THE JULIA SETS OF BASIC UNICREMER POLYNOMIALS OF ARBITRARY DEGREE

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Abstract. Call a polynomial P a *basic uniCremer polynomial* if no two periodic rays land at one point and there exists a Cremer periodic point. Using mostly topological tools we show that there are only the following two types of basic uniCremer Julia sets. The *red dwarf* Julia sets J are nowhere connected im kleinen and such that the intersection of all impressions of external angles is a continuum in J containing the Cremer point and the orbits of all critical images. The *solar* Julia sets J are such that every angle with dense orbit has a degenerate impression disjoint from other impressions and J is connected im kleinen at the landing point of its ray. We also show that any bi-accessible point is either precritical or pre-Cremer. The quadratic case had been considered before using different tools.

1. Introduction

Polynomial dynamics studies trajectories of points under a polynomial map $P : \mathbb{C} \to \mathbb{C}$ of the complex plane $\mathbb{C}$ into itself. The most interesting dynamics takes place on the Julia set J of P which can be defined as the closure of the set of all repelling periodic points of P. The set J can be either connected or disconnected, and in this paper we concentrate upon the case when J is connected.

Let $\widehat{\mathbb{C}}$ be the complex sphere, $P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a degree d polynomial with a connected Julia set J_P. Denote by K_P the corresponding filled-in Julia set. Let $\theta = z^d : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ ($\mathbb{D} \subset \mathbb{C}$ is the open unit disk). There exists a conformal isomorphism $\Psi : \mathbb{D} \to \widehat{\mathbb{C}} \setminus K_P$ with $\Psi \circ \theta = P \circ \Psi$ [DH85]. The Ψ-images of radii of $\mathbb{D}$ are called *external rays* (to the Julia set J) and are denoted R_α where α is the corresponding angle. If the Julia set is locally connected, the topology and dynamics of J_P

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are well described. Indeed, if J_P is locally connected, then Ψ extends to a continuous function $\overline{\Psi} : \overline{\mathbb{D}} \to \overline{\widehat{\mathbb{C}} \setminus K_P}$ and $\overline{\Psi} \circ \theta = P \circ \overline{\Psi}$. Let $\mathbb{S}^1 = \mathrm{Bd}(\mathbb{D})$, $\sigma_d = \theta|_{\mathbb{S}^1}$, $\psi = \overline{\Psi}|_{\mathbb{S}^1}$. Define an equivalence relation $\sim_P$ on $\mathbb{S}^1$ by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$. The equivalence $\sim_P$ is called the *(d-invariant) lamination (generated by P)*. The quotient space $\mathbb{S}^1/\sim_P = J_{\sim_P}$ is homeomorphic to J_P and the map $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ induced by σ_d is topologically conjugate to P. The set $J_{\sim_P}$ (with the map $f_{\sim_P}$) is a topological (combinatorial) model of $P|_{J_P}$ and is often called the *topological (combinatorial) Julia set*.

Let us call irrational neutral periodic points *CS-points*. In his fundamental paper [K04] Kiwi extended the above construction to *all* polynomials P with connected Julia set and no CS-points. For such polynomials he obtained a d-invariant lamination $\sim_P$ on $\mathbb{S}^1$. Then $J_{\sim_P} = \mathbb{S}^1/\sim_P$ is a locally connected continuum and $P|_{J_P}$ is semi-conjugate to the induced map $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ by a monotone map $m : J_P \to J_{\sim_P}$ (by *monotone* we mean a map whose point preimages are connected). The lamination $\sim_P$ generated by P provides a combinatorial description of the dynamics of $P|_{J_P}$. In addition Kiwi proved that at all periodic points p of P in J_P the set J_P is locally connected at p and $m^{-1} \circ m(p) = \{p\}$.

Consider a quadratic polynomial P with a *Cremer fixed point* (i.e. with a neutral non-linearizable fixed point $p \in J$ such that $P'(p) = e^{2\pi i\alpha}$ with α irrational); then P is said to be a *basic Cremer polynomial*, and its Julia set is called a *basic Cremer Julia set* [BO06a]. By a result of Schleicher and Zakeri [SZ99, Theorem 3] (see also [Zak00, Theorem 3]) *if* a basic Cremer Julia set contains a biaccessible point then this point eventually maps to a Cremer point; hence a basic Cremer polynomial has no repelling periodic points at which more than one ray lands. The results of [K04] do not apply to P (in fact, in [BO06b] we show that any monotone map of a basic Julia set J onto a locally connected continuum must collapse J to a point). Thus to study the dynamics of such polynomials one needs to develop different tools.

To an extent this is done in [BO06a] the dynamics of basic Cremer polynomials is studied with the help of continuum theory techniques. Observe that in the case of polynomials *without* CS-points the best description of the dynamics (when the map on the Julia set is conjugate to the map induced by z^d on the quotient space of $\mathbb{S}^1$ under the appropriate lamination) is possible exactly when the Julia set has nice topological properties (is locally connected). It turns out that in the case of basic Cremer polynomials very similar facts take place: a basic

Cremer Julia set J has nicer dynamics if and only if there are points at which J is *connected im kleinen* (see definitions below).

Let us have an overview of known results on the dynamics of basic Cremer polynomials and the topology of their Julia sets. By Sullivan [Sul83], a basic Cremer Julia set J is not locally connected. Since rays land at some points of J (e.g., repelling periodic points [DH85]), it makes sense to study the pattern in which this can occur. In this respect the following important question is due to C. McMullen [McM94]: can a basic Cremer Julia set contain points at which at least two rays land (so-called *biaccessible points*)? This question was partially answered by Schleicher and Zakeri in the cited papers [SZ99] and [Zak00], however it is still unknown if there exist basic Cremer Julia sets with biaccessible points. Another related paper is that of Sørensen [Sor98] where the author constructs Cremer polynomials with rays which accumulate on both the Cremer point and its preimage and thus gives examples of Cremer polynomials whose Julia sets have very interesting topological properties.

Fix a basic Cremer polynomial P with the Julia set J. Then J is connected and the Cremer point p belongs to $\omega(c)$ ([Mn93] and [Per97], see also [C05, Theorem 1.3]). If an angle α is such that its impression $\text{Imp}(\alpha)$ is disjoint from all other impressions then we call α a *K-separate* angle. A continuum X is *connected im kleinen at a point x* provided for each open set U containing x there exists a connected set $C \subset U$ with x in the interior of C. The main result of [BO06a] is that there are two types of basic Cremer Julia sets. The *red dwarf* Julia sets J are nowhere connected im kleinen and such that the intersection of the impressions of all external angles is a continuum in J containing the Cremer point and the orbits of all critical images. The *solar* Julia sets J are such that the set of K-separate angles with degenerate impressions contains all angles with dense orbits and a dense set of periodic angles, and the Julia set J is connected im kleinen at the landing points of their rays.

The aim of this paper is to extend results of [BO06a] to the higher degree case. To this end we need to define the class of polynomials of degrees greater than 2 analogous to basic Cremer polynomials. By [DH85] at every repelling periodic point at least one ray lands, and all such rays are rational. The existence of repelling periodic points at which two or more rays land plays an important role, e.g., in [K04]. On the other hand, by [SZ99], [Zak00] no repelling periodic point of a basic Cremer polynomial is biaccessible. The latter property is the defining property for the class of polynomials we want to study: a polynomial P is said to be a *basic uniCremer polynomial* if it has a Cremer periodic point and no repelling periodic point of P is biaccessible (by [K00] and [GM93] then the Cremer point must be fixed). Then the analog of the results of [BO06a] holds and following theorem can be proven.

Theorem 4.10. *For a basic uniCremer polynomial P the following facts are equivalent:*

(1) *there is an impression not containing the Cremer point;*
(2) *there is a degenerate impression;*
(3) *the set Y of all K-separate angles with degenerate impressions contains all angles with dense orbits and a dense set of periodic angles, and the Julia set J is connected im kleinen at landing points of the corresponding rays;*
(4) *there is a point at which the Julia set is connected im kleinen.*

Basic uniCremer Julia set with the properties from Theorem 4.10 are said to be *solar*. The remaining basic uniCremer Julia sets are called *red dwarf* Julia sets. They can be defined as basic uniCremer Julia sets such that all impressions contain p. The following lemma describes red dwarf Julia sets and complements Theorem 4.10.

Lemma 4.3. *If J is a red dwarf Julia set then the (non-empty) intersection of all impressions contains all forward images of all critical points, there exists $\varepsilon > 0$ such that the diameter of any impression is greater than ε, and there are no points at which J is connected im kleinen. Moreover, in this case no point of J is biaccessible and p is not accessible from $\mathbb{C} \setminus J$.*

2. Fixed points and impressions

In this section we will assume that $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic map and X is a non-separating plane continuum such that $f(X) \subset X$. Given a planar continuum Z denote by $T(Z)$, the *topological hull of Z*, the union of Z and all of its bounded complementary domains. Denote by $B(x, r)$ the open ball of radius r centered at $x \in \mathbb{C}$ and by $S(x, r)$ is boundary.

2.1. Fixed points.

Let S be a simple closed curve in $\mathbb{C}$ and suppose $f : S \to \mathbb{C}$ has no fixed points on S. Since f has no fixed points on S, the point $f(z) - z$ is never 0. Hence the unit vector $v(z) = \frac{f(z)-z}{|f(z)-z|}$ always exists. Let $z(t)$ be a convenient counterclockwise parametrization of S by $t \in \mathbb{S}^1 = \mathbb{R} \mod \mathbb{Z}$ and define the map $\overline{v} = v \circ z : \mathbb{S}^1 \to \mathbb{S}^1$ by

$$\overline{v}(t) = v(z(t)) = \frac{f(z(t)) - z(t)}{|f(z(t)) - z(t)|}.$$

Then Ind(f,S), *the index of f on S*, is the *degree* of $\overline{v}$. By the Argument Principle (see, e.g., [CB84]) applied to $f(z) - z$ it follows that if Ind(f,S) $= n$ then f has exactly n fixed points in $T(S)$ (counted with multiplicity). In order to compute the index of f on a simple closed curve S approximating X we will introduce Bell's notion of variation (see [A99] and [MOT02] for a more complete description of Bell's results).

The *standard junction* J_0 is the union of the three rays $R_i = \{z \in \mathbb{C} \mid z = re^{i\pi/2}, r \in [0,\infty)\}$, $R_+ = \{z \in \mathbb{C} \mid z = re^0, r \in [0,\infty)\}$, $R_- = \{z \in \mathbb{C} \mid z = re^{i\pi}, r \in [0,\infty)\}$, having the origin 0 in common. By U we denote the lower half-plane $\{z \in \mathbb{C} \mid z = x + iy, y < 0\}$. A *junction* J_v is the image of J_0 under any orientation-preserving homeomorphism $h : \mathbb{C} \to \mathbb{C}$ where $v = h(0)$. We will often suppress h and refer to $h(R_i)$ as R_i, and similarly for the remaining rays and the region $h(U)$.

Suppose S is a simple closed curve and $A \subset S$ is a subarc of S with endpoints a and b, with $a < b$ in the counter-clockwise orientation on S. We will usually denote such a subarc by $A = [a, b]$.

Definition 2.1 (Variation on an arc)**.** Let S be a simple closed curve such that $X \subset T(S)$ and $A = [a, b]$ a subarc of S such that $X \cap A = \{a, b\}$, $f(a), f(b) \in T(S)$ and $f(A) \cap A = \emptyset$. We define the *variation of f on A with respect to S*, denoted Var(f,A), by the following algorithm:

(1) Choose an orientation preserving homeomorphism h of $\mathbb{C}$ such that $h(0) = v \in A$ and $T(S) \subset h(U) \cup \{v\}$.
(2) Choose a convenient parametrization of $[a, b]$ with $a < b$.
(3) *Crossings:* Consider the set $K = [a, b] \cap f^{-1}(J_v)$. Each time a point of $[a, b] \cap f^{-1}(R_+)$ is followed immediately by a point of $[a, b] \cap f^{-1}(R_i)$ in K, count $+1$. Each time a point of $[a, b] \cap f^{-1}(R_i)$ is followed immediately by a point of $[a, b] \cap f^{-1}(R_+)$ in K, count -1. Count no other crossings.
(4) The sum of the crossings found above is the variation, denoted Var(f,A).

The following Theorem is due to Harold Bell. It first appeared in print in [A99] (see also [MOT02] for more details).

Theorem 2.2 (Harold Bell)**.** *Suppose that $g : \mathbb{C} \to \mathbb{C}$ is a map, X is a non-separating continuum and $g(X) \subset X$. Let S be a simple closed curve such that $X \subset T(S)$, $X \cap S$ is finite and if $\{S_i\}$ are the closures of all components of $S \setminus X$ then for each i, $g(S_i) \cap S_i = \emptyset$. Then*

$$Ind(g,S) = \sum_i Var(g,S_i) + 1.$$

In particular, if g is holomorphic then the number of fixed points of g in $T(S)$ counted with multiplicity is $\sum_i$ Var(g,S_i)$+1$.

Let X be a non-separating plane continuum. Then a *crosscut C of X* is a closed arc whose intersection with X consists of the endpoints of C. The *shadow of C*, denoted by $Sh(C)$, is the bounded component of $\mathbb{C} \setminus [X \cup C]$.

Lemma 2.3. *Suppose C is a crosscut of the continuum X. Let $v \in C$ be a point which is not an endpoint of C, and J_v be a junction such that $J_v \cap (X \cup C) = \{v\}$. Then there exists an arc I such that $S = I \cup C$ is a simple closed curve, $X \subset T(S)$ and $f(I) \cap J_v = \emptyset$.*

Proof. Left to the reader. □

Definition 2.4 (Winding number)**.** Let $f : U \to \mathbb{C}$ be a holomorphic map from a simply connected domain U into the plane, S be a simple closed curve in U, and $v \in \mathbb{C} \setminus f(S)$ be a point. Define $f_v : S \to \mathbb{S}^1$ by

$$f_v(x) = \frac{f(x) - v}{|f(x) - v|}.$$

Then the *winding number* of $f|_S$ about v, denoted $\mathrm{Win}(f, S, v)$, is defined as the *degree* of f_v. Since f is holomorphic, $\mathrm{Win}(f, S, v) \geq 0$.

Corollary 2.5. *For each crosscut C of X with $f(C) \cap C = \emptyset$ we have Var(f,C) ≥ 0*

Proof. Suppose that C is a crosscut of X such that $f(C) \cap C = \emptyset$ and Var(f,C) $\neq 0$. Choose a junction J_v such that $J_v \cap (X \cup C) = \{v\}$ and $v \in C \setminus X$. By Lemma 2.3, there exists an arc I such that $S = I \cup C$ is a simple closed curve, $X \subset T(S)$ and $f(I) \cap J_v = \emptyset$. Hence Var($f$,$\overline{C}$) $=$ $\mathrm{Win}(f, S, v) \geq 0$ [MOT02]. □

Suppose that R_θ is an external ray of the non-separating continuum X. Choose an order $<$ on R_θ such that points close to infinity are large, and for $x \in R_\theta$ let $(-\infty, x) = \{z \in R_\theta \mid z < x\}$. Then we say that R_θ *crosses a crosscut C of X essentially* if there exists $x \in R$ such that C separates $(-\infty, x)$ from infinity in $C \setminus X$.

Lemma 2.6. *Suppose that $x \in X$ is a repelling fixed point and R_θ is a fixed external ray of X landing on x. Then there exist arbitrary small simple closed curves S with $x \in T(S) \setminus S$, $S \subset$ Int($T(f(S))$ and there exists a component C of $S \setminus X$, which crosses R_θ essentially, such that Var(f,C) ≥ 1.*

Proof. Choose a neighborhood U of x and $\delta > 0$ so that $f|_U$ is an orientation preserving homeomorphism and if $r < \delta$ and $S = S(0, r)$, then $S' \subset \text{Int}(T(f(S))$ (in particular, $S \cap f(S) = \emptyset$). By Lemma 2.2 of [BO06a], there exists a component C of $S \setminus X$ such that $\overline{C}$ crosses R_θ essentially. Let z, b be the endpoints of C. Then R_θ separates a, b in $U \setminus X$. Since $f|_U$ is a homeomorphism, $f(R_\theta) = R_\theta$ separates $f(a)$ from $f(b)$ in $f(U) \setminus X$. Hence $\text{Var}(f,C) \neq 0$. By Corollary 2.5 $\text{Var}(f,C) \geq 1$. □

Theorem 2.7. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic map, X is a non-separating continuum or a point such that $f(X) \subset X$, X contains only repelling fixed points and for each fixed point $x_i \in X$ there exists an external ray R_i of X, landing at x_i, such that $f(R_i) = R_i$. Then X is a single point.*

Proof. By Theorem 2.2 (see [A99]), f has $n \geq 1$ fixed point in X. Let $x_1, \dots, x_n$ be all fixed points of f in X (since they all are repelling the multiplicity at each fixed point is 1). Choose a closed simply connected neighborhood V of X so that the only fixed points of f in V are $x_1, \dots, x_n$. By Lemma 2.6 we can find for each i arbitrary small simple closed curves S_i around x_i so that $f(S_i) \subset V$ and S_i is contained in the interior of $T(f(S_i))$. In particular $S_i \cap f(S_i) = \emptyset$ and $T(X \cup f(S_i)) \subset V$. By Lemma 2.6 there exists a component C_i of $S_i \setminus X$, which crosses R_i essentially, such that $\text{Var}(f,C_i) \geq 1$. By choosing S_i sufficiently small we may assume that $Sh(C_i) \cap Sh(C_j) = \emptyset = T(f(S_i)) \cap T(f(S_j))$ for all $i \neq j$. Choose for each i a junction J_i with $v_i \in C_i \setminus X$ such that $J_i \cap J_j = \emptyset$ for all $i \neq j$. Then there exists $\delta > 0$ such that for each $x \in X \setminus \cup T(S_i)$, $d(x, f(x)) > \delta$, $d(X, \cup J_i) > \delta$, $d(T(f(S_i), T(f(S_j)) > \delta$, for all $i \neq j$. Moreover, $d(S_i, f(S_i)) > \delta$ for all i.

Choose $0 < \eta < \delta/5$ such that for every crosscut D of X of diameter less than η:

(1) $D \cup f(D) \subset V$,
(2) $[D \cup f(D)] \cap \bigcup J_i = \emptyset$,
(3) if $D \cap [\cup_i S_i] \neq \emptyset$, then $d(D, f(D)) > \delta/3$,
(4) diam $(f(D)) < \delta/3$.

Choose a simple closed curve Z such that $X \subset T(Z)$, $Z \cap X$ is finite, each component Z_j of $Z \setminus X$ is a crosscut of diameter less than η, for each C_i we have $\{Z \cap C_i\} = \{l_i, r_i\}$, $\text{Sh}(C_i) \cap Z \neq \emptyset$ and the diameters of the two components C_i^l, C_i^r of $C_i \setminus \{l_i, r_i\}$ non-disjoint from X are less than η. Replace the subarc of Z contained in $\text{Sh}(C_i)$ by the corresponding subarc of C_i with the same endpoints l_i, r_i. If we do this for all the C_i's we get a new simple closed curve Z' such that $Z' \cap X$ is finite and it is easy to check that, by the choice of constants, $H \cap f(H) = \emptyset$ for the closures H of every component of $Z' \setminus X$. Let H_i be the closure of the component of $Z'\setminus X$ which contains v_i. Observe that by (2) and by the choice of η we have $f(Z\bigcup_j(C_j^l\cup C_j^r))\cap(\bigcup_i J_i)=\emptyset$. Hence $\mathrm{Var}(f,H_i)=\mathrm{Var}(f,C_i)\geq 1$. Let G_k denote all components of $Z'\setminus X$. Then by Theorem 2.2 and Corollary 2.5,

$$\mathrm{Ind}(f,Z')=\sum_k \mathrm{Var}(f,G_k)+1\geq\sum_{i=1}^n \mathrm{Var}(f,C_i)+1\geq n+1.$$

Then f has at least $n+1$ fixed points in $T(Z')$. Since $T(Z')\subset V$, this is a contradiction. □

2.2. **Impressions and connectedness im kleinen.** Now we prove a few technical lemmas which can be of independent interest. Mostly they deal with the topological properties of impressions. In this subsection X is a non-separating one-dimensional plane continuum. The assumption that X is non-separating is not essential but simplifies the arguments (e.g., in this case each subcontinuum of X is also a non-separating, one-dimensional plane continuum and the intersection of any two subcontinua of X is connected). Speaking of *points* we mean points in the (dynamic) plane while *angles* mean arguments of external rays. Given an external ray R_α and a family C_n of crosscuts C_n who cross R_α essentially and are such that $\mathrm{diam}(C_n)\to 0$ we define the impression $\mathrm{Imp}(\alpha)$ by $\mathrm{Imp}(\alpha)=\cap_n\overline{Sh(C_n)}$. This is equivalent to the standard definition; also, $\mathrm{Imp}(\alpha)$ is independent of the choice of the sequence of crosscuts [Pom92]. A continuum K is said to be *decomposable* if there exist two continua $A\subsetneqq K, B\subsetneqq K$ such that $A\cup B=K$ and *indecomposable* otherwise. Theorem 2.8 holds for *all* polynomials.

Theorem 2.8. [CMR05, Theorem 1.1] *The Julia set of a polynomial P is indecomposable if and only if there exists an angle γ whose impression has non-empty interior in $J(P)$; in this case the impressions of all angles coincide with $J(P)$.*

From the topological standpoint, if the Julia set is indecomposable then one cannot use impressions to further study its structure: representing $J(P)$ as the union of smaller more primitive continua is impossible in this case. Besides, if $J(P)$ is indecomposable then the results of the paper are immediate. Thus, in the lemmas below we assume that no impression has interior in X.

Definition 2.9. A continuum (or a point) $K\subset J$ is said to be a *ray continuum* if there exists a non-empty set of angles $\mathcal{A}\subset\mathbb{S}^1$ such that for each $\alpha\in\mathcal{A}$ the *principal set* $\overline{R_\alpha}\setminus R_\alpha\subset K$; we say that the set of angles $\mathcal{A}$ (and their rays) is *connected to* K and denote it by $\mathcal{A}\vee K$.

The union of K and a finite set of rays connected to K is a closed connected set. Examples of ray continua are continua which are unions of impressions (or principal sets). By $\mathrm{Ch}(A)$ we denote the convex hull of a planar set A. Lemma 2.10 studies how ray continua intersect. If $A_1, A_2 \subset \mathbb{S}^1$ are such that $\mathrm{Ch}(A_1) \cap \mathrm{Ch}(A_2) = \emptyset$ then they are said to be *unlinked*.

Lemma 2.10. *Suppose that K_1, K_2 are disjoint ray continua connected to finite sets of angles A_1, A_2 respectively. Then A_1 and A_2 are unlinked.*

Proof. Without loss of generality $A_1 \cap A_2 = \emptyset$. If A_1 and A_2 are not unlinked, then there exists $\alpha_1, \alpha_2 \in A_1$ such that A_2 separates α_1 and α_2. This clearly implies that $K_1 \cap K_2 \neq \emptyset$, a contradiction. □

We need a few topological notions. A continuum X is *connected im kleinen at x* if for each open set U containing x there exists a connected set $C \subset U$ with x in the interior of C. A continuum X is *locally connected* at $x \in X$ provided for each neighborhood U of x there exists a connected and open set V such that $x \in V \subset U$. Observe that sometimes different terminology is used (see the discussion in [BO06a]).

Lemma 2.11 contains a sufficient condition for a continuum X to be connected im kleinen at some point x. The idea is to establish "short connections" among impressions which cut x off the rest of X and apply it to prove that X is connected im kleinen at some points. If $\mathrm{Imp}(\theta)$ is disjoint from all other impressions then we call θ a *K-separate* angle.

Lemma 2.11. *Suppose that θ is a K-separate angle and $Imp(\theta) = \{x\}$ is a singleton. Then arbitrarily close to θ there are angles $s < \theta < t$ such that $Imp(s) \cap Imp(t) \neq \emptyset$. Also, J is connected im kleinen at x.*

Proof. First of all, let us show that there must exist non-disjoint impressions. Indeed, suppose that all impressions are pairwise disjoint. Then by the Moore theorem ([M25]) the map ϕ which collapses all impressions to points and leaves the rest of the plane untouched maps the plane $\mathbb{C}$ onto a plane $\mathbb{C}$ while mapping X onto a circle. This means that X is not a non-separating continuum, a contradiction.

Now, suppose that the first claim of the lemma fails. Consider all angles s, t such that $\mathrm{Imp}(s) \cap \mathrm{Imp}(t) \neq \emptyset$. Then there are angles $l_1 \leq l_2 < \theta < r_1 \leq r_2$ such that the following holds:

(1) $\mathrm{Imp}(l_1) \cap \mathrm{Imp}(r_1) \neq \emptyset$;
(2) $\mathrm{Imp}(l_2) \cap \mathrm{Imp}(r_2) \neq \emptyset$;
(3) if $l \in (l_2, \theta)$ and $r \in (\theta, r_2)$ then $\mathrm{Imp}(l) \cap \mathrm{Imp}(r) = \emptyset$;
(4) if $l \in (l_1, \theta)$ and $r \in (\theta, r_1)$ then $\mathrm{Imp}(l) \cap \mathrm{Imp}(r) = \emptyset$.

Set $N = \mathrm{Imp}(l_1) \cup \mathrm{Imp}(l_2) \cup \mathrm{Imp}(r_1) \cup \mathrm{Imp}(r_2)$. By Lemma 2.10 N is a continuum. Set $A = N \cup (\cup_{z\in[l_2,\theta]} \mathrm{Imp}(z)), B = N \cup (\cup_{z\in[\theta,r_1]} \mathrm{Imp}(z))$. Then A and B are subcontinua of J, while $A \cap B = N \cup \{x\}$ is not connected because $x \notin N$ by the assumptions, a contradiction.

Suppose now that U is an open set in J containing x. Since impressions are upper semicontinuous, there exist $s < \theta < t$ such that $x \notin \mathrm{Imp}(s) \cup \mathrm{Imp}(t), \mathrm{Imp}(s) \cap \mathrm{Imp}(t) \neq \emptyset$ and for all $\gamma \in [s,t]$, $\mathrm{Imp}(\gamma) \subset U$. Let $C = \cup_{a\in[s,t]} \mathrm{Imp}(a)$. Then $C \subset U$ is connected. We claim that x is in the interior of C.

Indeed, set $E = R_s \cup \mathrm{Imp}(s) \cup R_t \cup \mathrm{Imp}(t)$. Then $\mathbb{C} \setminus E$ consists of two components. Denote the one containing x by W and the other one V. Let $d(x, E \cup \mathrm{Bd}(U)) = \varepsilon$. Consider the $\varepsilon/2$-disk D centered at x. Then $D \subset W$ is disjoint from $E \cup V$, and $D \cap J \subset C$ since points of $D \cap J$ cannot belong to impressions of angles not from (s,t). Thus, x belongs to the interior of C as desired. □

On the other hand, under some conditions X is nowhere connected im kleinen, or connected im kleinen at very few points.

Lemma 2.12. *The following holds.*

1. *If X is connected im kleinen at x then for any ε there exists θ such that Imp$(\theta) \subset B(x,\varepsilon)$ (in particular, there are angles with impressions of arbitrarily small diameter).*
2. *Suppose that there exists $\delta > 0$ such that for each $\theta \in \mathbb{S}^1$, diam(Imp$(\theta)) > \delta$. Then X is nowhere connected im kleinen.*
3. *Suppose that the intersection Z of all impressions is not empty. Then the only case when X is connected im kleinen at a point is (possibly) when $Z = \{z\}$ is a singleton and X is connected im kleinen at z.*

Proof. (1) Choose $\varepsilon > 0$ and a continuum K containing x such that $\mathrm{diam}(K) < \varepsilon$ and $0 < \delta < \varepsilon$ such that $B(x,\delta) \cap X \subset K$. Choose a crosscut $C \subset B(x, \delta/2)$, then $C \cap X \subset K$. Hence, $\mathrm{Sh}(C) \subset B(x,\varepsilon)$ and so there are angles whose impressions are contained in $\mathrm{Sh}(C)$ and hence in $B(x,\varepsilon)$. These angles have impressions of diameter less than 2ε.

(2) Immediately follows from (1).

(3) Suppose that X is connected im kleinen at z. Then by (1) there is a sequence of impressions converging to $\{z\}$ which implies that $Z = \{z\}$. The example of a Cantor bouquet X with the vertex z shows that X can even be locally connected at z. □

3. Wandering continua for uniCremer polynomials

In Section 3 we use Thurston's invariant geometric laminations defined in [Thu85]. A *geometric lamination* is a compact set $\mathcal{L}$ of chords in $\overline{\mathbb{D}}$ and points in $\mathbb{S}^1$ such that any two distinct chords can meet, at most, in an end-point (i.e., the intersection of any two distinct chords is either empty or a point in $\mathbb{S}^1$). We refer to a non-degenerate chord $\ell \in \mathcal{L}$ as a *leaf* and to a point in $\mathcal{L}$ as a *degenerate leaf* (by "leaves" we mean non-degenerate leaves, and by "(degenerate) leaves" we mean both types of leaves). A degenerate leaf may be an endpoint of a leaf. If $\ell \cap \mathbb{S}^1 = \{a, b\}$ for a leaf ℓ, we write $\ell = ab$. We denote by $\mathcal{L}^*$ the union of all leaves in $\mathcal{L}$. Then $\mathcal{L}^* \cup \mathbb{S}^1$ is a continuum. We can extend $\sigma_d : \mathbb{S}^1 \to \mathbb{S}^1$ over $\mathcal{L}^*$ by mapping $\ell = ab$ linearly onto the chord $\sigma_d(a)\sigma_d(b)$ and denoting this chord by $\sigma_d(\ell)$. A *gap* G is the closure of a complementary domain of $\mathbb{D} \setminus \mathcal{L}^*$. A geometric lamination $\mathcal{L}$ is *d-invariant* if σ_d preserves gaps and leaves of $\mathcal{L}$ in the following sense:

1. (Leaf invariance) For each leaf $\ell \in \mathcal{L}$, $\sigma_d(\ell)$ is a (degenerate) leaf in $\mathcal{L}$ and there exist d pairwise disjoint leaves $\ell_1, \dots, \ell_d$ in $\mathcal{L}$ such that for each i, $\sigma_d(\ell_i) = \ell$.
2. (Gap invariance) For each gap G of $\mathcal{L}$, $\sigma_d(\mathrm{Bd}(G))$ is a (degenerate) leaf or the boundary of a gap G' of $\mathcal{L}$. We denote by $\sigma_d(G)$ the (degenerate) leaf or the gap G', respectively. If $\sigma_d(G) = G'$ is a gap then we also require that $\sigma_d|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(G')$ be the composition of a monotone map and a positively oriented covering map.

We show that if a leaf of a d-invariant lamination $\mathcal{L}$ maps in some "direction" then $\mathcal{L}$ contains an invariant leaf or gap located in the same "direction". From now on the following applies: 1) $d \geq 2$ is fixed and omitted from the notation; 2) we define *non-trivial* laminations as laminations containing at least one leaf and consider only non-trivial laminations; clearly, if $\mathcal{L}$ is non-trivial then the endpoints of leaves are dense in $\mathbb{S}^1$ and every point of $\mathbb{S}^1$ belongs to a (degenerate) leaf.

Theorem 3.1. *Let $\mathcal{L}$ be an invariant lamination with a leaf ℓ_0. Suppose that $\mathbb{D} \setminus \ell_0 = A \cup B$ with A and B open, disjoint and connected, and $\sigma(\ell_0) \subset A$. Then $\overline{A}$ must contain either:*

1. *a leaf ℓ such that $\sigma(\ell) \subset \ell$,*
2. *a gap G such that $\sigma(G) \subset G$.*

Proof. For leaves $\ell, \ell' \in \overline{A}$ we say that $\ell \leq \ell'$ if ℓ separates $\ell' \setminus \ell$ from B in $\mathbb{D}$ (loosely, leaves in $\overline{A}$ grow in the sense of this ordering as they move away from B). Since $\sigma(\ell_0) \subset A$, $\ell_0 \leq \sigma(\ell_0)$. Let $\mathcal{M}$ be a maximal linearly ordered collection of leaves in $\overline{A}$ containing ℓ_0 and such that for all $\ell \in \mathcal{M}$, $\ell \le \sigma(\ell)$ (by the Zorn Lemma such collections exist).

Since $\mathcal{L}$, and hence $\mathcal{M}$, has a countable basis, there exists $\ell_1 \le \ell_2 \le \dots$ in $\mathcal{M}$ such that for each $\ell \in \mathcal{M}$ one can find ℓ_i with $\ell_i \ge \ell$. Since $\mathcal{L}$ is closed, there exists $uv = \ell_\infty \in \mathcal{L}$ such that $\lim \ell_i = \ell_\infty$. Let us consider several possibilities for ℓ_∞. By continuity $\ell_\infty \le \sigma(\ell_\infty)$. Below we will need the following claim given here without proof.

Claim. *Suppose that $G \subset \overline{A}$ is a gap, $\sigma(G) \not\subset G$, and $\ell \le \ell'$ are two leaves in Bd(G). Then if $\sigma(\ell) \ge \ell'$ then $\sigma(\ell') \ge \ell'$.*

Consider now some cases. Suppose first that ℓ_∞ is a leaf. If $\sigma(\ell_\infty) \subset \ell_\infty$, then we are done. If there exists a sequence of leaves ℓ'_i separating ℓ_∞ from $\sigma(\ell_\infty)$ which converge to ℓ_∞, then $\lim \sigma(\ell'_i) = \sigma(\ell_\infty)$ and hence there exists a i_0 such that $\ell_\infty < \ell'_{i_0}$ and $\ell'_{i_0} \le \sigma(\ell'_{i_0})$, contradicting the maximality of $\mathcal{M}$. Hence there exists a gap G, which contains ℓ_∞ and is located "between" ℓ_∞ and $\sigma(\ell_\infty)$. If $\sigma(\ell_\infty)$ is not in Bd(G) then there exists a leaf $\ell' \in$ Bd(G) separating $\ell_\infty \setminus \ell'$ from $\sigma(\ell_\infty) \setminus \ell'$. If $\sigma(\ell') \subset \ell'$ then we are done. Otherwise by the Claim $\ell_\infty < \ell' \le \sigma(\ell')$ contradicting the maximality of $\mathcal{M}$. Hence ℓ_∞ and $\sigma(\ell_\infty)$ are contained in Bd(G). We may assume that $\sigma(G) \not\subset G$ and $\sigma(\ell_\infty) \not\subset \ell_\infty$. Then there must exist a leaf $\ell' \ne \ell_\infty$ in the boundary of G such that ℓ' separates $G \setminus \ell'$ from $\sigma(G) \setminus \ell'$. By the Claim $\ell_\infty < \ell' \le \sigma(\ell')$ contradicting the maximality of $\mathcal{M}$.

Hence we may assume that ℓ_∞ is a degenerate leaf and there are no invariant leaves or gaps in $\overline{A}$. Since $\ell_\infty \le \sigma(\ell_\infty)$, ℓ_∞ is a σ-fixed point. Since ℓ_∞ is repelling, it is not a limit of leaves in $\mathcal{M}$. Hence there exists a maximal leaf ℓ' of $\mathcal{M} \setminus \ell_\infty$. Let us show that ℓ_∞ is an endpoint of ℓ'. Indeed, otherwise there exists a gap G' which contains ℓ_∞, ℓ' in its boundary. Since $\sigma(G') \not\subset G'$, there exists a boundary leaf $\ell'' \in$ Bd(G') containing ℓ_∞ and separating $\sigma(G') \setminus \ell''$ from $G' \setminus \ell''$. Then $\ell' < \ell''$. Since $\sigma(\ell') \ge \ell'$, by the Claim $\sigma(\ell'') \ge \ell''$. Hence ℓ'' can be added to $\mathcal{M}$ contradicting the maximality of $\mathcal{M}$, and so ℓ_∞ is an endpoint of ℓ'. Since by the assumption ℓ' is not invariant, then $\sigma(\ell') > \ell'$. Suppose that G'' is a gap containing ℓ' and separated by ℓ' from B. Since $\sigma(G'') \not\subset G''$, there exists a boundary leaf $\ell'' \in$ Bd(G'') containing ℓ_∞ and separating $\sigma(G'') \setminus \ell''$ from $G'' \setminus \ell''$. Then $\ell' < \ell''$. Since $\sigma(\ell') \ge \ell'$, by the Claim $\sigma(\ell'') \ge \ell''$. Hence ℓ'' can be added to $\mathcal{M}$ contradicting the maximality of $\mathcal{M}$. Hence ℓ' is a limit of leaves all containing ℓ_∞ and separated by ℓ' from B. By continuity some of them can be added to $\mathcal{M}$, a contradiction completing the proof. □

A set $M \subset \mathbb{C}$ is *wandering* if all its iterates are pairwise disjoint. Ray continua are defined in Definition 2.9. One can iterate the set of angles connected to a wandering ray continuum by taking on each step the convex hull of the current iteration of this set of angles; by Lemma 2.10 these convex hulls are disjoint.

Observe that a set of angles connected to a wandering ray continuum K is not well-defined. Also, it may happen that K is contained in a larger wandering ray continuum K' such that $\mathcal{A}' \vee K', \mathcal{A} \vee K$ and $\mathcal{A}' \supset \mathcal{A}$. However the growth of the set of angles connected to wandering ray continua is limited as the following theorem shows (the result is due to Kiwi [K02], see also [BL02]).

Theorem 3.2. [K02] *A set of angles such that all its images under σ are unlinked consists of no more than 2^d angles.*

By Theorem 3.2 given a wandering ray continuum K connected to the set of angles $\mathcal{A}$ there is always a maximal set of angles $\mathcal{A}'$ connected to K of cardinality at most 2^d. Theorem 3.3 shows that wandering ray continua connected to a non-trivial set of angles give rise to laminations.

Theorem 3.3. *Suppose P is a polynomial of degree d with connected Julia set J which contains a wandering ray continuum K' connected to a set of angles $\mathcal{A}'$ of cardinality more than 1. Then there exists a d-invariant lamination $\mathcal{L}(K')$ such that $\mathcal{A}'$ is contained in a leaf or a gap of $\mathcal{L}(K')$.*

Proof. First we show that any pullback A of any forward iterate of K' is tree-like. Indeed, A is wandering since so is K'. On the other hand, if A is not tree-like then it contains the boundary of a Fatou domain, and by the Sullivan theorem [Sul85] cannot be wandering. Hence all pullbacks of images of K' are tree-like. Then by a theorem of J. Heath [Hea96] the map P on a pullback A can be not one-to-one only if A contains a critical point of P.

Choose $K = P^N(K')$ so that the following holds. Since K' is wandering then for each critical point c any image of K' may contain only one iteration of c. Choose a forward image K of K' so that for each critical point c either some forward image of c belongs to K, or the orbit of c is disjoint from the orbit of K. In particular, $K, P(K), \dots$ do not contain critical points. Let us show that then there are d^m pairwise disjoint pullbacks of $P^n(K)$ of order $m \le n$ none of which contains a critical point. Indeed, suppose that a pullback A of $P^n(K)$ of order m contains a critical point c. Since by the choice of K there exists $i > 0$ such that $P^i(c) \in K$ we then get that $P^n(K)$ contains both $P^{n+i}(c)$ and $P^m(c)$, a contradiction. By [Hea96] this implies that all powers of P restricted onto a pullback A of $P^n(K)$ of order $m \le n$ are one-to-one.

This implies that the pullbacks of images of K are in fact well-defined as sets: any two pullbacks A, B are either the same or disjoint. Indeed, suppose that A is a pullback of $P^n(K)$ of order m and B is a pullback of $P^r(K)$ of order s. Suppose that $A \cap B \neq \emptyset$ and show that then $A = B$. Choose a point $x \in A \cap B$ and consider several cases. For definiteness suppose that $m > s$. First let us show that $n - m = r - s$. Indeed, $P^m(x) \in P^n(K)$ and $P^s(x) \in P^r(K)$. Then the latter implies that $P^m(x) = P^{m-s}(P^s(x)) \in P^{m-s+r}(K)$. Hence $P^n(K)$ and $P^{m-s+r}(K)$ are not disjoint (both contain $P^m(x)$) and hence $n = m - s + r$ because K is wandering. Now, by the above there is only one pullback of $P^m(K)$ containing $P^s(x)$, namely $P^r(K)$. Hence $P^s(A) = P^r(K)$ and A is the pullback of $P^r(K)$ of order s along the orbit $x, P(x), P^s(x)$. Since the same holds for B we conclude that $A = B$.

Choose a maximal set of angles $\mathcal{A}$ connected to K and containing $\sigma^N(\mathcal{A}')$ (clearly, all angles from $\sigma^N(\mathcal{A}')$ are connected to $P^N(K)$). Denote a pullback of $P^n(K)$ by P of order m by $K(m, n, i)$ where different numbers i correspond to different pullbacks of the same order m of the same image $P^n(K)$ of K. In other words, $K(m, n, i)$ is the i-th component of the set $P^{-m}(P^n(K))$. To $K(m, n, i)$ we associate the set of angles $\Theta(m, n, i)$ which are all the angles from $\sigma^{-m}(\sigma^n(\mathcal{A}))$ connected to $K(m, n, i)$. Let us show that the sets $\Theta(m, n, i)$ have the same properties as their "generating" sets $K(m, n, i)$: two such sets of angles either coincide or are disjoint.

First let us show that if $m \leq n$ then it is impossible to have two angles $\alpha, \beta \in \Theta(m, n, i)$ such that $\sigma(\alpha) = \sigma(\beta)$. Indeed, if we apply the result of [Hea96] to the set $K(m, n, i)$ united with the rays R_α, R_β we will see that $K(m, n, i)$ must contain a critical point, a contradiction with the above. This easily implies that the cardinality of the set $\Theta(m, n, i)$ is the same as that of $\mathcal{A}$; together with the fact that all pullbacks coincide or are disjoint this implies that such sets of angles either coincide or are disjoint too. The case when $m > n$ can be proven analogously to the proof of a similar claim dealing with pullbacks and is left to the reader. If we denote K by $K(0, 0, 0)$ then $\Theta(0, 0, 0) = \mathcal{A}$. Set $G = \mathrm{Ch}(\mathcal{A})$.

If $G(m, n, i) = \mathrm{Ch}(\Theta(m, n, i))$ then by Lemma 2.10 all the sets $G(m, n, i)$ are pairwise disjoint. By Theorem 3.2 all the sets $\Theta(m, n, i)$ are finite. Hence all $G(m, n, i)$ are pairwise disjoint finite gaps, leaves and points mapped onto each other by σ and its powers. Moreover, σ restricted on the sets $G(m, n, i)$ satisfies all the properties described in the definition of the lamination because this corresponds to the action of the map P on the plane. Hence $\cup_{m,n,i} G(m, n, i)$ is a σ-invariant non compact lamination $\mathcal{L}(G)$. It follows easily that the closure of this non-compact lamination is a σ-invariant lamination $\mathcal{L}(K)$. Observe that by the construction $\mathcal{A}'$ is contained in a leaf or a gap, and hence $\mathcal{L}(K)$ is non-trivial (because the cardinality of $\mathcal{A}$ is more than 1). □

Theorem 3.4 describes properties of the lamination $\mathcal{L}(K)$. The construction from Theorem 3.3 allows one to talk about the sets from the grand orbit of $\mathcal{A}, \mathcal{A} \vee K$, under σ. In Theorem 3.4 we use the notation from the proof of Theorem 3.3.

Theorem 3.4. *The lamination $\mathcal{L}(K)$ has the following properties.*

(1) *If $\theta\theta' \in \mathcal{L}(K) \setminus \mathcal{L}(G)$ is a leaf, then $Imp(\theta) \cap Imp(\theta') \neq \emptyset$.*
(2) *For any gap H of $\mathcal{L}(K)$ and any $\theta \in H \cap \mathbb{S}^1$, let $i(\theta) = K(n,m,i) \cup Imp(\theta)$ if $\theta \in \Theta(n,m,i)$ and $i(\theta) = Imp(\theta)$ otherwise. Then*

$$Imp^+(H) = \bigcup\{i(\theta) \mid \theta \in H \cap S^1\} \text{ is connected.}$$

Proof. Note first that (2) holds in case $H = G(n,m,i)$, for some (n,m,i) since in this case $\mathrm{Imp}^+(H) = K(n,m,i) \cup \bigcup\{\mathrm{Imp}(\theta) \mid \theta \in H \cap S^1\}$ is connected by construction. We show that (1) holds. By assumption $\theta\theta'$ is a limit of gaps or leaves $G(n_j, m_j, i_j) \in \mathcal{L}(G)$. Suppose that $\mathrm{Imp}(\theta) \cap \mathrm{Imp}(\theta') = \emptyset$. Assume, by taking a subsequence if necessary, that $L = \lim(\mathrm{Imp}^+(G(n_j, m_j, i_j)))$ is a sub-continuum of J. Then L meets both $\mathrm{Imp}(\theta)$ and $\mathrm{Imp}(\theta')$. Choose a crosscut C (C') such that R_θ ($R_{\theta'}$) crosses C (C', respectively) essentially and such that $\mathrm{Sh}(C) \cap Sh(C') = \emptyset$. Choose a continuum I (I') in $\mathrm{Bd}(Sh(C)) \cap J$ ($\mathrm{Bd}(Sh(C')) \cap J$) containing the endpoints of C (C', respectively). Choose $z \in L \setminus Sh(C) \cup Sh(C')$ and $z_j \in \mathrm{Imp}^+(G(n_j, m_j, i_j))$ such that $z = \lim z_j$. For j large, $\mathrm{Imp}^+(G(n_j, m_j, i_j)) \setminus [Sh(C) \cup Sh(C')] \subset K(n_j, m_j, i_j)$. Hence there exist three disjoint continua in J joining I to I'. By Kuratowski's "θ-curve theorem" (Theorem 2 from [Kur68, vol. 2, Chapter 10, §61, II, p. 511]), one of these continua contains points which are not in the closure of the unbounded component of $\mathbb{C} \setminus J$. This contradiction shows that (1) must hold.

Suppose next that H is a gap in $\mathcal{L}(K) \setminus \mathcal{L}(G)$. All leaves $\theta\theta'$ in the boundary of H are either leaves in $\mathcal{L}(K) \setminus \mathcal{L}(G)$ or leaves in the intersection of some $G(n,m,i)$ and H. In the former case, by (1), $\mathrm{Imp}(\theta) \cup \mathrm{Imp}(\theta')$ is connected. In the latter case, the continuum $K(n,m,i)$ is a continuum in $\mathrm{Imp}^+(H)$ meeting both $\mathrm{Imp}(\theta)$ and $\mathrm{Imp}(\theta')$. Hence for all leaves $\theta\theta'$ in the boundary of H, there exists a sub-continuum of $\mathrm{Imp}^+(H)$ meeting both $\mathrm{Imp}(\theta)$ and $\mathrm{Imp}(\theta')$.

Suppose that $\mathrm{Imp}^+(H) = A \cup B$, with A and B disjoint and closed in $\mathrm{Imp}^+(H)$. Let $S_A = \{\theta \in H \mid i(\theta) \cap A \neq \emptyset\}$ and $S_B = \{\theta \in H \mid i(\theta) \cap B \neq \emptyset\}$, then S_A and S_B are disjoint. If $\theta\theta'$ is a leaf in the boundary of H, then $\theta \in S_A$ if and only if $\theta' \in S_A$. We claim that S_A and S_B are closed sets. To see this suppose that $\theta_j \in S_A$ and $\lim \theta_j = \theta_\infty$. Then $i(\theta_j) \subset A$ and hence $\mathrm{Imp}(\theta_j) \subset A$. By upper semi-continuity of impressions, $\limsup \mathrm{Imp}(\theta_j) \subset \mathrm{Imp}(\theta_\infty)$. Since A is closed, $\mathrm{Imp}(\theta_\infty) \subset A$, $i(\theta_\infty) \cap A \neq \emptyset$ and $\theta_\infty \in S_A$ as desired. If S_A^+ is the union of S_A and all leaves in the boundary of H which meet S_A, and if S_B^+ is defined similarly, then $\mathrm{Bd}(H) = S_A^+ \cup S_B^+$, where S_A^+ and S_B^+ are disjoint and closed. This contradicts the fact that $\mathrm{Bd}(H)$ is connected and (2) holds. □

In Theorem 3.5 we use the notation from the proof of Theorem 3.3.

Theorem 3.5. *Suppose that J is the Julia set of a basic uniCremer polynomial P of degree d. Suppose that $K \subset J$ is a wandering ray continuum connected to a set of angles $\mathcal{A}'$ which contains at least two angles. Then there exists an n such that $|\sigma^n(\mathcal{A}')|$ is a singleton and for every $\theta \in \mathcal{A}'$, $\cup_{n\geq 0}\sigma^n(\theta)$ is not dense in $\mathbb{S}^1$.*

In particular, K is pre-critical and if a point $x \in J$ is bi-accessible then it is either pre-critical or pre-Cremer.

Proof. By Theorem 3.3, there exists a d-invariant lamination $\mathcal{L}(K)$ such that $\mathcal{A}$ is contained in a gap or leaf of $\mathcal{L}(K)$. Let p be the fixed Cremer point of P. Denote by $A(p)$ the set of all angles whose impressions contain p. Let us show that there is a gap or a (degenerate) leaf containing $A(p)$. Suppose otherwise. Then there exists a leaf $\theta\theta' \in \mathcal{L}(K)$ and angles α, β such that $\alpha, \beta \in A(p)$, all four angles are distinct, and the chord $\alpha\beta$ crosses the leaf $\theta\theta'$. If $\theta\theta'$ is a leaf from $\mathcal{L}(K) \setminus \mathcal{L}(G)$ then it is a limit of leaves of elements of $\mathcal{L}(G)$. Hence there exists a set $G(m, n, i)$ and two angles $\gamma, \gamma' \in G(m, n, i)$ such that $R_\gamma \cup R_{\gamma'} \cup K(m, n, i)$ separates R_α from R_β. Since $p \notin K(m, n, i)$, at least one of impressions $\mathrm{Imp}(\alpha), \mathrm{Imp}(\beta)$ does not contain p, a contradiction. Suppose $\theta\theta' \in \mathcal{L}(G)$. Then we set $\gamma = \theta, \gamma' = \theta'$ and repeat the same argument. Thus, there exists a gap or a (degenerate) leaf containing $A(p)$. Let H be a gap or a (degenerate) leaf containing $A(p)$ of maximal cardinality, and study possibilities for H.

If H is a gap then there is only one gap containing $A(p)$. Indeed, suppose two gaps H, H' contain $A(p)$. Then $A(p)$ consists of one or two points. If it consists of one point then there is a leaf ℓ of H and a leaf ℓ' of H' containing $A(p)$. For geometric reasons these leaves cannot be limit leaves of $\mathcal{L}(K)$, hence they are leaves from $\mathcal{L}(G)$, a contradiction (all leaves of $\mathcal{L}(G)$ are wandering while $\sigma(A(p)) \subset A(p)$). If $A(p)$ consists of two points then these points must be the endpoints of a common leaf of H and H'. However in this case as before the leaf in question cannot be a limit leaf of $\mathcal{L}(G)$, so it has to be a leaf from

$\mathcal{L}(G)$, and since it is not wandering we get a contradiction. So, H is the unique gap containing $A(p)$.

Let us show that regardless of whether H is a gap or not, $\sigma(H) \subset H$. Let us prove it by way of contradiction. First assume that H is a gap. Since $\sigma(A(p)) \subset A(p)$ then $\sigma(H) \cap H \supset \sigma(A(p))$ and $\sigma(A(p))$ consists of one or two points none of which is wandering. Then for geometric reasons there exist non-limit leaves of $\mathcal{L}(K)$ coming out of points of $\sigma(A(p))$. As above, this is impossible since all non-limit leaves belong to $\mathcal{L}(G)$, are wandering and hence disjoint from $\sigma(A(p))$. Now, if H is a leaf then by the definition of H we see that H is not a boundary leaf of a gap. Since by the assumption $\sigma(H) \not\subset H$ then $H, \sigma(H)$ must be leaves containing $\sigma(A(p))$. If $A(p)$ consists of two points then they must be the endpoints of H, and since $\sigma(A(p)) \subset A(p)$ then $\sigma(H) \subset H$. If $A(p)$ consists of one point then $\sigma(H) \not\subset H$ implies that the leaves $H, \sigma(H)$ meet only at their common endpoint $A(p) = \sigma(A(p))$. Since $\mathcal{L}(G)$ is dense in $\mathcal{L}(K)$, there exists a gap located between H and $\sigma(H)$ with the vertex coinciding with $A(p)$, contradicting the choice of H. Finally, if H is a degenerate leaf then it must be disjoint from other leaves or gaps, and $\sigma(H) = H$.

Let us prove that the only possible dynamics in the described situation is that when H is a gap and there exists (m, n, i) such that $G(m, n, i) \cap \mathrm{Bd}(H) \neq \emptyset$. Assume that this is not true and show that then there exists a leaf $\ell \in \mathcal{L}(G)$ such that ℓ separates $\sigma^n(\ell)$ from H, and then $\sigma^n(\ell)$ separates $\sigma^{2n}(\ell)$ from ℓ (in other words, ℓ is repelled from H by two iterations of σ^n). First suppose that H is a (degenerate) leaf. Then $H \in \mathcal{L}(K) \setminus \mathcal{L}(G)$, and there exist $n > 0$ and a leaf $\ell \in \mathcal{L}(G)$ such that ℓ is repelled from H by two iterations of σ^n. Now let H be a gap, but the grand orbit of $\mathcal{A} = \Theta(0, 0, 0)$ is disjoint from $\mathrm{Bd}(H)$. Then every leaf in $\mathrm{Bd}(H)$ is a limit of leaves from $\mathcal{L}(G)$. Suppose that ℓ' is a boundary leaf of H. Since $\sigma(H) \subset H$ then ℓ' cannot be pre-critical since if $\sigma^k(\ell')$ is a point, with k minimal, then $\sigma^{k-1}(\ell') = \lim G(n_j, m_j, i_j)$ and so for geometric reasons $\sigma(G(n_j, m_j, i_j)) \cap G \neq \emptyset$, a contradiction. Hence all boundary leaves of H are preperiodic. Suppose that ℓ' is a boundary leaf of H with $\sigma^n(\ell') = \ell'$. Since ℓ' is a limit of a sequence of leaves from $\mathcal{L}(G)$ and σ^n repels leaves from ℓ', it follows that there exists a leaf $\ell \in \mathcal{L}(G)$ repelled from ℓ' by two iterations of σ^n.

Then by Theorem 3.1, there exists a σ^n-invariant leaf or gap H' such that ℓ separates H' from H. Let $Z = \overline{\mathrm{Imp}^+(H')}$. Note that $p \notin Z$. Since H' is σ^n-invariant then by Theorem 3.4, Z is a P^n-invariant continuum. By Theorem 2.7 and because of the properties of basic uniCremer polynomials, Z is a periodic point at which at least two rays land (because H' is a leaf or a gap), a contradiction.

So, H is a gap and there exists (m, n, i) such that $G(m, n, i) \cap \mathrm{Bd}(H) \neq \emptyset$. If $G(n, m, i) \cap \mathrm{Bd}(H) = \{a\}$ is a point, then a is an endpoint of a leaf aa' in the boundary of G. But then aa' must be a limit leaf of $\mathcal{L}(G)$ which is impossible for geometric reasons. Hence $G(n, m, i) \cap H$ is a leaf ℓ. Since $\{\sigma^k(\ell)\}$ are pairwise disjoint and all contained in $\mathrm{Bd}(H)$ for all $k \geq 0$, there exists the least l such that $\sigma^l(\ell)$ is a point, and hence $\sigma^{l-1}(\ell)$ is a *critical* leaf, i.e. a leaf whose image is a point. If $\sigma^l(G(m, n, i))$ is a point then we are done with the proof of the first claim of the theorem. If $\sigma^l(G(m, n, i))$ is not a point then the same arguments can be repeated. However there are only finitely many critical leaves in our lamination. Hence $\sigma^r(G(n, m, i))$ is a point for some r. Since by Theorem 3.3 $\mathcal{A}$ is contained in some $G(m, n, i)$, we are done with the first claim. The claim that for any $\theta \in \mathcal{A}$ its σ-orbit is not dense easily follows from this description, and the proof of the first part of the theorem is complete.

By [Hea96] K is pre-critical. Suppose that x is a bi-accessible point which is neither pre-critical nor pre-Cremer. Then by the above it maps into a repelling periodic point y and since x is bi-accessible then y is a periodic cutpoint, a contradiction. □

4. Main results

First we establish some facts which may be of independent interest and serve as an additional motivation for us. As was explained in the Introduction, Kiwi's results [K04] do not apply to uniCremer polynomials. Still, one could hope to model (topologically) a basic uniCremer Julia set J by monotonically mapping J onto a locally connected continuum. It turns out that this is impossible. In the quadratic case we proved in [BO06b] that a monotone map of J onto a locally connected continuum collapses J to a point. However the proofs in [BO06b] rely also upon results of [GMO99] not known for basic uniCremer polynomials. Using a new argument we fill this gap and prove Theorem 4.2.

Let us state some results of [BO06b]. An *unshielded* continuum $K \subset \mathbb{C}$ is a continuum which coincides with the boundary of the infinite complementary component to K. Given an external (conformal) ray R, a crosscut C is said to be R-*transversal* if $\overline{C}$ intersects $\overline{R}$ only once and the intersection is transverse and contained in $C \cap R$; if $t \in R$ then by C_t we always denote a R-transversal crosscut such that $C_t \cap \overline{R} = \{t\}$. Given an external ray R we define the *(induced) order* on R so that $x <_R y\, (x, y \in R)$ if and only if the point x is "closer to K **on the ray**

R than y". Given an external ray R, we call a family of R-transversal crosscuts C_t, $t \in R$ an *R-defining* family of crosscuts if for each $t \in R$ there exists a R-transversal crosscut C_t such that $\mathrm{diam}(C_t) \to 0$ as $t \to K$ and $\mathrm{Sh}(C_t) \subset \mathrm{Sh}(C_s)$ if $t <_R s$.

Lemma 4.1. [BO06b, Lemma 2.1] *Let K be an unshielded continuum and R be an external ray to K. Then there exists an R-defining family of R-transversal crosscuts C_t, $t \in R$.*

Now we can prove Theorem 4.2.

Theorem 4.2. *Suppose that P is a basic uniCremer polynomial and $\varphi : J \to A$ is a monotone map of J onto a locally connected continuum A. Then A is a singleton.*

Proof. By way of contradiction suppose that $\varphi : J \to A$ is a monotone map onto a locally connected non-degenerate continuum A. Define the map Φ on the complex plane $\mathbb{C}$ so that it identifies precisely *fibers* (point-preimages) of φ and does not identify any points outside J. Since the decomposition of $\mathbb{C}$ into fibers of φ and points of $\mathbb{C} \setminus J$ is upper-semicontinuous, the map Φ is continuous. Since J and hence all its subcontinua are non-separating then by the Moore Theorem [M25] the map Φ maps $\mathbb{C}$ onto $\mathbb{C}$, and so $\Phi(J) = \varphi(J) = A$ is a *dendrite* (locally connected continuum containing no simple closed curve). External (conformal) rays R_α in the J-plane are then mapped into continuous pairwise disjoint curves $\Phi(R_\alpha)$ in the A-plane called below *A-rays*. Clearly, if $R_\alpha = R$ lands then so does $\Phi(R)$. Let us show that $\Phi(R)$ *always* lands, and in fact the impression $\mathrm{Imp}(\alpha)$ maps under φ to the landing point of the A-ray $\Phi(R)$. By Lemma 4.1 there exists an R-defining family of crosscuts C_t. Since Φ is continuous then $\mathrm{diam}(\Phi(C_t)) \to 0$ as $t \to J$. Consider two cases.

Suppose that $\Phi(C_t)$ is an arc for all $t \in R$, and hence a crosscut of A. Since A is locally connected then by Carathéodory theory $\Phi(C_t)$ converges to a unique point $x \in A$ which implies that $\Phi(R)$ lands. Also, by Carathéodory theory the Φ-images of the closures of the shadows $\mathrm{Sh}(C_t)$ converge to the same point x which belongs to them all. Since the intersection of the closures of the shadows $\mathrm{Sh}(C_t)$ is the impression $\mathrm{Imp}(\alpha)$ of α then $\varphi(\mathrm{Imp}(\alpha)) = \{x\}$ as desired. Otherwise there exists $t \in R$ such that $\Phi(C_t)$ is a simple closed curve. This can only happen if the endpoints of C_t map under φ to the same point, say, z. It follows that $\Phi(C_s)$ is a simple closed curve for all $s <_R t$, and these curves all contain z. Thus, $\Phi(R)$ lands at z. Moreover, in this case the shadows $\mathrm{Sh}(C_s)$ are mapped inside the simple closed curves $\Phi(C_s)$, $s <_R t$ which as before implies that $\varphi(\mathrm{Imp}(\alpha)) = \{z\}$.

Since impressions are upper-semicontinuous, the family of A-rays is continuous, and the map ψ associating to every angle α the landing point of the A-ray $\Phi(R_\alpha)$ is a continuous map of $\mathbb{S}^1$ onto A. Define the *valence* of a point $y \in A$ as the number of components of the set $A \setminus \{y\}$ if it is finite and infinity otherwise. Let $\mathcal{B}_y$ be the set of A-rays landing at y. Then the number of components of $\mathbb{S}^1 \setminus \mathcal{B}_y$ equals the valence of y. Indeed, if (α, β) is a component of $\mathbb{S}^1 \setminus \mathcal{B}_y$ then the A-rays of angles in (α, β) land in a component of $A \setminus \{y\}$ and for distinct components of $\mathbb{S}^1 \setminus \mathcal{B}_y$ we get distinct components of $A \setminus \{y\}$. In fact there is exactly one component of $A \setminus \{y\}$ contained in the appropriate wedge in the plane formed by the A-rays $\Phi(R_\alpha)$ and $\Phi(R_\beta)$. Indeed, otherwise choose angles $\gamma, \theta \in (\alpha, \beta)$ so that their A-rays $\Phi(R_\gamma)$ and $\Phi(R_\theta)$ land at points x, z from distinct components of $A \setminus \{y\}$. Then the path $\psi([\gamma, \theta])$ connects points x and z inside A and hence must pass through y. On the other hand by the construction $y \notin \psi([\gamma, \theta])$, a contradiction.

We show that except for a countable set of points there are no more than two A-rays landing at $y \in A$. Let Q' be the set of all points $y \in A$ with finite valence for which there are infinitely many A-rays landing at y. Then by the previous paragraph $\mathcal{B}_y$ has a non-empty interior for any point $y \in Q'$, and so Q' is countable. On the other hand, by Theorem 10.23 of [Nad92] the set Q'' of all branch points of A is countable (a *branch point* is a point of valence greater than 2). Hence, for any point $y \in A \setminus (Q' \cup Q'')$ such that its valence is greater than 1 (such points are called *cutpoints*) exactly two A-rays land at y.

Choose a point $y \in A \setminus (Q' \cup Q'')$ using a bit of dynamics. Let H be the union of grand orbits of p and all critical points of P and set $Q''' = \varphi(H)$. Then $\hat{Q} = Q' \cup Q'' \cup Q'''$ is countable. Choose $y \in A \setminus \hat{Q}$ to be a cutpoint of A. Then exactly two A-rays land at y. Moreover, let $\varphi^{-1}(y) = K$; then by the choice of y forward images of K avoid p and critical points of P. Since impressions are mapped by φ into points, there are exactly two angles α, β with $K = \mathrm{Imp}(\alpha) \cup \mathrm{Imp}(\beta)$ and the impressions of other angles are disjoint from K.

By Theorem 3.5 there are integers $0 \le l < m$ such that $P^l(K)$ and $P^m(K)$ intersect. Since forward images of K avoid critical points, $\sigma^r(\alpha) \neq \sigma^r(\beta)$ for any r and hence both $P^l(K) = \mathrm{Imp}(\sigma^l(\alpha)) \cup \mathrm{Imp}(\sigma^l(\beta))$ and $P^m(K) = \mathrm{Imp}(\sigma^m(\alpha)) \cup \mathrm{Imp}(\sigma^m(\beta))$ are unions of impressions of two distinct angles. Now, if the pair of angles $\sigma^l(\alpha), \sigma^l(\beta)$ maps into itself by σ^{m-l} then $P^l(K)$ is a P^{m-l}-fixed continuum not containing p, hence by Theorem 2.7 it is a repelling periodic point at which two rays land, a contradiction. Hence there exists an angle $\gamma \in \{\sigma^m(\alpha), \sigma^m(\beta)\} \setminus \{\sigma^l(\alpha), \sigma^l(\beta)\}$ such that $\mathrm{Imp}(\gamma)$ non-disjoint from the $P^l(K)$. If we now pull $P^l(K)$ back to K we will get an angle $\gamma' \notin \{\alpha, \beta\}$ whose impression is non-disjoint from K, a contradiction. □

Since we assume that J is decomposable, all impressions are proper and have empty interior. In particular no countable union of impressions coincides with J. We begin by studying *red dwarf* Julia sets, i.e. uniCremer Julia sets such that impressions of all angles contain the Cremer point p.

Lemma 4.3. *If J is a red dwarf Julia set then* (1) *the intersection K of all impressions contains all forward images of all critical points,* (2) *there exists $\varepsilon > 0$ such that the diameter of any impression is greater than ε,* (3) *there are no points at which J is connected im kleinen, and* (4) *no point of J is biaccessible and p is not accessible from $\mathbb{C} \setminus J$.*

Proof. Let us show that for any angle α its impression $\mathrm{Imp}(\alpha)$ contains all critical images. Indeed, otherwise there exists a critical point c such that $P(c) \notin \mathrm{Imp}(\alpha)$ which implies that $c \notin K$. Choose a curve T starting at $P(c)$, going to infinity and bypassing $\mathrm{Imp}(\alpha)$. Then the pullback of T containing c cuts the plane into at least two components each of which contains at least one pullback of $\mathrm{Imp}(\alpha)$, a contradiction. Hence K contains all critical values, and since K is forward invariant, K contains all forward images of all critical points.

Clearly, (1) implies (2) (by [Mn93] there is a recurrent critical point whose forward orbit avoids p). By Lemma 2.12(2) J is nowhere connected im kleinen. Moreover, no point $x \in J$ is biaccessible from $\mathbb{C} \setminus J$. Indeed, if $x \neq p$ is biaccessible then one of the two half-planes into which x and rays landing at x cut the plane will not contain p, hence all rays contained in that half-plane will not contain p in their impressions, a contradiction. On the other hand, if $x = p$ we can take a preimage of p and get the same contradiction. Hence p cannot be accessible from $\mathbb{C} \setminus J$ because if it is then the corresponding ray is not fixed (periodic rays cannot land on p by Douady and Hubbard [DH85]) and hence this ray and its image show that p is biaccessible, a contradiction. □

Lemma 4.4 follows from Theorem 2.7.

Lemma 4.4. *If $K \subset J$ is a P^n-invariant continuum or singleton not containing p then K is a singleton. In particular, if γ is a periodic angle and $p \notin Imp(\gamma)$ then $Imp(\gamma)$ is a singleton.*

Proof. All P^n-fixed points in K are repelling, and by the definition of a basic uniCremer polynomial at each of them exactly one P^n-fixed ray lands. Hence by Theorem 2.7 K is a singleton. □

For $x \in J$ let $A(x)$ be the set of all angles whose impressions contain x, and let $B(x)$ be the union of these impressions. Then $A(x)$ and $B(x)$ are closed sets. In Lemma 4.5 we study these sets for a periodic x.

Lemma 4.5. *If x is periodic then one of the following holds:*

(1) *$x \notin B(p)$, then $\{x\} = B(x) = Imp(\theta)$ for a periodic K-separate angle θ, $A(x) = \{\theta\}$, and x is a repelling periodic point;*
(2) *$x \in B(p)$, then $B(x)$ is non-degenerate and no angle $\theta \in A(x)$ is K-separate.*

In particular, no angle $\theta \in A(p)$ is K-separate, and $B(p)$ is non-degenerate.

Proof. Consider first the case when $x = p$. We need to show that then the case (2) holds. First let us show that $B(p)$ is non-degenerate. Suppose that $A(p)$ is infinite. Since $A(p)$ is invariant it follows from a well-known result from the topological dynamics of locally expanding maps that $\sigma|_{A(p)}$ is not one-to-one. Hence by [Hea96] $B(p)$ contains a critical point and cannot coincide with p. Suppose that $A(p)$ is finite. Then $A(p)$ contains periodic angles, hence $B(p)$ contains periodic points distinct from p and hence again $B(p)$ is not degenerate.

Let us show that no angle $\theta \in A(p)$ is K-separate. We may assume that $A(p) = \{\theta\}$ consists of just one angle and need to show that θ is not K-separate. Clearly, $\sigma(\theta) = \theta$. Denote the landing point of R_θ by x, and show that there is a critical point $c \in B(p) = \text{Imp}(\theta)$. Indeed, otherwise choose a neighborhood U of $B(p)$ such that no critical points belong to $\overline{U}$, consider the set of all points never exiting $\overline{U}$, and then the component K of this set containing p. Such sets are called *hedgehogs* (see papers by Perez-Marco [Per94] and [Per97]) and have a lot of important properties. In particular, by [Per94] and [Per97] K cannot contain a periodic point other than p, a contradiction (clearly, $B(p) \subset K$ and $B(p)$ contains x). Hence $c \in B(p)$ for some critical point c. This implies that for some integer $i, 1 \le i \le d-1$ we have $c \in \text{Imp}(\theta + i/d)$ and hence $B(p) = \text{Imp}(\theta)$ is not disjoint from the impression of another angle and θ is not K-separate as desired.

Suppose now that $x \notin B(p)$ is a periodic point of period m. Then $p \notin B(x)$ and hence by Lemma 4.4 $B(x)$ is degenerate. By the above quoted topological result this implies that $A(x)$ is finite and therefore, by the assumptions on P, $A(x) = \{\theta\}$ where θ is periodic and K-separate. Now assume that $x \in B(p)$. Then there is an angle $\alpha \in A(p)$

with $p, x \in \text{Imp}(\alpha)$ and hence $p \in B(x)$ and $B(x)$ is not degenerate. Now, if there are more than one angle in $A(x)$ then all such angles are not K-separate and we are done. If however there is only one angle in $A(x)$ then this angle is $\alpha = \theta$ which belongs to $A(p)$ and by the previous paragraph is not K-separate either. □

Lemma 4.5 implies a few facts: e.g., if θ is a K-separate periodic angle then $\text{Imp}(\theta)$ is a repelling periodic point. Lemma 4.6 shows that in some cases there are lots of such angles. For $F \subset \mathbb{S}^1$ denote by $\text{Imp}(F)$ the set $\cup_{\theta \in F} \text{Imp}(\theta)$. Let E be the set of all K-separate periodic angles; by Lemma 4.5 each angle in E has degenerate impression.

Lemma 4.6. *Suppose $p \notin Imp(\theta)$ for some angle θ. Then* (1) *$B(p)$ is a nowhere dense subset of J,* (2) *the set E is dense in $\mathbb{S}^1$,* (3) *the set $Imp(E)$ is dense in J,* (4) *for a closed set of angles $F \neq \mathbb{S}^1$ the set $Imp(F)$ is a proper subset of J,* (5) *J is connected im kleinen at every point $y \in Imp(E)$,* (6) *in any arc $W \subset \mathbb{S}^1$ there are two angles whose impressions meet.*

Proof. To prove the first claim it is enough to show that $B(p) \neq J$. Suppose otherwise. Since by our standing assumption J is decomposable then no finite union of impressions can coincide with J, and $A(p)$ is infinite. Then in any open arc V there are angles whose impressions meet. Indeed, we can find a big integer N such that $\sigma^{-N}(A(p)) \cap V$ is infinite. Each angle from $\sigma^{-N}(A(p))$ contains a P^N-preimage of p in its impression. Since there are only finitely many P^N-preimages of p then there are two angles $\gamma, \gamma' \in \sigma^{-N}(A(p)) \cap V$ which contain the same preimage of p and therefore meet as desired.

Denote by U an open arc containing θ such that for any angle in U its impression does not contain p (then by Lemma 4.4 periodic angles from U have degenerate impressions). Assume that $(\gamma, \gamma') \subset U$ is an arc such that the impressions of γ, γ' meet. The union $Z = R(\gamma) \cup \text{Imp}(\gamma) \cup \text{Imp}(\gamma') \cup R(\gamma')$ cuts the plane into two half-planes H and G; assume for the sake of definiteness that $p \in G$. Since no finite union of impressions coincides with J then the union of impressions of angles from (γ, γ') is not contained in Z. Hence there exists a point $h \in J \cap H$. Then the impression of an angle from (γ', γ) cannot contain h because Z separates this angle's ray from h. On the other hand, the impression of an angle from $[\gamma, \gamma']$ cannot contain p either. Hence no impression contains h and p simultaneously, implying that $B(p) \neq J$.

By Lemma 4.5 E is the set of all periodic angles such that landing points of their rays do not belong to $B(p)$. By the upper semi-continuity of impressions, E is open in the set of all periodic angles. Since E is invariant, E is dense in $\mathbb{S}^1$ which proves (2). We claim that $\text{Imp}(E)$ is dense in J. Indeed, otherwise there exists an open set $U \subset J$ disjoint from $\text{Imp}(E)$. Since periodic points are dense in J, they are dense in U, and by the definition of E all these periodic points belong to $B(p)$. Hence $B(p)$ has non-empty interior, a contradiction with (1). Thus, $\text{Imp}(E)$ is dense in J which proves (3). The claim (2) implies (4). By Lemma 2.11 the rest of the lemma follows. □

So far the results of this section use mainly topological tools. This changes in the lemmas below where we rely upon both continuum theory and dynamics. Our aim is to prove that the angles with dense in $\mathbb{S}^1$ orbits have degenerate impressions. Problems of this kind are often related to the dynamics of critical points. The result obtained in Lemma 4.8 enables us to apply some standard tools and seems to be interesting by itself. However first we need a simple lemma.

Lemma 4.7. *Suppose that there exists an angle θ whose impression does not contain p. Then the following statements are equivalent.*

(1) *An angle α has a dense in $\mathbb{S}^1$ orbit.*
(2) *Any point of Imp(α) has a dense in J orbit.*
(3) *There exists a point in Imp(α) which has a dense in J orbit.*

Proof. Suppose that α has a dense in $\mathbb{S}^1$ orbit and choose $x \in \text{Imp}(\alpha)$. Let $y \in J$ and $\varepsilon > 0$. By Lemma 4.6 there exists $\gamma \in \mathbb{S}^1$ such that $\text{Imp}(\gamma) = \{z\}$ and $d(z, y) < \varepsilon/2$. Since the orbit of α is dense in $\mathbb{S}^1$ and impressions are upper semi-continuous, there exists $n > 0$ such that $\sigma^n(\alpha)$ is so close to γ that $\text{Imp}(\sigma^n(\alpha)) \subset B(z, \varepsilon/2)$, and hence $d(P^n(x), y) < \varepsilon$. Therefore, $\omega(x) = J$ and (1) implies (2). Clearly, (2) implies (3). Now, suppose that (3) holds. Then $\text{Imp}(\omega(\alpha)) = J$ which by Lemma 4.6(4) implies that $\omega(\alpha) = \mathbb{S}^1$ as desired. □

Now we show that if there exists an angle θ whose impression does not contain p then no critical point has a dense orbit in J.

Lemma 4.8. *Suppose that there exists an angle θ whose impression does not contain p. Suppose that α is an angle with a dense orbit in $\mathbb{S}^1$. Then α is K-separate and Imp(α) does not contain a critical point of P. In particular, no critical point can have a dense orbit in J.*

Proof. The proof consists of several steps. Suppose that there are finitely many angles $\alpha = \alpha_0, \alpha_1, \dots, \alpha_k$ such that the union I of their impressions is a continuum. Let us show that then I has to be wandering (all its iterates are pairwise disjoint). By way of contradiction and without loss of generality we may assume that $P(I) \cap I \neq \emptyset$. Consider the set Q of all angles whose impressions are not disjoint from $B(p)$ and the set $\mathrm{Imp}(Q)$. Then $Q \neq \mathbb{S}^1$ by Lemma 4.5. Hence by Lemma 4.6(4) $\mathrm{Imp}(Q) \neq J$. Let us show that we may assume that *all* images of I intersect $\mathrm{Imp}(Q)$.

By Lemma 4.5 the set Q contains two angles with non-disjoint impressions. Denote these angles γ and β. Then the rays of these angles together with their impressions cut the plane into two components H and G so that the periodic K-separate angles from (γ, β) have point-impressions in H and the periodic K-separate angles from (β, γ) have point-impressions in G. Choose periodic K-separate angles $\theta' \in (\gamma, \beta)$ and $\theta'' \in (\beta, \gamma)$, and then choose $k < l$ so that $\sigma^k(\alpha) \in (\gamma, \beta)$ is very close to θ' and $\sigma^l(\alpha) \in (\beta, \gamma)$ is very close to θ''. Then the continuum $Z = P^k(I) \cup P^{k+1}(I) \cup \dots \cup P^l(I)$ connects points from H to points from G. Hence by Lemma 2.10 $Z \cap \mathrm{Imp}(Q) \neq \emptyset$. Thus, from some time on the images of I are non-disjoint from $\mathrm{Imp}(Q)$, and we may assume that in fact $I \cap \mathrm{Imp}(Q) \neq \emptyset$ and hence *all* images of I intersect $\mathrm{Imp}(Q)$.

Just like the set $B(p)$ was constructed as the union of all impressions non-disjoint from p, and the set $\mathrm{Imp}(Q)$ was constructed as the union of all the impressions non-disjoint from $B(p)$, this process can be continued for $k+1$ more steps resulting into a union of impressions which we will denote by T. Clearly, T is a closed; moreover, since no point of $\mathrm{Imp}(E)$ can belong to T then T is a proper subset of J. Since α has a dense orbit, by Lemma 4.7 the orbit of any point of $\mathrm{Imp}(\alpha)$ is dense in J. On the other hand, by the previous paragraph the orbit of $\mathrm{Imp}(\alpha)$ is contained in T and $T \neq J$, a contradiction. So, the assumption that I is not wandering leads to a contradiction which implies that I is wandering. Observe that then by Theorem 3.2 $k \leq 2^d$.

Let us now show that α is K-separate. Indeed, otherwise let $\alpha' \neq \alpha$ be such that $\mathrm{Imp}(\alpha) \cap \mathrm{Imp}(\alpha') \neq \emptyset$. Then by the above the maximal finite collection of angles $\alpha_0 = \alpha, \alpha_1 = \alpha', \dots, \alpha_k$ such that the union I of their impressions is connected consists of $k+1 > 1$ angles and is such that I is wandering. By maximality the impressions of other angles are disjoint from I. However the orbit of α is dense which contradicts Theorem 3.5. Hence α is K-separate as desired. To complete the proof it remains to notice that if a critical point c belongs to $\mathrm{Imp}(\alpha)$ then, because locally around c the map P is not one-to-one, there exists an angle α' such that $\sigma(\alpha) = \sigma(\alpha')$ and $c \in \mathrm{Imp}(\alpha')$ implying that α is not K-separate. Hence $\mathrm{Imp}(\alpha)$ does not contain critical points. Since this holds for any angle with dense orbit we conclude that by Lemma 4.7 no critical point can have a dense orbit in J. □

Lemma 4.9 completes a series of claims made in Lemma 4.7 and Lemma 4.8.

Lemma 4.9. *Suppose that there exists an angle θ whose impression does not contain p. Let α be an angle such that the σ-orbit of α is dense in $\mathbb{S}^1$. Then Imp(α) is a singleton, and, moreover, the angle α is K-separate.*

Proof. By Mañé [Mn93] the closure B' of the union of orbits of all critical points of P contains p. By Lemma 4.8 B' is nowhere dense in J. By Lemma 4.6 there is an angle $\gamma \in E$ such that the singleton $\text{Imp}(\gamma)$ is not contained in B'. By the upper semi-continuity of impressions we can find an arc U around γ so that the union of impressions of angles from $\overline{U}$ is positively distant from B'. By Lemma 2.11 we can find two angles $\tau' < \gamma < \tau''$ in U (the order is considered within U) such that $\text{Imp}(\tau') \cap \text{Imp}(\tau'') \neq \emptyset$. Set $U' = (\tau', \tau'')$.

Consider the two connected open components of $\mathbb{C} \setminus R_{\tau'} \cup \text{Imp}(\tau') \cup \text{Imp}(\tau'') \cup R_{\tau''}$; let V be the component containing rays of angles from U'. Then there are points of J in V. Indeed, otherwise $\text{Imp}(\tau') \cup \text{Imp}(\tau'')$ contains the impressions of all angles from U' which yields that a forward σ-image of τ' or τ'' will coincide with J implying by Theorem 2.8 that J is indecomposable, a contradiction with the standing assumption. Let us prove that α is K-separate and its impression is a point. Indeed, by the previous paragraph V is positively distant from B'. Since V is simply connected, we can find two Jordan disks $W' \supset \overline{W''} \supset J \cap V$ which are both positively distant from B'. Therefore all pull-backs of W' and W'' are univalent. By Mañé [Mn93] this implies that the diameter of the pull-backs of W'' converge to 0 as the power of the map approaches infinity. Observe that as σ-images of α approach γ, the corresponding P-images of $\text{Imp}(\alpha)$ get closer and closer to $\text{Imp}(\gamma)$ (because of the upper semi-continuity of impressions) and thus we may assume that infinitely many P-images of $\text{Imp}(\alpha)$ are contained in W''. Pulling W'' back along the orbit of $\text{Imp}(\alpha)$ for longer and longer time we see that the diameter of $\text{Imp}(\alpha)$ cannot be positive, and hence $\text{Imp}(\alpha) = \{y\}$ is a point as claimed. Moreover, by Lemma 4.8 the angle α is K-separate. This completes the proof.

□

We can now prove our main theorem.

Theorem 4.10. *For a uniCremer polynomial P the following facts are equivalent:*

1. *there is an impression not containing the Cremer point;*
2. *there is a degenerate impression;*
3. *the set Y of all K-separate angles with degenerate impressions contains all angles with dense orbits and a dense set of periodic*

angles, and the Julia set J is connected im kleinen at landing points of the corresponding rays;

(4) *there is a point at which the Julia set is connected im kleinen.*

Proof. Let us prove that (1) implies (2). Indeed, suppose that there is an angle not containing p in its impression. Then by Lemma 4.6 there exist angles with degenerate impressions.

We show that (2) implies (3). Indeed, let $\mathrm{Imp}(\alpha)$ be a point. Then so are the impressions of the angles $\alpha+1/d, \dots, \alpha+(d-1)/d$. At least one of them is not p, so we may assume that $\mathrm{Imp}(\alpha) \neq \{p\}$. Then by Lemma 4.6 the set E is dense in $\mathbb{S}^1$. Let us now consider an angle β whose σ-orbit is dense in $\mathbb{S}^1$. By Lemma 4.9 $\mathrm{Imp}(\beta)$ is a point and β is K-separate. By Lemma 2.11 J is connected im kleinen at the landing points of the rays with arguments either from E, or from the set of angles with dense in $\mathbb{S}^1$ orbits. This shows that indeed (2) implies (3).

Clearly, (3) implies (4). It remains to show that (4) implies (1). Indeed, if all impressions contain p then by Lemma 4.3 J is nowhere connected im kleinen, a contradiction. The proof is complete. □

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