FIXED POINTS FOR POSITIVELY ORIENTED MAPPINGS OF THE PLANE

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Abstract. Cartwright and Littlewood showed that each non-separating invariant continuum, under an orientation preserving homeomorphism of the plane, contains a fixed point. Bell announced in 1984 an extension of this result to holomorphic mappings of the plane.

In this paper we introduce the class of oriented maps of the plane. We show that among perfect,onto maps of the plane the oriented maps are exactly the compositions of open maps and monotone maps. We extend the above fixed point theorems to the class of positively oriented, perfect,continuous surjections of the plane.

Introduction

The classical fixed point problem asks whether each map of a non separating plane continuum must have a fixed point. Cartwright and Littlewood [CL51] showed that the answer is yes if the map can be extended to an orientation preserving homeomorphism of the plane. It took over 20 years until Bell [Bel78] extended this to the class of all homeomorphisms of the plane. Our ultimate goal is to extend these results to a natural but larger class of plane maps. Bell announced in 1984 (see also Akis [Aki99]) that the Cartwright -Littlewood Theorem can be extended to the class of all holomorphic maps of the plane. These maps behave like orientation preserving homeomorphisms in the sense that they preserve local orientation. We will show that the class consisting of compositions of open and of monotone maps of the plane are oriented and naturally decomposes into two classes, one of which preserves and the other of which reverses local orientation. Moreover, any map in either of these classes is itself a composition of a monotone and a light open map. We will also show that such maps induce a continuous extension to the circle of prime ends of a saturated invariant subcontinuum to the circle of prime ends of its image. Finally we will show that each invariant non-separating plane continuum, under a positively oriented map of the plane, must contain a fixed point.

Bell's above mentioned extension to holomorphic maps made use of several unpublished results which he obtained in the 1970's. These results are also critical in our arguments. The necessary tools are established and proved in [BMOT02b].

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1. Oriented maps

In this paper all maps are perfect (i.e., closed mappings with compact fibers). It is well know that each homeomorphism of the plane is either orientation preserving or orientation reversing. In this section we will establish an appropriate extension of this result for confluent and onto mappings of the plane (Theorem 1.6) by showing that such maps either preserve or reverse local orientation. We will call such maps positively or negatively oriented maps, respectively. A mapping $f : X \to Y$ is called *confluent* provided for each continuum $K \subset Y$ and each component C of $f^{-1}(K)$, $f(C) = K$. For perfect mappings of the plane, confluent is equivalent to the composition of open and monotone maps [LR74]. Holomorphic maps are prototypes of positively oriented maps but positively oriented maps, unlike holomorphic maps, do not have to be light (i.e., they don't necessarily have 0-dimensional point inverses).

We will denote the complex plane by $\mathbb{C}$ and for a continuum $X \subset \mathbb{C}$ we will denote by $T(X)$ the union of X and all of its bounded complementary domains. Clearly $T(X)$ is always a non-separating plane continuum.

Definition 1.1 (Bell)**.** A map $f : U \to \mathbb{C}$ from a simply connected domain U is *positively (negatively, respectively) oriented* provided for each simple closed curve S in U and each point $p \in U \setminus f^{-1}(f(S))$, the degree of f_p, degree$(f_p) \geq 0$ (degree$(f_p) \leq 0$, respectively) where

$$f_p(x) = \frac{f(x) - f(p)}{|f(x) - f(p)|} : S \to S^1.$$

If for each $p \in T(S) \setminus f^{-1}(f(S))$, the degree of f_p, degree$(f_p) > 0$ (degree$(f_p) < 0$, respectively), we say that f is *strictly positively (strictly negatively, respectively) oriented.*

Note that degree(f_p) is also the winding number, denoted by Win$(f,f|_S,f(p))$, of $f|_S$ about $f(p)$. Below we only consider perfect and onto mappings of the plane.

Definition 1.2. A perfect surjection $f : \mathbb{C} \to \mathbb{C}$ is *oriented* provided for each simple closed curve S and each $x \in T(S)$, $f(x) \in T(f(S))$.

Clearly every strictly positively or strictly negatively oriented map is oriented.

It is well known that both open and monotone maps (and hence compositions of such maps) of continua are confluent. It will follow (Lemma 1.5) from a result of Lelek and Read [LR74] that each perfect, onto and confluent mapping of the plane is the composition of a monotone map and a light open map (i.e., point inverses are 0-dimensional). The following Lemmas are in preparation of Theorem 1.6:

Lemma 1.3. *Let $f : \mathbb{C} \to \mathbb{C}$ be a perfect and onto map. Then f is confluent if and only if f is oriented.*

Proof. Suppose that f is oriented. Let A be an arc in $\mathbb{C}$ and let C be a component of $f^{-1}(A)$. Suppose that $f(C) \neq A$. Let $a \in A \setminus f(C)$. Since $f(C)$ does not separate a from infinity, we can choose a simple closed curve S with $C \subset T(S)$, $S \cap f^{-1}(A) = \emptyset$ and $f(S)$ is so close to $f(C)$ that $f(S)$ does not separate a from ∞. This contradicts the fact that f is oriented.

Now suppose that K is an arbitrary continuum in $\mathbb{C}$ and let L be a component of $f^{-1}(K)$. Let $x \in L$ and let A_i be a sequence of arcs in $\mathbb{C}$ such that $\lim A_i = K$ and $f(x) \in A_i$ for each i. Let M_i be the component of $f^{-1}(A_i)$ containing the point x. By the previous paragraph $f(M_i) = A_i$. Since f is perfect, $M = \limsup M_i \subset L$ is a continuum and $f(M) = K$. Hence f is confluent.

Suppose next that $f : \mathbb{C} \to \mathbb{C}$ is a perfect confluent surjection which is not oriented. Then there exists a a simple closed curve S in $\mathbb{C}$ and $p \in T(S) \setminus f^{-1}(f(S))$ such that $f(p) \notin T(f(S))$. Let L be a half-line with end-point $f(p)$ running to infinity in $\mathbb{C} \setminus f(S)$. Let L^* be an arc in L with endpoint $f(p)$ and diameter greater than the diameter of the continuum $f(T(S))$. Let K be the component of $f^{-1}(L^*)$ which contains p. Then $K \subset T(S)$ since $p \in T(S)$ and $L \cap f(S) = \emptyset$. Hence $f(K) \neq L^*$ contradicting the assumption that f is confluent. Thus, f is oriented.

□

Lemma 1.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be a light, open and perfect surjection. Then there exists an integer k and a finite subset $B \subset \mathbb{C}$ such that f is a local homeomorphism at each point of $\mathbb{C} \setminus B$ and for each point $y \in \mathbb{C} \setminus f(B)$, $|f^{-1}(y)| = k$.*

Proof. Let S^2 be the one point compactification of $\mathbb{C}$ and extend f to a map of S^2 onto S^2 so that $f^{-1}(\infty) = \infty$. By abuse of notation we also denote the extended map by f. Then f is a continuous and open surjection of S^2. The result now follows from Whyburn [Why42, X.6.3]. □

Lemma 1.5. *Let $f : \mathbb{C} \to \mathbb{C}$ be a confluent and perfect surjection. Then $f = g \circ h$, where $h : \mathbb{C} \to \mathbb{C}$ is a monotone and perfect surjection with acyclic fibers and $g : \mathbb{C} \to \mathbb{C}$ is a light, open and perfect surjection.*

Proof. By the monotone and light factorization theorem, $f = g \circ h$, where $h : \mathbb{C} \to X$ is monotone, $g : X \to \mathbb{C}$ is light, and X is the quotient space obtained from $\mathbb{C}$ by identifying each component of $f^{-1}(y)$ to a point for each $y \in \mathbb{C}$. Let $y \in \mathbb{C}$ and let C be a component of $f^{-1}(y)$. If C were to separate $\mathbb{C}$, then $f(C) = y$ would be a point while $f(T(C))$ would be a non-degenerate continuum. Choose an arc $A \subset \mathbb{C} \setminus \{y\}$ which meets both $f(T(C))$ and its complement and let $x \in T(C) \setminus C$ such that $f(x) \in A$. If K is the component of $f^{-1}(A)$ which contains x, then $K \subset f(T(C))$. Hence $f(K)$ cannot map onto A contradicting the fact that f is confluent. Thus for each $y \in \mathbb{C}$, each component of $f^{-1}(y)$ is acyclic.

By Moore's Theorem X is homeomorphic to $\mathbb{C}$. Since f is confluent, it is easy to see that g is confluent. By a theorem of Lelek and Read [LR74] g is open since it is confluent and light, f extends to a confluent map of the sphere S^2 and S^2 is locally connected. The fact that both h and g are perfect is proved in Engelking [Eng89, 3.7.5] □

Theorem 1.6. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a perfect and onto map. Then the following are equivalent:*

(1) *f is either strictly positively or strictly negatively oriented.*
(2) *f is oriented,*
(3) *f is confluent,*

Proof. It is clear that (1) implies (2). By Lemma 1.3 every oriented map is confluent. Hence suppose that $f : \mathbb{C} \to \mathbb{C}$ is a perfect, confluent surjection. By Lemma 1.5, $f = g \circ h$, where

$h : \mathbb{C} \to \mathbb{C}$ is a monotone and perfect surjection with acyclic fibers and $g : \mathbb{C} \to \mathbb{C}$ is a light, open and perfect surjection. By Stoilow's Theorem [Why64] there exists a homeomorphism $j : \mathbb{C} \to \mathbb{C}$ such that $g \circ j$ is an analytic map. Then $f = g \circ h = (g \circ j) \circ (j^{-1} \circ h)$. Since $k = j^{-1} \circ h$ is a monotone surjection of $\mathbb{C}$ with acyclic fibers, it is a near homeomorphism. Hence there exists a sequence k_i of homeomorphisms of $\mathbb{C}$ such that $\lim k_i = k$. We may assume that all of the k_i have the same orientation. Let $f_i = (g \circ j) \circ k_i$, S a simple closed curve and $p \in T(S) \setminus f^{-1}(f(S))$. Note that $\lim f_i^{-1}(f_i(S)) \subset f^{-1}(f(S))$. Hence $p \in T(S) \setminus f_i^{-1}(f_i(S))$ for i sufficiently large. Moreover, since f_i converges to f, $f_i|_S$ is homotopic to $f|_S$ in the complement of $f(p)$ for i large. Thus for large i, degree$((f_i)_p) =$ degree(f_p), where

$$(f_i)_p(x) = \frac{f_i(x) - f_i(p)}{|f_i(x) - f_i(p)|} \text{ and } f_p(x) = \frac{f(x) - f(p)}{|f(x) - f(p)|}.$$

Since $g \circ j$ is an analytic map, it is positively oriented and degree$((f_i)_p) =$ degree$(f_p) > 0$ if k_i is orientation preserving and degree$((f_i)_p) =$ degree$(f_p) < 0$ if k_i is orientation reversing. Thus, f is positively oriented if each k_i is orientation preserving and f is negatively oriented if each k_i is orientation reversing. □

We shall need the following three results in the next section.

Lemma 1.7. *Let X and Y be non-degenerate acyclic plane continua and $f : \mathbb{C} \to \mathbb{C}$ a perfect and onto map such that $f^{-1}(Y) = X$ and $f|_{\mathbb{C}\setminus X}$ is confluent. Then for each $y \in \mathbb{C} \setminus Y$, each component of $f^{-1}(y)$ is acyclic.*

Proof. Suppose there exists $y \in \mathbb{C} \setminus Y$ such that some component C of $f^{-1}(y)$ is not acyclic. Then there exists $z \in T(C) \setminus f^{-1}(y) \cup X$. By unicoherence of $\mathbb{C}$, $Y \cup \{y\}$ does not separate $f(z)$ from infinity in $\mathbb{C}$. Let L be a ray in $\mathbb{C} \setminus [Y \cup \{y\}]$ from $f(z)$ to infinity. Then $L = \cup L_i$, where each $L_i \subset L$ is an arc with end-point $f(z)$. For each i the component M_i of $f^{-1}(L_i)$ containing z maps onto L_i. Then $M = \cup M_i$ is a connected closed subset in $\mathbb{C} \setminus f^{-1}(y)$ from z to infinity. This is a contradiction since $z \in T(f^{-1}(y))$. □

Theorem 1.8. *Let X and Y be non-degenerate acyclic plane continua and $f : \mathbb{C} \to \mathbb{C}$ a perfect and onto map such that $f^{-1}(Y) = X$ and $f|_{\mathbb{C}\setminus X}$ is confluent. If A and B are cross cuts of X such that $B \cup X$ separates $A \setminus f^{-1} \circ f(B)$ form ∞ in $\mathbb{C}$ then $f(B) \cup Y$ separates $f(A) \setminus f(B)$ from ∞.*

Proof. Suppose not. Then there exists a half-line L joining $f(A)$ to infinity in $\mathbb{C} \setminus f(B) \cup Y$. As in the proof of Lemma 1.7, there exists a closed and connected set $M \subset \mathbb{C} \setminus B \cup X$ joining A to infinity, a contradiction. □

Corollary 1.9. *Under the conditions of Theorem 1.8, if L is a ray irreducible from Y to infinity, then each component of $f^{-1}(L)$ which meets $\mathbb{C} \setminus X$ is a closed and connected set from X to infinity.*

2. Induced maps of prime ends

Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an oriented, perfect and onto map and $f^{-1}(Y) = X$, where X and Y are acyclic continua. We will show that in this case the map f induces a confluent map F of the prime end circle of X to the prime end circle of Y. This result was announced by Mayer in the early 1980's but never appeared in print. It was also used by Cartwright and Littlewood in [CL51]. There are easy counter examples that show if f is not confluent then it may not induce a continuous function between the circles of prime ends. We will denote by $\widehat{\mathbb{C}}$ the complex 2-sphere and by $\mathbb{D}$ the closed unit ball in $\mathbb{C}$. Then $\mathbb{D} \subset \mathbb{C} \subset \widehat{\mathbb{C}}$.

Theorem 2.1. *Let X and Y be non-degenerate acyclic plane continua and $f : \mathbb{C} \to \mathbb{C}$ a perfect and onto map such that:*

1. *Y has no cut point,*
2. *$f^{-1}(Y) = X$ and*
3. *$f|_{\mathbb{C}\setminus X}$ is confluent.*

Let $\varphi : \widehat{\mathbb{C}}\setminus X \to \widehat{\mathbb{C}}\setminus\mathbb{D}$ and $\psi : \widehat{\mathbb{C}}\setminus Y \to \widehat{\mathbb{C}}\setminus\mathbb{D}$ be conformal mappings. Define $\hat{f} : \widehat{\mathbb{C}}\setminus\mathbb{D} \to \widehat{\mathbb{C}}\setminus\mathbb{D}$ by $\hat{f} = \psi \circ f \circ \varphi^{-1}$.

Then $\hat{f}$ extends to a map $\bar{f} : \overline{\widehat{\mathbb{C}} \setminus \mathbb{D}} \to \overline{\widehat{\mathbb{C}} \setminus \mathbb{D}}$. Moreover, $\bar{f}^{-1}(S^1) = S^1$ and $F = \bar{f}|_{S^1}$ is a confluent map.

Proof. Note that f takes accessible points of X to accessible points of Y. For if P is a path in $[\mathbb{C} \setminus X] \cup \{p\}$ with end point $p \in X$, then by (2), $f(P)$ is a path in $[\mathbb{C} \setminus Y] \cup \{f(p)\}$ with endpoint $f(p) \in Y$.

Let A be a cross cut of X such that the diameter of $f(A)$ is less than half of the diameter of Y and let U be the bounded component of $\mathbb{C} \setminus (X \cup A)$. Let the endpoints of A be $x, y \in X$ and suppose that $f(x) = f(y)$. If x and y lie in the same component of $f^{-1}(f(x))$ then each cross cut $B \subset U$ of X is mapped to a generalized return cut of Y based at $f(x)$ (i.e., the endpoints of B map to $f(x)$). Note that in this case by Theorem 1.8, $\mathrm{Bd}(f(U)) \subset f(A) \cup \{f(x)\}$.

Now suppose that $f(x) = f(y)$ and x and y lie in distinct components of $f^{-1}(f(x))$. Then by unicoherence of $\mathbb{C}$, $\mathrm{Bd}(U) \subset A \cup X$ is a connected set and $\mathrm{Bd}(U) \not\subset \bar{A} \cup f^{-1}(f(x))$. Now $\mathrm{Bd}(U) \setminus (\bar{A} \cup f^{-1}(f(x))) = \mathrm{Bd}(U) \setminus f^{-1}(f(\bar{A}))$ is an open set in $\mathrm{Bd}(U)$. Thus there is a cross cut $B \subset U \setminus f^{-1}(f(\bar{A}))$ of X with $\bar{B} \setminus B \subset \mathrm{Bd}(U) \setminus f^{-1}(f(\bar{A}))$. Now $f(B)$ is contained in a bounded component of $\mathbb{C} \setminus (Y \cup f(A)) = \mathbb{C} \setminus (Y \cup f(\bar{A}))$ by Theorem refconfeq. Since $Y \cap f(\bar{A}) = \{f(x)\}$ is connected and Y does not separate $\mathbb{C}$, it follows by unicoherence that $f(B)$ lies in a bounded component of $\mathbb{C} \setminus f(\bar{A})$. Since $Y \setminus \{f(x)\}$ meets $f(\bar{B})$ and misses $f(\bar{A})$ and $Y \setminus f(x)$ is connected, $Y \setminus \{f(x)\}$ lies in a bounded complementary component of $f(\bar{A})$. This is impossible as the diameter of $f(A)$ is smaller than the diameter of Y. It follows that there exists a $\delta > 0$ such that if the diameter of A is less than δ and $f(x) = f(y)$, then x and y must lie in the same component of $f^{-1}(f(x))$.

In order to define the extension $\bar{f}$ of f over the boundary S^1 of $\overline{\widehat{\mathbb{C}} \setminus \mathbb{D}}$, let C_i be a chain of cross cuts of $\widehat{\mathbb{C}} \setminus \mathbb{D}$ which converge to a point $p \in S^1$ such that $A_i = \varphi^{-1}(C_i)$ is a null chain of cross cuts of X with end points a_i and b_i which converge to a point $x \in X$. There are three cases to consider:

Case 1. f identifies the end points of A_i for some A_i with diameter less than δ. In this case the chain of cross cuts is mapped by f to a chain of generalized return cuts based at $f(a_i) = f(b_i)$. Hence $f(a_i)$ is an accessible point of Y which corresponds (under ψ) to a unique point $q \in S^1$. Define $\bar{f}(p) = q$.

Case 2. Case 1 does not apply and there exists an infinite subsequence A_{i_j} of cross cuts such that $f(\bar{A}_{i_j}) \cap f(\bar{A}_{i_k}) = \emptyset$ for $j \neq k$. In this case $f(A_{i_j})$ is a chain of generalized cross cuts which converges to a point $f(x) \in Y$. This chain corresponds to a unique point $q \in S^1$ since Y has no cut points. Define $\bar{f}(p) = q$.

Case 3. Cases 1 and 2 do not apply. Without loss of generality suppose there exists an i such that for $j > i$ $f(\bar{A}_i) \cap f(\bar{A}_j)$ contains $f(a_i)$. In this case $f(A_j)$ is a chain of generalized cross cuts based at the accessible point $f(a_i)$ which corresponds to a unique point q on S^1 as above. Define $\bar{f}(p) = q$.

It remains to be shown that $\bar{f}$ is a continuous extension of $\hat{f}$ and F is confluent. For continuity it suffices to show continuity at S^1. Let $p \in S^1$ and let C be a small cross cut whose endpoints are on opposite sides of p such that $A = \varphi^{-1}(C)$ has diameter less than δ and such that the endpoints of A are two accessible points of X. Since f is uniformly continuous near X, the diameter of $f(A)$ is small and since ψ is uniformly continuous with respect to connected sets in the complement of Y ([UY51]), the diameter of $B = \psi \circ f \circ \varphi^{-1}(C)$ is small. Also B is either a generalized cross cut or generalized return cut. Since $\hat{f}$ preserves separation of cross cuts, it follows that the image of the domain U bounded by C which does not contain ∞ is small. This implies continuity of $\bar{f}$ at p.

To see that F is confluent let $K \subset S^1$ be a subcontinuum and let H be a component of $\bar{f}^{-1}(K)$. Choose a chain of cross cuts C_i such that $\varphi^{-1}(C_i) = A_i$ is a cross cut of X meeting X in two accessible points a_i and b_i, $C_i \cap \bar{f}^{-1}(K) = \emptyset$ and $\lim C_i = H$. It follows from the preservation of cross cuts (see Theorem 1.8) that $\hat{f}(C_i)$ separates K from ∞. Hence $\hat{f}(C_i)$ must meet S^1 on both sides of K and $\lim \bar{f}(C_i) = K$. Hence $F(H) = \lim \bar{f}(C_i) = K$ as required. □

Corollary 2.2. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a perfect, confluent and onto mapping of the plane, $X \subset \mathbb{C}$ is a subcontinuum without cut points and $f(X) = X$. Let $\hat{X}$ be the component of $f^{-1}(f(X))$ containing X. Let $\varphi : \widehat{\mathbb{C}} \setminus T(\hat{X}) \to \widehat{\mathbb{C}} \setminus \mathbb{D}$ and $\psi : \widehat{\mathbb{C}} \setminus T(f(X)) \to \widehat{\mathbb{C}} \setminus \mathbb{D}$ be conformal mappings. Define $\hat{f} : \widehat{\mathbb{C}} \setminus [\mathbb{D} \cup \varphi(f^{-1}(X))] \to \widehat{\mathbb{C}} \setminus \mathbb{D}$ by $\hat{f} = \psi \circ f \circ \varphi^{-1}$. Put $S^1 = \overline{\widehat{\mathbb{C}} \setminus \mathbb{D}} \setminus [\widehat{\mathbb{C}} \setminus \mathbb{D}]$.*

Then $\hat{f}$ extends over S^1 to a map $\bar{f} : Cl(\widehat{\mathbb{C}} \setminus \mathbb{D}) \to Cl(\widehat{\mathbb{C}} \setminus \mathbb{D})$. Moreover $\bar{f}^{-1}(S^1) = S^1$ and $F = \bar{f}|_{S^1}$ is a confluent map.

Proof. By Lemma 1.5 $f = g \circ m$ where m is a monotone perfect and onto mapping of the plane with acyclic point inverses, and g is an open and perfect surjection of the plane to itself. By Lemma 1.4, $f^{-1}(X)$ has finitely many components. Let S in $\mathbb{C} \setminus f^{-1}(f(X))$ be a simple closed curve separating $\hat{X}$ from infinity and all other components of $f^{-1}(X)$ and let U be the component of $\mathbb{C} \setminus S$ which contains $\hat{X}$. Then U is simply connected and hence homeomorphic to $\mathbb{C}$. By [Lel66] $f(U)$ is also simply connected. Then $f|_U : U \to f(U)$ is a locally confluent map. By [LR74], $f|_{U \setminus \hat{X}}$ is confluent. The result now follows from Theorem 2.1 applied to f restricted to U. □

3. Fixed points for positively oriented maps

In this section we will consider a positively oriented map of the plane. As we shall see below, a straight forward application of the tools developed in [BMOT02b] will give us the desired fixed point result. The more difficult case of negatively oriented maps will be considered in a subsequent paper [BMOT02a].

In this section we will assume by way of contradiction that $f : \mathbb{C} \to \mathbb{C}$ is a positively oriented map, X is a non-separating plane continuum such that $f(X) \subset X$ and X contains no fixed points of f.

Let S be a simple closed curve in $\mathbb{C}$ and suppose $g : S \to \mathbb{C}$ has no fixed points on S. Since g has no fixed points on S, the point $z - g(z)$ is never 0. Hence the unit vector $v(z) = \frac{z-g(z)}{|z-g(z)|}$ always exists. Let $z(t)$ be a convenient counterclockwise parameterization of S by $t \in S^1 = \mathbb{R} \mod \mathbb{Z}$ and define the map $\overline{v} = v \circ z : S^1 \to S^1$ by

$$\overline{v}(t) = v(z(t)) = \frac{z(t) - g(z(t))}{|z(t) - g(z(t))|}.$$

Then Ind(g,S) the index of g on S is the *degree* of $\overline{v}$. It is well known that if Ind(g,S) $\neq 0$ and $f : T(S) \to \mathbb{C}$ is a continuous extension of g, then f has a fixed point in $T(S)$. In order to compute the index of f on a simple closed curve S approximating X we will introduce Bell's notion of variation.

The *standard junction* J_0 is the union of the three rays $R_i = \{z \in \mathbb{C} \mid z = re^{i\pi/2}, r \in [0, \infty)\}$, $R_+ = \{z \in \mathbb{C} \mid z = re^0, r \in [0, \infty)\}$, $R_- = \{z \in \mathbb{C} \mid z = re^{i\pi}, r \in [0, \infty)\}$, having the origin 0 in common. By U we denote the lower half-plane $\{z \in \mathbb{C} \mid z = x + iy, y < 0\}$. A *junction* J_v is the image of J_0 under any orientation-preserving homeomorphism $h : \mathbb{C} \to \mathbb{C}$ where $v = h(0)$. We will often suppress h and refer to $h(R_i)$ as R_i, and similarly for the remaining rays and the region $h(U)$.

Suppose S is a simple closed curve and $A \subset S$ is a subarc of S with endpoints a and b, with $a < b$ in the counter-clockwise orientation on S. We will usually denote such a subarc by $A = [a, b]$.

Definition 3.1 (Variation on an arc)**.** Let S be a simple closed curve such that $X \subset T(S)$ and $A = [a, b]$ a subarc of S such that $X \cap A = \{a, b\}$, $f(a), f(b) \in T(S)$ and $f(A) \cap A = \emptyset$. We define the *variation of f on A with respect to S*, denoted Var(f,A), by the following algorithm:

(1) Choose an orientation preserving homeomorphism h of $\mathbb{C}$ such that $h(0) = v \in A$ and $T(S) \subset h(U) \cup \{v\}$.
(2) Choose a convenient parameterization of $[a, b]$ with $a < b$.
(3) *Crossings:* Consider the set $K = [a, b] \cap f^{-1}(J_v)$. Each time a point of $[a, b] \cap f^{-1}(\mathbb{R}^+)$ is followed immediately by a point of $[a, b] \cap f^{-1}(\mathbb{R}^i)$ in K, count +1. Each time a point of $[a, b] \cap f^{-1}(\mathbb{R}^i)$ is followed immediately by a point of $[a, b] \cap f^{-1}(\mathbb{R}^+)$ in K, count -1. Count no other crossings.
(4) The sum of the crossings found above is the variation, denoted Var(f,A). It is shown in [BMOT02b] that the variation of a cross cut is well-defined. Moreover, it follows from that paper (see Theorem 2.12 and Remark 2.19) that if S is simple closed curve such that $X \subset T(S)$ and each component Q_i of $S \setminus X$ is sufficiently small,

$$\text{Ind(f,S)} = \sum_{Q_i} \text{Var}(\text{f}, Q_i) + 1.$$

Lemma 3.2. *Let $f : \mathbb{C} \to \mathbb{C}$ be a map and X a nonseparating continuum such that $f(X) \subset X$. Suppose $C = (a, b)$ is a cross cut of the continuum X. Let $v \in (a, b)$ be a point and J_v be a junction such that $J_v \cap (X \cup C) = \{v\}$. Then there exists an arc I such that $S = I \cup C$ is a simple closed curve, $X \subset T(S)$ and $f(S) \cap J_v = f(C) \cap J_v$.*

Proof. Since $f(X) \subset X$ and $J_v \cap X = 0$, it is clear that there exists an arc I with endpoints a and b near X such that $I \cup C$ is a simple closed curve, $X \subset T(I \cup C)$ and $f(I) \cap J_v = \emptyset$. This completes the proof. □

Corollary 3.3. *Suppose X is an invariant continuum for a positively oriented map $f : \mathbb{C} \to \mathbb{C}$. Then for each cross cut C such that $f(C) \cap C = \emptyset$, Var(f,C) ≥ 0*

Proof. Suppose that $C = (a, b)$ is a cross cut of X such that $f(C) \cap C = \emptyset$ and $\mathrm{Var}(f,C) \neq 0$. Choose a junction J_v such that $J_v \cap (X \cup C) = \{v\}$ and $v \in C \setminus X$. By Lemma 3.2, there exists an arc I such that $S = I \cup C$ is a simple closed curve and $f(S) \cap J_v = f(C) \cap J_v$. Moreover, by choosing I sufficiently close to X, we may assume that $v \in \mathbb{C} \setminus f(S)$. Hence $\mathrm{Var}(f,C) = \mathrm{Win}(f, S, v) \geq 0$ [BMOT02b]. □

Theorem 3.4. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a positively oriented map and X is a non-separating continuum such that $f(X) \subset X$. Then there exists a point $x_0 \in X$ such that $f(x_0) = x_0$.*

Proof. Suppose we are given a non-separating continuum X and $f : \mathbb{C} \to \mathbb{C}$ a positively oriented map such that $f(X) \subset X$. Assume that $f|_X$ is fixed point free. Choose a simple closed curve S such that $X \subset T(S)$ and points $a_0 < a_1 < \ldots < a_n$ in $S \cap X$ such that for each i $C_i = (a_i, a_{i+1})$ is a sufficiently small cross cut of X, $f(C_i) \cap C_i = \emptyset$ and $f|_{T(S)}$ is fixed point free. By Corollary 3.3, Var(f,C_i) ≥ 0 for each i. Hence, Ind(f,S) $= \sum$ Var(f,C_i)$+1 \geq 1$ (see [BMOT02b]). This contradiction completes the proof. □

Corollary 3.5. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a perfect and onto confluent map, and X is a non-separating continuum such that $f(X) \subset X$. Then there exists a point $x_0 \in X$ of period 2.*

Proof. By Theorem 1.6, f is either positively or negatively oriented. In either case, the second iterate f^2 is positively oriented and must have a fixed point in X by Theorem 3.4. □

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