k : \prod_{i=0}^{k} \mathbb{Z}/n_i \to \prod_{i=0}^{k} \mathbb{Z}/n_i$ by restricting h_N to the first $k+1$ coordinates, where $N(k) \subset N$. Hence, informally, $h_k(x_0, \dots, x_k)$ is defined as $(x_0, x_1, \dots, x_k) + (1, 0, \dots, 0)$ by adding modulo n_i in each coordinate and carrying.

We are ready to prove the above mentioned result about the ω-limit sets of a self homeomorphism defined on a dendrite.

Theorem 3.8. *Let X be a dendrite and $f \colon X \to X$ be a homeomorphism from X onto itself. If $x \in X$ then $\omega(x)$ is either a periodic orbit or a Cantor set. Moreover if $\omega(x)$ is a Cantor set, then $f_{|\omega(x)}$ is an adding machine.*

Proof. Let 0_m and 0_∞ denote the m-tuple of zeros and the infinite sequence of zeros, respectively. Take $x \in X$ and assume that $\omega(x)$ is not a periodic orbit. We will construct a decreasing sequence of subcontinua of X which contain x, as follows. First, by Lemma 3.7, $x \in C(0)$ where $C(0)$ is an end-periodic component of $X \setminus \mathrm{Fix}(f)$. Put $\mathrm{cl}_X(C(0)) \cap \mathrm{Fix}(f) = \{d\}$ and let $n_0 > 1$ be minimal such that $f^{n_0}(C(0)) = C(0)$. Put $D(0) = \mathrm{cl}_X(C(0))$ and note that $D(0) = C(0) \cup \{d\}$ and $f^{n_0}(D(0)) = D(0)$. Put $C(i) = f^i(C(0))$ and $D(i) = f^i(D(0))$ for $1 \le i < n_0$. Let $N(0) = \{n_0\}$. Since $h_0 : \mathbb{Z}/n_0 \to \mathbb{Z}/n_0$ is defined as $h_0(m) = m + 1 \mod (n_0)$, we can also write $D(i) = D(h_0^i(0)) = f^i(D(0))$ for any $0 \le i < n_0$. Then $C(i)$ is an end-periodic component of $X \setminus \mathrm{Fix}(f)$ and $D(i) \cap \mathrm{Fix}(f) = \{d\}$.

Now define $f_0 = (f^{n_0})_{|D(0)}$ and note that $f_0 \colon D(0) \to D(0)$ is a homeomorphism from the dendrite $D(0)$ onto itself. Moreover $\mathrm{Fix}(f_0) \neq \emptyset$ and, by Lemma 3.7, $x \in C(0,0) = C(0_2)$, where $C(0_2)$ is an end-periodic component of $D(0) \setminus \mathrm{Fix}(f_0)$. Put $D(0_2) = \mathrm{cl}_X(C(0_2))$, $D(0_2) \cap \mathrm{Fix}(f_0) = \{d(0)\}$ and let

$n_1 > 1$ be minimal such that $f_0^{n_1}(D(0_2)) = D(0_2)$. Note that $D(0_2) \subsetneq D(0)$ since $d \in D(0) \setminus D(0_2)$. Let $N(1) = \{n_0, n_1\}$. Put $D(h_1^i(0_2)) = f^i(D(0_2))$ for $1 \le i < n_0 \cdot n_1 - 1$, and $d(i) = d(h_0^i(0)) = f^i(d(0))$ for $1 \le i < n_0 - 1$. Let $f_1 = (f_0^{n_1})_{|D(0_2)}$ and note that $f_1 \colon D(0_2) \to D(0_2)$ is a homeomorphism from the dendrite $D(0_2)$ onto itself.

Now we proceed by induction for constructing the subcontinuum $D(0_{j+1})$ from the subcontinuum $D(0_j)$ that contains x. Put $f_{j-1} = (f_{j-2}^{n_{j-1}})_{|D(0_j)}$ and note that $f_{j-1} \colon D(0_j) \to D(0_j)$ is a homeomorphism. Hence $\mathrm{Fix}(f_{j-1}) \neq \emptyset$ and, since $\omega(x)$ is not a periodic orbit, $x \in D(0_j) \setminus \mathrm{Fix}(f_{j-1})$. Thus, by Lemma 3.7, x belongs to an end-periodic component $C(0_{j+1})$ of $D(0_j) \setminus \mathrm{Fix}(f_{j-1})$. Put $D(0_{j+1}) = \mathrm{cl}_X(C(0_{j+1}))$, $D(0_{j+1}) \cap \mathrm{Fix}(f_{j-1}) = \{d(0_j)\}$ and let $n_j > 1$ be minimal such that $f_{j-1}^{n_j}(D(0_{j+1})) = D(0_{j+1})$. Let $N(j) = \{n_0, n_1, \ldots, n_j\}$. Put $D(h_j^i(0_{j+1})) = f^i(D(0_{j+1}))$ for $1 \le i < n_0 n_1 \cdots n_j - 1$, and $d(h_{j-1}^i(0_j)) = f^i(d(0_j))$ for $1 \le i < n_0 n_1 \cdots n_{j-1} - 1$.

In this way, for $k_i \in \{0, 1, \ldots, n_i - 1\}$ and $i \in \{0, 1, \ldots, m\}$, we have constructed a subcontinuum $D(k_0, k_1, \ldots, k_m)$ of X, such that

$$D(k_0, k_1, \ldots, k_m, k_{m+1}) \subsetneq D(k_0, k_1, \ldots, k_m)$$

for every $k_{m+1} \in \{0, 1, \ldots, n_{m+1} - 1\}$. Define

$$D(k_0, k_1, k_2, \ldots) = \bigcap_{m=0}^{\infty} D(k_0, k_1, \ldots, k_m)$$

and note that $D(k_0, k_1, k_2, \ldots)$ is the intersection of a decreasing sequence of subcontinua of X, thus is a subcontinuum of X as well. Also define

$$d(k_0, k_1, k_2, \ldots) = \lim_{m \to \infty} d(k_0, k_1, \ldots, k_m).$$

The limit exists because the sequence of points $(d(k_0, k_1, \ldots, k_m))_m$ forms a monotone sequence contained in an arc in X.

Define

$$K = \{d(k_0, k_1, k_2, \ldots) \colon k_i \in \{0, 1, \ldots, n_i - 1\} \text{ for all } i\}$$

and note that $K \subset X$. Put $N = \{n_0, n_1, n_2, \ldots\}$ and $C_N = \prod_i \mathbb{Z}/n_i$. Let $\varphi \colon K \to C_N$ be defined by $\varphi(d(k_0, k_1, \ldots)) = (k_0, k_1, \ldots)$. We claim that φ is a homeomorphism. To see this, let τ be the topology on X and τ_s the topology on K as a subspace of X. If τ_p is the product topology on C_N, then we must show that $\tau_s = \tau_p$. Assume first that U is a basic open set in τ_p. Let $d(k_0, k_1, k_2, \ldots) \in U$. Then there is m such that

$$U = \{k_0\} \times \{k_1\} \times \cdots \times \{k_m\} \times \prod_{i > m} \mathbb{Z}/n_i$$

Let $V = D(k_0, k_1, \ldots, k_m) \setminus \{d(k_0, k_1, \ldots, k_{m-1})\}$. Note that $d(k_0, k_1, k_2, \ldots) \in V \cap K$ and that V is a component of

$$V' = X \setminus \{d, d(k_0), d(k_0, k_1), \ldots, d(k_0, k_1, \ldots, k_{m-1})\}.$$

Since $V' \in \tau$ and X is locally connected, it follows that $V \in \tau$, so $V \cap K \in \tau_s$. Since $V \cap K \subset U$ it follows that $U \in \tau_s$. This shows that $\tau_p \subset \tau_s$.

To prove the other inclusion let $U \in \tau_s$. Then $U = V \cap K$, for some $V \in \tau$. Let $y = d(k_0, k_1, \ldots) \in U$. For simplicity put $D_\infty = D(k_0, k_1, \ldots)$ and, for each i, $D_i = D(k_0, k_1, \ldots, k_i)$, $d_i = d(k_0, k_1, \ldots, k_i)$ and $I_i = D_i \setminus D_\infty$. Then I_i is arcwise connected. To see this we will show that every point $z \in I_i$ can be joined to $d_{i-1} \in I_i$ by an arc lying entirely in I_i. Let zd_{i-1} be the arc in D_i joining z to d_{i-1}. Since y separates d_{i-1} from $D_\infty \setminus \{y\}$, it suffices to show that y does not lie on zd_{i-1}. Note that $d_j \in d_{i-1}y$ for all $j > i-1$. If $y \in zd_{i-1}$, then $d_j \in zd_{i-1}$ for all $j > i-1$. This implies that $z \in D_\infty$, a contradiction. Hence I_i is arcwise connected for all i. Since $(D_i)_i$ is a decreasing sequence it follows that $(I_i)_i$ is a decreasing sequence as well, and since $\bigcap_i I_i = \emptyset$, it follows that $\operatorname{diam}(I_i) \to 0$. Then there is n such that $I_n \subset V$.

Note that

$$D_n \cap K = \{k_0\} \times \{k_1\} \times \cdots \times \{k_n\} \times \prod_{i>n} \mathbb{Z}/n_i$$

so $D_n \cap K \in \tau_p$. Moreover $y \in D_n \cap K$ and

$$D_n \cap K = (I_n \cap K) \cup (D_\infty \cap K) \subset (V \cap K) \cup \{y\} = U \cup \{y\} = U.$$

This implies that $U \in \tau_p$ and then $\tau_s \subset \tau_p$. Thus $\tau_s = \tau_p$ and since C_N is a Cantor set in the product topology, K is a Cantor set as well in the subspace topology τ_s.

Since $d(h_j^i(0_{j+1})) = f^i(d(0_{j+1}))$ and

$$d(k_0, k_1, k_2, \dots) = \lim_{m\to\infty} d(k_0, k_1, k_2, \ldots, k_m),$$

it follows that $f(d(k_0, k_1, \dots)) = d(h_N(k_0, k_1, \dots))$. In other words, $f_{|K} = \varphi^{-1} \circ h_N \circ \varphi$ and $f_{|K}$ is an adding machine. In particular the orbit of any point in K is dense in K. Now, by [Nad92, Theorem 10.4], $\operatorname{diam}(f^n(D(0_\infty)) \to 0$ and since $x, d(0_\infty) \in D(0_\infty)$ it follows that $f^n(x) \to f^n(d(0_\infty))$. Therefore $\omega(x) = \omega(d(0_\infty))$ and since the orbit of $d(0_\infty)$ is dense in K, we have $\omega(d(0_\infty)) = K$. This shows that $\omega(x)$ is a Cantor set and $f_{|\omega(x)}$ is an adding machine. □

Corollary 3.9. *If $f\colon X \to X$ is a homeomorphism from a dendrite X onto itself, then the entropy of f is zero.*

Proof. Let $h_N : C_N \to C_N$ be an adding machine. Then h_N is an isometry in the natural metric on C_N and, hence, the entropy of h_N is zero. Moreover, if the entropy of f is positive, then there exists $x \in X$ such that the entropy of $f_{|\omega(x)}$ is positive. Hence the result follows from Theorem 3.8. □

4. Open maps between dendrites

Consider spaces X, Y, M and maps $f\colon X \to Y$ and $u\colon M \to Y$. Then a map $v\colon M \to X$ is said to be a *lifting of u with respect to f* provided that $u = f \circ v$. Denote by $C(X,Y)$ the space of all maps from X into Y. In Section 1 of [K00] the following result is proved.

Theorem 4.1. *Let $f\colon X \to Y$ be a confluent and light map from a compact space X onto Y. Let $w\colon D \to Y$ be a map from a dendrite D and let $x_0 \in X$ and $\theta \in D$ be such that $f(x_0) = w(\theta)$. Then*

(4.1.1) *there is a lifting $v\colon D \to X$ of w with respect to f such that $v(\theta) = x_0$;*

(4.1.2) *all liftings of w with respect to f constitute a zero-dimensional compact subset of $C(D,X)$.*

For proving Corollary 4.3 we will use the following reformulation of the conclusion of Theorem 4.1.

Corollary 4.2. *Let $f\colon X \to Y$ be a confluent and light map from a compact space X onto Y. Let $w\colon D \to Y$ be a map from a dendrite D and let $x_0 \in X$ and $\theta \in D$ be such that $f(x_0) = w(\theta)$. Then there exist a compact zero-dimensional space Z, a point $z_0 \in Z$, and a map $q\colon Z \times D \to X$ such that*

(4.2.1) *$q(z_0, \theta) = x_0$,*

(4.2.2) *$f(q(z,t)) = w(t)$ for each $(z,t) \in Z \times D$,*

(4.2.3) *for each lifting $\lambda\colon D \to X$ of w with respect to f, there is a uniquely determined element $z \in Z$ such that $\lambda(t) = q(z,t)$ for each $t \in D$.*

Proof. Let Z be the set of all $z \in C(D,X)$ such that z is a lifting of w with respect to f. By (4.1.2) Z is compact and zero-dimensional. Let z_0 be the lifting v guaranteed in (4.1.1) and define $q\colon Z \times D \to X$ as $q(z,t) = z(t)$. Then it is easy to show that properties (4.2.1), (4.2.2) and (4.2.3) are satisfied. □

Corollary 4.3. *Suppose that $f\colon X \to Y$ is an open and onto map between dendrites X and Y. Then there is a compact and zero-dimensional set Z and an onto map $q\colon Z \times Y \to X$ such that if $\pi_2\colon Z \times Y \to Y$ is the map given by $\pi_2(z,y) = y$ for any $(z,y) \in Z \times Y$, then $f \circ q = \pi_2$. Additionally we have the following properties*

(4.3.1) *if $q(z_1, y_1) = q(z_2, y_2)$, then $y_1 = y_2$.*

(4.3.2) *if $z \in Z$ and $R = q(\{z\} \times Y)$, then the maps $q_{|\{z\}\times Y}\colon \{z\} \times Y \to R$ and $f_{|R}\colon R \to Y$ are homeomorphisms.*

Proof. Open maps between dendrites are confluent and light, so we can use Corollary 4.2 with the map f as given in the hypothesis, $D = Y$ and w as the identity map on Y. To show that the map $q\colon Z \times Y \to X$ is onto let $x \in X$. By (4.1.1) there is a lifting $\lambda\colon Y \to X$ of w with respect to f such that $\lambda(f(x)) = x$. By (4.2.3) there is an element $z \in Z$ such that $\lambda(y) = q(z,y)$ for any $y \in Y$. In particular $q(z, f(x)) = \lambda(f(x)) = x$, so f is onto. Properties (4.3.1) and (4.3.2) are easy to prove. □

For a natural number n we write $I_n = \{1, 2, \ldots, n\}$.

Theorem 4.4. *Let $f\colon X \to Y$ be an open map from a dendrite X onto a dendrite Y, and let $\mathfrak{C}$ be the set of critical points of f. Then there exist $n \in \mathbb{N}$ and n subcontinua $X_1, X_2, \ldots, X_n$ of X with the following properties*

(4.4.1) $X = X_1 \cup X_2 \cup \cdots \cup X_n$;

(4.4.2) *for any $i, j \in I_n$ with $i \neq j$ the set $X_i \cap X_j$ contains at most one element. Moreover if $x \in X_i \cap X_j$ then $x \in \mathfrak{C} \setminus E(X)$ and $f(x) \in E(Y)$;*

(4.4.3) *for each $i \in I_n$, the map $f_i = f_{|X_i} : X_i \to Y$ is open and onto;*

(4.4.4) *for each $i \in I_n$, if $f(\mathfrak{C} \cap X_i) \subset E(Y)$, then the map $f_i = f_{|X_i} : X_i \to Y$ is a homeomorphism;*

(4.4.5) *for each $i \in I_n$, if $f(\mathfrak{C} \cap X_i) \setminus E(Y) \neq \emptyset$, it follows that*

(4.4.5.1) *if c is a critical point of f_i and $c \notin E(X_i)$ then $f_i(c) \notin E(Y)$;*

(4.4.5.2) *there is a compact and zero-dimensional set Z_i and an onto map $q_i\colon Z_i \times Y \to X$ such that if $\pi_2\colon Z_i \times Y \to Y$ is the map given by $\pi_2(z, y) = y$ for any $(z, y) \in Z_i \times Y$, then $f_i \circ q_i = \pi_2$. Additionally we have properties* (4.3.1) *and* (4.3.2) *of* Corollary 4.3 *when Z, X, q and f are replaced by Z_i, X_i, q_i and f_i, respectively.*

Proof. Put $M = f^{-1}(E(Y))$ and consider the sets $O_M = O(X) \cap M$ and $B_M = B(X) \cap M$. Then $M = E(X) \cup O_M \cup B_M$ by (2.1.5). Moreover, the sets $E(X)$, O_M and B_M are pairwise disjoint and, by (2.1.8), the set $M \setminus E(X)$ is finite. Clearly $M \setminus E(X) = O_M \cup B_M$. Now consider the family

$$\mathcal{C} = \{ C \subset X : C \text{ is a component of } X \setminus M \}.$$

In the following lines we establish some properties of the family $\mathcal{C}$.

1) If $C \in \mathcal{C}$ then $f(C) = Y \setminus E(Y)$ and $f(\mathrm{cl}_X(C)) = Y$.

To show this let $C \in \mathcal{C}$ and $c \in C$. If $f(c) \in E(Y)$, then $c \in M$, a contradiction to the fact that $C \cap M = \emptyset$. Hence $f(C) \subset Y \setminus E(Y)$. To show the other inclusion fix a point $x \in C$ and let $y \in Y \setminus E(Y)$. Put $z = f(x)$. Note that the set $Y \setminus E(Y)$ is arcwise connected and that $yz \cap E(Y) = \emptyset$. Then for the component K of $f^{-1}(yz)$ that contains x, we have $K \cap M = \emptyset$. Hence $K \subset C$. Since f is confluent we have $f(K) = yz$, so there is $c \in K$ such that $f(c) = y$. This shows that $Y \setminus E(Y) \subset f(C)$ and the first part of 1) holds. Since f is closed we have

$$f(\mathrm{cl}_X(C)) = \mathrm{cl}_Y(f(C)) = \mathrm{cl}_Y(Y \setminus E(Y)) = Y.$$

Hence 1) holds. Now we claim that

2) If $C, D \in \mathcal{C}$ and $C \neq D$, then $\mathrm{cl}_X(C) \cap D = \emptyset$.

To show this let $C, D \in \mathcal{C}$ be such that $C \neq D$. Note that M is a subset of X such that $E(X) \subset M$ and $M \setminus E(X)$ is finite. Then, by Theorem 2.2, C is open and closed in $X \setminus M$. Thus $\mathrm{cl}_{X\setminus M}(C) \cap D = C \cap D = \emptyset$, so

$$\emptyset = \mathrm{cl}_{X\setminus M}(C) \cap D = \mathrm{cl}_X(C) \cap (X \setminus M) \cap D = \mathrm{cl}_X(C) \cap D.$$

This shows 2).

3) If $C \in \mathcal{C}$, then $\mathrm{cl}_X(C) \setminus C \subset M$.

To see this let $C \in \mathcal{C}$ and take a point $x \in \mathrm{cl}_X(C) \setminus C$. If $x \notin M$, then $x \in D$ for some $D \in \mathcal{C}$. Note that $\mathrm{cl}_X(C) \cap D \neq \emptyset$ and $D \neq C$. This contradicts 2), so 3) holds.

4) If $C, D \in \mathcal{C}$, $C \neq D$ and $B = \mathrm{cl}_X(C) \cap \mathrm{cl}_X(D)$, then either $B = \emptyset$ or B is a one-point set and $B \subset O_M \cup B_M$.

To show this let C, D and B be as assumed. Consider that B is nonempty. Then B is a subcontinuum of X, so $f(B)$ is a subcontinuum of Y. Let us assume that there is a point $b \in B \setminus M$. Then, by 3), $b \in \mathrm{cl}_X(C) \setminus M \subset C$ and $b \in \mathrm{cl}_X(D) \setminus M \subset D$. This implies that $C = D$, which is a contradiction. Hence $B \subset M$. Thus $f(B) \subset f(M) \subset E(Y)$. Since $E(Y)$ is zero-dimensional and $f(B)$ is connected, it follows that $f(B)$ is a one-point set. Hence, by (2.1.2), B is a one-point set too.

Put $B = \{x\}$ and note that $x \in M$. Then $x \in E(X) \cup O_M \cup B_M$. Let us assume that $x \in E(X)$. Fix points $c \in C$ and $d \in D$ and consider the arcs $cx \subset \mathrm{cl}_X(C)$ and $dx \subset \mathrm{cl}_X(D)$. Then $cx \cap dx \subset B = \{x\}$, so the set $cx \cup dx$ is an arc in X with end-points c and d. Since $x \in E(X)$ either $x = c$ or $x = d$. Hence either $\mathrm{cl}_X(C) \cap D \neq \emptyset$ or $\mathrm{cl}_X(D) \cap C \neq \emptyset$. In any situation we contradict assertion 2), so 4) holds.

5) If $C \in \mathcal{C}$, then $E(\mathrm{cl}_X(C)) = \mathrm{cl}_X(C) \cap M$.

To show this note first that $\mathrm{cl}_X(C) \cap M \subset \mathrm{cl}_X(C) \setminus C \subset E(\mathrm{cl}_X(C))$. On the other hand suppose $x \in E(\mathrm{cl}_X(C))$ and $x \notin M$. Then $x \notin E(X)$ so $X \setminus \{x\}$ has at least two components A and B. Assume, without loss of generality, that $\mathrm{cl}_X(C) \setminus \{x\} \subset A$. Choose $a \in A \setminus E(X)$ and $b \in B \setminus E(X)$, then $x \in ab$ and $ab \cap E(X) = \emptyset$. By (2.1.8), $ab \cap M$ is finite and there exists an open sub-arc pq of ab which contains x such that $pq \cap M = \emptyset$. Then $pq \subset C$ which contradicts the assumption that $x \in E(\mathrm{cl}_X(C))$, and 5) holds.

6) The family $\mathcal{C}$ is finite.

To see this fix a point $y \in Y \setminus E(Y)$. By 1), $f^{-1}(y) \cap C \neq \emptyset$ for each $C \in \mathcal{C}$. By (2.1.7), $f^{-1}(y)$ is finite. Hence $\mathcal{C}$ is finite and 6) holds.

By 6) there exists $n \in \mathbb{N}$ such that $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ and $C_i \neq C_j$ for every $i, j \in I_n$ with $i \neq j$. Given $i \in I_n$ put $X_i = \mathrm{cl}_X(C_i)$. Clearly X_i is a subcontinuum of X. Moreover if $i, j \in I_n$ and $i \neq j$ then, by 4), the set $X_i \cap X_j$ is either empty or it is a one-point set whose only element belongs to $O_M \cup B_M$. By 5) we have $E(X_i) = X_i \cap M$ for any $i \in I_n$. We claim that

7) $X = X_1 \cup X_2 \cup \cdots \cup X_n$.

To see this put $X_0 = X_1 \cup X_2 \cup \cdots \cup X_n$ and note that $X \setminus M \subset X_0$. Suppose that there is a point $x \in M \setminus X_0$. Then $X \setminus X_0$ is an open subset of X that contains x. Then $f(X \setminus X_0)$ is an open subset of Y that contains $f(x) \in E(Y)$. Hence there exists $y \in f(X \setminus X_0)$ such that $y \notin E(Y)$. Let

$a \in X \setminus X_0$ be such that $f(a) = y$. Note that $a \in X \setminus M$ so $a \in X_0$. This contradiction shows that $M \subset X_0$, so 7) holds.

By 7) assertion (4.4.1) holds. Assertion (4.4.2) follows from 1), 3) and 4). To show (4.4.3) let i and f_i be as assumed. By 1) $f_i(X_i) = f(\mathrm{cl}_X(C_i)) = Y$ so f_i is onto. Since f is open, f_i is interior at any point of $X_i \setminus \bigcup_{j \neq i} X_j$. Hence to show that f_i is open it suffices to show that

8) f_i is interior at any point of $X_i \cap X_j$ for $j \neq i$.

To show this let $j \neq i$ and take a point $x \in X_i \cap X_j$. By 4) $x \in O_M \cup B_M$. Since $O_M \cup B_M$ is finite, there exists an open and connected subset V of X_i such that $V \cap (O_M \cup B_M) = \{x\}$. Note that f_i is interior at any point of $V \setminus \{x\}$. We claim that $y = f(x) \in \mathrm{int}_Y(f(V))$. For suppose that there exists $y_n \in Y \setminus f(V)$ such that $y_n \to y$. Then $\mathrm{Lim}\, y_n y = \{y\}$. Since $\dim f^{-1}(y) = 0$, there exists $a \in V \setminus f^{-1}(y)$. Then $ax \subset V$ and $f(ax) \subset f(V)$. Since $y \in E(Y)$ and $f(ax) \subset f(V)$ is a subcontinuum of Y containing y, there exist a first point w_n of $y_n y$ (from y_n) such that $w_n \in f(\mathrm{cl}_X(V))$ and a first point z_n of $w_n y$ (from w_n) such that $z_n \in f(ax)$. Choose $v_n \in ax$ such that $f(v_n) = z_n$ and let K_n be the component of $f^{-1}(w_n z_n)$ containing v_n. Then $f(K_n) = w_n z_n$ and since $\dim f^{-1}(y) = 0$ we have $\mathrm{Lim}\, K_n = \{x\}$. Hence $K_n \subset V$ for sufficiently large n. Choose n such that $K_n \subset V$ and let $u_n \in K_n$ be such that $f(u_n) = w_n$. Since w_n is the first point of $y_n y \cap f(\mathrm{cl}_X(V))$ we have $w_n \notin \mathrm{int}_Y(f(V))$, contradicting the fact that f is interior at u_n. This completes the proof of 8) and, hence, (4.4.3) holds.

To show assertion (4.4.4) let $i \in I_n$ and assume that $f(\mathfrak{C} \cap X_i) \subset E(Y)$. Then f_i is an open and onto map with no critical points. By Corollary 3.3, f_i is a homeomorphism.

To show assertion (4.4.5) of the theorem, let $i \in I_n$ and assume that $f(\mathfrak{C} \cap X_i) \setminus E(Y) \neq \emptyset$. Let c be a critical point of f_i such that $c \notin E(X_i)$. If $f_i(c) \in E(Y)$, then $c \in X_i \cap M = E(X_i)$ according to 5). This contradiction shows that $f_i(c) \notin E(Y)$, so (4.4.5.1) holds. Finally (4.4.5.2) follows from Corollary 4.3. □

Corollary 4.5. *Suppose that $f : X \to Y$ is an open map from the dendrite X onto the dendrite Y such that $f(\mathfrak{C}) \subset E(Y)$, where $\mathfrak{C}$ is the set of critical points of f. Then there exist $n \in \mathbb{N}$ and n subcontinua $X_1, \ldots, X_n$ such that $X = \bigcup_{i=1}^{n} X_i$, $X_i \cap X_j$ is at most one critical point of f and for each $i \in I_n$, $f_{|X_i} : X_i \to Y$ is a homeomorphism.*

5. Open maps on dendrites

It is easy to see that the set of critical points $\mathfrak{C}$ of an open map $f : X \to Y$ between two dendrites can be uncountable. In this section we will show that for an arc $A \subset X$, the critical set of the restricted map $f_{|A}$ is finite. We always assume that $f \colon X \to Y$ is an open map from a dendrite X onto a dendrite Y.

Theorem 5.1. *Let A be a subcontinuum of X such that $f_{|A}$ is one-to-one. Then there is a subcontinuum B of X such that $A \subset B$ and $f_{|B} : B \to Y$ is a homeomorphism.*

Proof. Let C be a component of $Y \setminus f(A)$. By [Nad92, Theorem 5.6] $\mathrm{cl}_Y(C) \cap f(A) \neq \emptyset$. Let $a_C \in A$ be such that $f(a_C) \in \mathrm{cl}_Y(C)$. Since X contains no simple closed curves, we have $\mathrm{cl}_Y(C) \cap f(A) = \{f(a_C)\}$. Moreover, by (2.1.10) there is a subcontinuum A_C of X such that $a_C \in A_C$ and $f_{|A_C} : A_C \to \mathrm{cl}_Y(C)$ is a homeomorphism. Then

$$B = A \cup \left(\bigcup \{A_C \colon C \text{ is a component of } Y \setminus f(A)\}\right)$$

satisfies the required conditions. □

In the next theorem we show that on a given arc $A \subset X$, the map $f_{|A}$ has only finitely many critical points.

Theorem 5.2. *Let A be an arc in X from a point $a \in X$ to a point $b \in X$. Order A by $\leq$ in such a way that $a \leq b$. Then there are $a = a_0 < a_1 < \cdots < a_k = b$ such that $f_{|a_i a_{i+1}}$ is one-to-one, for any $i \in \{0, 1, \ldots, k-1\}$ and the set of critical points of $f_{|A}$ is $\{a_1, a_2, \ldots, a_{k-1}\}$*

Proof. First assume that $A \setminus \{a, b\}$ contains infinitely many critical points a_i of the map $f_{|A}$. Given $i \in \mathbb{N}$ note that f is not one-to-one in any neighborhood of a_i in A. By compactness of A it follows that there is a subarc B of A such that B contains infinitely many a_n and $f(B) \neq Y$.

Since we can replace A by B we may assume that $f(A) \neq Y$. Fix an ordinary point $y \in Y \setminus f(A)$. Given $i \in \mathbb{N}$, by (2.1.10), there is a subcontinuum A_i of X such that $a_i \in A_i$ and $f_{|A_i} : A_i \to Y$ is a homeomorphism. Consider the first point map $r \colon X \to A$ from X to A and note that f is one to one in the arc xa_i, for any $x \in f^{-1}(y) \cap A_i$. By (2.1.7) the set $f^{-1}(y)$ is finite. Moreover $f^{-1}(y) \cap A = \emptyset$. Put $f^{-1}(y) = \{x_1, x_2, \ldots, x_n\}$ and note that $f^{-1}(y) \cap A_i \neq \emptyset$ for every $i \in \mathbb{N}$. Since A contains infinitely many a_i there exist $s \in \{1, 2, \ldots, n\}$ and $N \subset \mathbb{N}$ infinite such that $x_s \in A_i$ for any $i \in N$. Put $c = r(x_s)$. Since X is uniquely arcwise connected and N is infinite, there must exist $i, j \in N$ such that either $a_i < a_j < c$ or $c < a_j < a_i$. Hence $a_j \in ca_i \subset A_i$, contradicting that $f_{|A_i}$ is a homeomorphism. This contradiction shows that $f_{|A}$ has finitely many critical points $a_1, a_2, \ldots, a_{k-1}$.

Put $a_0 = a$, $a_k = b$ and assume, without loss of generality, that $a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k$ and that the set $A \setminus \{a_0, a_1, \ldots, a_k\}$ contains no critical points of $f_{|A}$. Given $i \in \{0, 1, \ldots, k-1\}$ suppose that $f_{|a_i a_{i+1}}$ is not one-to-one. Then there exist $p, q \in a_i a_{i+1}$ such that $f(p) = f(q)$. We can assume that $a_i \leq p < q \leq a_{i+1}$. By (2.1.2) $f(pq)$ is a non degenerate subcontinuum of Y, so we can take an end-point y_0 of $f(pq)$ different than $f(p)$. Let $x_0 \in pq$ be such that $f(x_0) = y_0$. Note that $x_0 \in A \setminus \{a_0, a_1, \ldots, a_{k+1}\}$. Moreover, it is not difficult to see that x_0 is a critical point of $f_{|A}$. This contradiction shows that $f_{|a_i a_{i+1}}$ is one-to-one. □

Remark 5.3. *Note that* Theorem 5.2 *does not state that* $\mathfrak{C} \cap A$ *is finite, where* $\mathfrak{C}$ *denotes the set of critical points of* f. *Indeed, easy examples show that this may not be true.*

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