Abstract. In this paper we try to present in a coherent fashion proofs of basic results developed so far (primarily by H. Bell) for the plane fixed point problem. Some of these results have been announced much earlier but without formal proof. We define the concept of the variation of a map on a simple closed curve and relate it to the index of the map on that curve: Index = Variation + 1. We define the concept of an outchannel for a fixed point free map which carries a nonseparating plane continuum into itself. We then prove Bell’s Lollipop Lemma and use it to show that such a map has a unique outchannel, and that outchannel must have variation = −1. We also define a special class of straight line crosscuts and show that these suffice for a satisfactory treatment of prime-ends of a non-separating plane continuum.

1. Introduction

By \( \mathbb{C} \) we denote the plane and by \( \mathbb{C}_\infty \) the Riemann sphere. Let \( X \) be a plane continuum. By \( T(X) \) we denote the topological hull of \( X \) consisting of \( X \) union all of its bounded complementary domains. Thus, \( \mathbb{C}_\infty \setminus T(X) \) is a simply-connected domain containing \( \infty \). The following is a long-standing question in topology.

**Fixed Point Question:** “Does a continuous function taking a nonseparating plane continuum into itself always have a fixed point?”

We study the slightly more general question, “Is there a plane continuum \( Z \) and a continuous function \( f : \mathbb{C} \to \mathbb{C} \) taking \( Z \) into \( T(Z) \) with no fixed points in \( T(Z) \)?” A Zorn’s Lemma argument shows that if one assumes the answer is “yes,” then there is a subcontinuum \( X \subset Z \) minimal with respect to these properties. Therefore, we will assume the following throughout this paper:

1.1. **Standing Hypotheses.** We assume that \( f : \mathbb{C} \to \mathbb{C} \) is a map and \( X \) is a plane continuum such that \( f(X) \subset T(X) = Y \), \( f \) has no fixed points in \( Y \), and \( X \) is minimal with respect to these properties.

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It will follow from Theorem 3.23 that for such a minimal continuum, \( f(X) = X = \partial Y \) (though it may not be the case that \( f(Y) \subset Y \)). By results of Cartwright-Littlewood and Bell, if one replaces “map” by “homeomorphism of the plane” in the fixed point question, the answer is yes. By results of Bell [3] (see also Sieklucki [11], and Iliadis [8]), the only unsolved general case (with no special assumptions on the map) is where the boundary of \( X \) is indecomposable (with a dense channel, explained later). In this paper we use tools first developed by Bell to elucidate the action of a fixed point free map (should one exist). Theorem 4.1 (Unique Outchannel) is new and complete proofs of Theorems 2.13 and 2.14 appear in print for the first time. In subsequent papers [6], we will apply the tools developed here to prove some additional special cases (with restrictions on the map) of a “fixed point theorem.”

2. Tools

Let \( S^1 \) denote the unit circle in the complex plane and let \( p : \mathbb{R} \to S^1 \) denote the covering map \( p(x) = e^{2\pi i x} \). Let \( g : S^1 \to S^1 \) be a map. By the degree of the map \( g \), denoted by degree(\( g \)), we mean the number \( \hat{g}(1) - \hat{g}(0) \), where \( \hat{g} : \mathbb{R} \to \mathbb{R} \) is a lift of the map \( g \) to the universal covering space \( \mathbb{R} \) of \( S^1 \) (i.e., \( p \circ \hat{g} = g \circ p \)). It is well-known that degree(\( g \)) is independent of the choice of the lift.

2.1. Index. Let \( g : S^1 \to \mathbb{C} \) be a map and let \( S = g(S^1) \). Suppose \( f : S \to \mathbb{C} \) has no fixed points on \( S \). Then for all \( z \in S \), the vector \( z - f(z) \neq 0 \). Hence the unit vector \( v(z) = \frac{z - f(z)}{|z - f(z)|} \) always exists. Define the map \( \overline{v} = v \circ g : S^1 \to S^1 \) by

\[
\overline{v}(t) = v(g(t)) = \frac{g(t) - f(g(t))}{|g(t) - f(g(t))|}.
\]

Then the map \( \overline{v} : S^1 \to S^1 \) lifts to a map \( \hat{v} : \mathbb{R} \to \mathbb{R} \) such that \( p \circ \hat{v} = \overline{v} \circ p \). Define the index of \( f \) with respect to \( g \), denoted \( \text{ind}(f, g) \) by

\[
\text{ind}(f, g) = \hat{v}(1) - \hat{v}(0) = \text{degree}(\overline{v}).
\]

More generally, for any parameters \( 0 \leq a < b \leq 1 \) in \( S^1 = \mathbb{R}/\mathbb{Z} \), define the fractional index of \( f \) on the path \( g_{[a,b]} \) in \( S \) by

\[
\text{ind}(f, g)_{[a,b]} = \hat{v}(b) - \hat{v}(a).
\]

While necessarily, the index of \( f \) with respect to \( g \) is an integer, the fractional index of \( f \) on \( g_{[a,b]} \) need not be. We shall have occasion to use fractional index in the proof of Theorem 2.13. Note that (fractional) index is the net change in argument of the vector \( g(t) - f(g(t)) \) as \( t \) runs along \( S^1 \) from \( a \) to \( b \).

Proposition 2.1. Let \( g : S^1 \to \mathbb{C} \) be a map with \( g(S^1) = S \), and suppose \( f : S \to \mathbb{C} \) has no fixed points on \( S \). Let \( a \neq b \in S^1 \) with \([a,b] \) denoting
the counterclockwise subarc on $S^1$ from $a$ to $b$ (so $S^1 = [a, b] \cup [b, a]$). Then $\text{ind}(f, g) = \text{ind}(f, g|_{[a, b]}) + \text{ind}(f, g|_{[b, a]})$.

2.2. Stability of Index. The following standard theorems and observations about the stability of index under fixed-point-free homotopy are consequences of the fact that index is continuous and integer-valued.

**Theorem 2.2.** Suppose $f : S \to \mathbb{C}$ is a map and $g_1 : S^1 \to S$ and $g_2 : S^1 \to S$ are homotopic maps in $\mathbb{C}$ such that the homotopy misses the fixed point set of $f$. Then $\text{ind}(f, g) = \text{ind}(f, g_1)$.

An embedding $g : S^1 \to S \subset \mathbb{C}$ is orientation preserving if $g$ is isotopic to the identity map $id|_{S^1}$. In particular, the index of $f$ on a simple closed curve $S$ missing the fixed point set of $f$ is independent of choice of parameterizations of $S$ with the same orientation. If $g_1$ and $g_2$ are orientation-preserving embeddings of $S^1$ with the same image set $g_1(S^1) = S = g_2(S^1)$, then we have a well-defined index of $f$ on $S$, namely $\text{ind}(f, S) = \text{ind}(f, g_1) = \text{ind}(f, g_2)$.

**Theorem 2.3.** Suppose $g : S^1 \to \mathbb{C}$ is a map with $g(S^1) = S$, and $f_1, f_2 : S \to \mathbb{C}$ are homotopic maps such that each level of the homotopy is fixed-point-free on $S$. Then $\text{ind}(f_1, g) = \text{ind}(f_2, g)$.

In particular, if $S$ is a simple closed curve and $f_1, f_2 : S \to \mathbb{C}$ are maps such that there is a homotopy $h_t : S \to \mathbb{C}$ from $f_1$ to $f_2$ with $h_t$ fixed-point free on $S$ for each $t \in [0, 1]$, then $\text{ind}(f_1, S) = \text{ind}(f_2, S)$.

**Corollary 2.4.** Suppose $g : S^1 \to \mathbb{C}$ is an orientation preserving embedding with $g(S^1) = S$, and $f : T(S) \to \mathbb{C}$ is a map such that $f$ has no fixed points on $S$ and $f(S) \subset T(S)$. Then $\text{ind}(f, g) = 1$.

**Proof.** Since $f(S) \subset T(S)$ which is a disk with boundary $S$ and $f$ has no fixed point on $S$, there is a fixed point free homotopy of $f|_S$ to a constant map $c : S \to \mathbb{C}$ taking $S$ to a point in $T(S) \setminus S$. By Theorem 2.3, $\text{ind}(f, g) = \text{ind}(c, g) = 1$. \hfill $\Box$

**Theorem 2.5.** Suppose $g : S^1 \to \mathbb{C}$ is a map with $g(S^1) = S$, and $f : T(S) \to \mathbb{C}$ is a map such that $\text{ind}(f, g) \neq 0$, then $f$ has a fixed point in $T(S)$.

**Proof.** Notice that $T(S)$ is a locally connected non-separating plane continuum and, hence, contractible. Suppose $f$ has no fixed point in $T(S)$. Choose point $q \in T(S)$. Let $c : S^1 \to \mathbb{C}$ be the constant map to $q$. Let $H$ be a homotopy from $g$ to $c$ with image in $T(S)$. Since $H$ misses the fixed point set of $f$, Theorem 2.2 implies $\text{ind}(f, g) = \text{ind}(f, c) = 0$. \hfill $\Box$

2.3. Variation. In this section we introduce the notion of variation of a map on an arc and relate it to winding number.

**Definition 2.6** (Junctions). The standard junction $J_0$ is the union of the three rays $R_i = \{z \in \mathbb{C} \mid z = re^{i\pi/2}, \ r \in [0, \infty)\}$, $R_+ = \{z \in \mathbb{C} \mid z =$
$re^0$, $r \in [0, \infty)$, $R_\infty = \{ z \in \mathbb{C} \mid z = re^{i\alpha}, \alpha \in [0, \infty) \}$, having the origin 0 in common. By $U$ we denote the lower half-plane $\{ z \in \mathbb{C} \mid z = x + iy, x \in \mathbb{R}, y < 0 \}$. A junction $J_0$ is the image of $J_0$ under any orientation-preserving homeomorphism $h : \mathbb{C} \to \mathbb{C}$ where $v = h(0)$.

We will often suppress $h$ and refer to $h(R_\infty)$ as $R_\infty$, and similarly for the remaining rays and the region $h(U)$. When needed we will write $R_{w^\pm}$ etc. when we want to refer to a particular $h(R_{w^\pm})$ of a junction $J_\alpha$ based at $v = h(0)$.

Suppose $S$ is a simple closed curve and $A \subset S$ is a subarc of $S$ with endpoints $a$ and $b$, with $a < b$ in the counter-clockwise orientation on $S$. We will usually denote such a subarc by $A = [a, b]$ and by $(a, b)$ its interior in $S^1$.

**Definition 2.7** (Variation on an arc). Let $S$ be a simple closed curve and $A = [a, b]$ a subarc of $S$ such that $f(a), f(b) \in T(S)$ and $f(A) \cap A = \emptyset$. We define the variation of $f$ on $A$ with respect to $S$, denoted $\text{var}(f, A, S)$, by the following algorithm:

1. Choose an orientation-preserving homeomorphism $h$ of $\mathbb{C}$ such that $h(0) = v \in A$ and $T(S) \subset h(U) \cup \{v\}$.
2. As always we assume that $a < b$ in the counterclockwise order.
3. **Counting crossings:** Consider the set $M = f^{-1}(J_{v}) \cap [a, b]$. Each time a point of $f^{-1}(h(R^+)) \cap [a, b]$ is immediately followed in $M$, in the natural order on $[a, b]$, by a point of $f^{-1}(h(R^-))$ count +1 and each time a point of $f^{-1}(h(R^-)) \cap [a, b]$ is immediately followed in $M$, in the natural order on $[a, b]$, by a point of $f^{-1}(h(R^+))$ count −1. Count no other crossings.
4. The sum of the crossings found above is the variation, denoted $\text{var}(f, A, S)$.

Note that $f^{-1}(h(R^+)) \cap [a, b]$ and $f^{-1}(h(R^-)) \cap [a, b]$ are disjoint closed sets in $[a, b]$. Hence, in (3) in the above definition, we count only a finite number of crossings and $\text{var}(f, A, S)$ is a finite integer.

Let $g : S^1 \to \mathbb{C}$ be a map and $w \in \mathbb{C} \setminus g(S^1)$ be a point. By the winding number of $g$ about the point $w$, denoted by $\text{win}(g, w)$, we mean the number $\text{ind}(c, g)$, where $c : \mathbb{C} \to \mathbb{C}$ is the constant map $c(z) = w$. It is well-known that the winding number is invariant under homotopies of $g$ in $\mathbb{C} \setminus w$ and independent of the choice of the point $w$ in a particular component of $\mathbb{C} \setminus g(S^1)$. Note that if $B$ is the closure of $S \setminus A$ and $\alpha : S \to \mathbb{C}$ is any map such that $\alpha|_A = f|_A$ and $\alpha(B) \subset T(S) \setminus \{v\} \subset U$, then $\text{var}(f, A, S) = \text{win}(\alpha, v)$.

In case $A$ is an open arc $(a, b) \subset S$ such that $\text{var}(f, A, S)$ is defined, it will be convenient to denote $\text{var}(f, A, S)$ by $\text{var}(f, A, S)$.

The following Lemma follows immediately from the definition.

**Lemma 2.8.** Let $S$ be a simple closed curve. Suppose that $a < c < b$ are three points in $S$ such that $\{f(a), f(b), f(c)\} \subset T(S)$ and $f([a, b]) \cap [a, b] = \emptyset$. Then $\text{var}(f, [a, b], S) = \text{var}(f, [a, c], S) + \text{var}(f, [c, b], S)$.
2.4. **Stability of Variation.** By the above remark that variation is a winding number, the invariance of winding number under suitable homotopies implies that the variation \( \text{var}(f, A, S) \) also remains invariant under such homotopies. That is, even though the specific crossings in (3) in the algorithm may change, the sum remains invariant. We will state the required results about variation below without proof. Proofs can also be obtained directly by using the fact that \( \text{var}(f, A, S) \) is integer valued and continuous under suitable homotopies.

**Proposition 2.9 (Junction Straightening).** Any two junctions \( h_1, h_2 \) with \( v_1 = h_1(0) \in A, v_2 = h_2(0) \in A, T(S) \subset h_1(U) \cup \{v_1\}, \) and \( T(S) \subset h_2(U) \cup \{v_2\} \) give the same variation.

**Proposition 2.10.** Variation \( \text{var}(f, A, S) \) is an integer, well-defined, and independent of \( h \).

Since \( U \) is open for a given junction \( J_v \) for \( A = [a, b] \) with \( T(S) \subset U \cup \{v\} \), the computation of \( \text{var}(f, A, S) \) depends only upon the crossings of the junction coming from a proper compact subarc of the open arc \((a, b)\).

Consequently, \( \text{var}(f, A, S) \) remains invariant under homotopies \( h_t \) of \( f|_{[a,b]} \) such that \( h_t(a) \) and \( h_t(b) \) remain in \( U \) and \( v \notin h_t([a, b]) \) for all \( t \). Moreover, the computation is stable under an isotopy of the plane that moves the entire junction \( J_v \) (even off \( A \)), provided in the the isotopy \( v \) never crosses the image \( f(A) \) and, \( f(a) \) and \( f(b) \) remain in the corresponding domain \( U_t \).

**Definition 2.11 (Variation on a finite union of arcs).** Let \( S \) be a simple closed curve and \( A = [a, b] \) a subcontinuum of \( S \) with partition a finite set \( F = \{a = a_0 < a_1, \ldots, a_n = b\} \). For each \( i \) let \( A_i = [a_i, a_{i+1}] \). Suppose that \( f \) satisfies \( f(a_i) \in T(S) \) and \( f(A_i) \cap A_i = \emptyset \) for each \( i \). We define the variation of \( f \) on \( A \) with respect to \( S \), denoted \( \text{var}(f, A, S) \), by

\[
\text{var}(f, A, S) = \sum_{i=0}^{n-1} \text{var}(f, [a_i, a_{i+1}], S).
\]

In particular, we include the possibility that \( a_n = a_0 \) in which case \( A = S \).

By considering a common refinement of two partitions \( F_1 \) and \( F_2 \) of an arc \( A \subset S \) such that \( f(F_1) \cup f(F_2) \subset T(S) \) and satisfying the conditions in Definition 2.11, it follows from Lemma 2.8 that we get the same value for \( \text{var}(f, A, S) \) whether we use the partition \( F_1 \) or the partition \( F_2 \). Hence, \( \text{var}(f, A, S) \) is well-defined. If \( A = S \) we denote \( \text{var}(f, S, S) \) simply by \( \text{var}(f, S) \).

2.5. **Index and variation for finite partitions.** What links Theorem 2.5 with variation is Theorem 2.13 below, first obtained by Bell and announced in the mid 1980’s, and later by Akis [2]. Our proof is a modification of Bell’s unpublished proof. We first need a variant of Proposition 2.9. Let \( r : \mathbb{C} \to \mathbb{D} \) be radial retraction: \( r(z) = \frac{z}{|z|} \) when \( |z| \geq 1 \) and \( r|_\mathbb{D} = \text{id}|_\mathbb{D} \).
Lemma 2.12 (Curve Straightening). Suppose $f : S^1 \to \mathbb{C}$ is a map with no fixed points on $S^1$. If $[a, b] \subset S^1$ is a proper subarc with $f([a, b]) \cap [a, b] = \emptyset$, $f((a, b)) \subset \mathbb{C} \setminus T(S^1)$ and $f([a, b]) \subset S^1$, then there exists a map $h : S^1 \to \mathbb{C}$ homotopic to $f$ in $\mathbb{C} \setminus T(S)$ relative to $[a, b]$, with each level of the homotopy fixed-point-free, such that $r \circ h : [a, b] \to S^1$ is locally one-to-one. Moreover, $\text{var}(f, [a, b], S^1) = \text{var}(h, [a, b], S^1)$.

Note that if $\text{var}(f, [a, b], S^1) = 0$, then $r \circ h$ carries $[a, b]$ one-to-one onto the arc in $S^1 \setminus [a, b]$ from $f(a)$ to $f(b)$. If the $\text{var}(f, [a, b], S^1) = m > 0$, then $r \circ h$ wraps the arc $[a, b]$ counterclockwise about $S^1$ so that $h([a, b])$ meets each ray in $J_m = m$ times. A similar statement holds for negative variation.

Theorem 2.13 (Index = Variation + 1, Bell). Suppose $g : S^1 \to \mathbb{C}$ is an orientation preserving embedding onto a simple closed curve $S$ and $f$ has no fixed points on $S$. If $F = \{a_0 < a_1 < \cdots < a_n\}$ is a partition of $S$ and $A_i = [a_i, a_{i+1}]$ for $i = 1, \ldots, n$ with $a_{n+1} = a_0$ such that $F \subset T(S)$ and $f(A_i) \cap A_i = \emptyset$ for each $i$, then

$$\text{ind}(f, g) = \sum_{i=0}^{n} \text{var}(f, A_i, S) + 1 = \text{var}(f, S) + 1.$$  

Note that it is possible for index to be defined yet variation not to be defined on a simple closed curve $S$. For example, consider the map $z \to 2z$ with $S$ the unit circle.

Proof. By an appropriate conjugation of $f$ and $g$, we may assume without loss of generality that $S = S^1$ and $g = id$. Let $F$ and $A_i = [a_i, a_{i+1}]$ be as in the hypothesis. Consider the collection of arcs

$$\mathcal{K} = \{K \subset S \mid K \text{ is the closure of a component of } S \cap f^{-1}(f(S) \setminus T(S))\}.$$  

For each $K \in \mathcal{K}$, there is an $i$ such that $K \subset A_i$. Since $f(A_i) \cap A_i = \emptyset$, it follows from the remark after Proposition 2.10 that $\text{var}(f, A_i, S) = \sum_{K \subset A_i, K \in \mathcal{K}} \text{var}(f, K, S)$. In particular, we can compute $\text{var}(f, K, S)$ using one fixed junction for $A_i$ and it is now clear that there are at most finitely many such $K$ with $\text{var}(f, K, S) \neq 0$. Moreover, the images of the endpoints of $K$ lie on $S$.

Let $m$ be the cardinality of the set $\mathcal{K}_f = \{K \in \mathcal{K} \mid \text{var}(f, K, S) \neq 0\}$. By the above remarks, $m < \infty$ and $\mathcal{K}_f$ is independent of $F$. We prove the theorem by induction on $m$.

Suppose for a given $f$ we have $m = 0$. Observe that from the definition of variation and the fact that the computation of variation is independent of the choice of an appropriate partition, it follows that,

$$\text{var}(f, S) = \sum_{K \in \mathcal{K}} \text{var}(f, K, S) = 0.$$  

We claim that there is a map $f_1 : S \to \mathbb{C}$ with $f_1(S) \subset T(S)$ and a homotopy $H$ from $f|_S$ to $f_1$ such that each level $H_t$ of the homotopy is fixed-point-free and $\text{ind}(f_1, id|_S) = 1$.  


To see the claim, first apply the Curve Straightening Lemma 2.12 to each $K \in \mathcal{K}$ (if there are infinitely many, they form a null sequence) to obtain a fixed-point-free homotopy of $f|_S$ to a map $h: S \to \mathbb{C}$ such that $r \circ h$ is locally one-to-one on each $K \in \mathcal{K}$, where $r$ is radial retraction of $\mathbb{C}$ to $D$, and $\text{var}(h, K, S) = 0$ for each $K \in \mathcal{K}$. Let $K$ be in $\mathcal{K}$ with endpoints $x, y$. Since $h(K) \cap K = \emptyset$, $r \circ h$ is one-to-one, and $\text{var}(h, K, S) = 0$. Since $\mathcal{K}$ is a null family, we can do this for each $K \in \mathcal{K}$ so that we obtain the desired $f_1: S \to \mathbb{C}$ as the end map of a fixed-point-free homotopy from $f$ to $f_1$. Since $f_1$ carries $S$ into $T(S)$, Corollary 2.4 implies $\text{ind}(f_1, \text{id}|_S) = 1$.

Since the homotopy $f \simeq f_1$ is fixed-point-free, it follows from Theorem 2.3 that $\text{ind}(f, \text{id}|_S) = 1$. Hence, the theorem holds if $m = 0$ for any $f$ and any appropriate partition $F$.

By way of contradiction, consider the collection of all maps $f$ on $S^1$ which satisfy the hypotheses of the theorem, but not the conclusion. By the above $0 < |\mathcal{K}_f| < \infty$ for each. Let $f$ and partition $F$ be a counterexample for which $m = |\mathcal{K}_f|$ is minimal. By modifying $f$, we will show there exists another counterexample $f'$ with $|\mathcal{K}_{f'}| < m$, a contradiction.

Choose $K \in \mathcal{K}$ such that $\text{var}(f, K, S) \neq 0$. Then $K = [x, y] \subset A_i = [a_i, a_{i+1}]$ for some $i$. By the Curve Straightening Lemma 2.12 and Theorem 2.3, we may suppose $r \circ f$ is locally one-to-one on $K$. Define a new map $f_1: S \to \mathbb{C}$ by setting $f_1|_{S \setminus K} = f|_{S \setminus K}$ and setting $f_1|_K$ equal to the linear map taking $[x, y]$ to the subarc $f(x)$ to $f(y)$ on $S$ missing $[x, y]$. Figure 1 (left) shows an example of a (straightened) $f$ and the corresponding $f_1$ for a case where $\text{var}(f, K, S) = 1$, while Figure 1 (right) shows a case where $\text{var}(f, K, S) = -2$. 

![Figure 1](image-url)
Since on $S \setminus K$, $f$ and $f_1$ are the same map, we have
\[ \text{var}(f, S \setminus K, S) = \text{var}(f_1, S \setminus K, S). \]
Likewise for the fractional index,
\[ \text{ind}(f, S \setminus K) = \text{ind}(f_1, S \setminus K). \]
By definition (refer to the observation we made in the case $m = 0$),
\[ \text{var}(f, S) = \text{var}(f, S \setminus K, S) + \text{var}(f, K, S) \]
\[ \text{var}(f_1, S) = \text{var}(f_1, S \setminus K, S) + \text{var}(f_1, K, S) \]
and by Proposition 2.1,
\[ \text{ind}(f, S) = \text{ind}(f, S \setminus K) + \text{ind}(f, K) \]
\[ \text{ind}(f_1, S) = \text{ind}(f_1, S \setminus K) + \text{ind}(f_1, K). \]
Consequently,
\[ \text{var}(f, S) - \text{var}(f_1, S) = \text{var}(f, K, S) - \text{var}(f_1, K, S) \]
and
\[ \text{ind}(f, S) - \text{ind}(f_1, S) = \text{ind}(f, K) - \text{ind}(f_1, K). \]
We will now show that the changes in index and variation, going from $f$ to $f_1$ are the same (i.e., we will show that $\text{var}(f, K, S) - \text{var}(f_1, K, S) = \text{ind}(f, K) - \text{ind}(f_1, K)$). We suppose first that $\text{ind}(f, K) = n + \alpha$ for some nonnegative $n \in \mathbb{Z}$ and $0 < \alpha < 1$. That is, the vector $z - f(z)$ turns through $n$ full revolutions counterclockwise and $\alpha$ part of a revolution counterclockwise as $z$ varies from $x$ to $y$ in $K$. (See Figure 1 (left) for a case $n = 0$ and $\alpha$ about $\frac{2}{3}$.) Then as $z$ varies from $x$ to $y$, $f_1(z)$ goes along $S$ from $f(x)$ to $f(y)$ in the clockwise direction, so $z - f_1(z)$ turns through the angle $-(1 - \alpha) = \alpha - 1$. Hence, $\text{ind}(f_1, K) = -(1 - \alpha)$. It is easy to see that $\text{var}(f, K, S) = n + 1$ and $\text{var}(f_1, K, S) = 0$. Consequently,
\[ \text{var}(f, K, S) - \text{var}(f_1, K, S) = n + 1 - 0 = n + 1 \]
and
\[ \text{ind}(f, K) - \text{ind}(f_1, K) = n + \alpha - (\alpha - 1) = n + 1. \]
In Figure 1 on the left we assumed that $f(x) < x < y < f(y)$, The cases where $f(y) < x < y < f(x)$ and $f(x) = f(y)$ are treated similarly.

Thus when $n \geq 0$, in going from $f$ to $f_1$, the change in variation and the change in index are the same. However, in obtaining $f_1$ we have removed one $K \in \mathcal{K}$, reducing the minimal $m$ for $f_1$ by one, producing a counterexample for smaller $m$, a contradiction.

The cases where $\text{ind}(f, K) = n + \alpha$ for negative $n$ and $0 < \alpha < 1$ are handled similarly, and illustrated for $n = -2$ and $\alpha$ about $\frac{1}{2}$ in Figure 1 (right).
2.6. **Locating arcs of negative variation.** The principal tool in proving the main theorem in the next section is the following theorem first obtained by Bell. It provides a method for locating arcs of negative variation on a curve of index zero.

**Theorem 2.14** (Lollipop Lemma, Bell). Let $S \subset \mathbb{C}$ be a simple closed curve such that $f$ has no fixed points on $S$. Suppose $F = \{a_0 < \cdots < a_n < a_{n+1} < \cdots < a_m\}$ is a partition of $S$, $a_{m+1} = a_0$ and $I_i = [a_i, a_{i+1}]$ such that $f(F) \subset T(S)$ and $f(I_i) \cap A_i = \emptyset$ for $i = 0, \ldots, m$. Suppose $I$ is an arc in $T(S)$ meeting $S$ only at its endpoints $a_0$ and $a_{n+1}$. Let $J_{a_0}$ be a junction in $(\mathbb{C} \setminus T(S)) \cup \{a_0\}$ and suppose that $f(I) \cap (I \cup J_{a_0}) = \emptyset$. Let $R = T([a_0, a_{n+1}] \cup I)$ and $L = T([a_{n+1}, a_{m+1}] \cup I)$. Then one of the following holds

1. If $f(a_{n+1}) \in R$, then
   \[
   \sum_{i \leq n} \text{var}(f, I_i, S) + 1 = \text{ind}(f, I \cup [a_0, a_{n+1}]).
   \]

2. If $f(a_{n+1}) \in L$, then
   \[
   \sum_{i > n} \text{var}(f, I_i, S) + 1 = \text{ind}(f, I \cup [a_{n+1}, a_{m+1}]).
   \]

(Note that in (1) in effect we compute $\text{var}(f, \partial R)$ but technically, we have not defined $\text{var}(f, A_i, \partial R)$ since the endpoints of $A_i$ do not have to map inside $R$ but they do map into $T(S)$. Similarly in Case (2).)

**Proof.** Without loss of generality, suppose $f(a_{n+1}) \in L$. Let $C = [a_{n+1}, a_{m+1}] \cup I$ (so $T(C) = L$). We want to construct a map $f' : C \to \mathbb{C}$, fixed-point-free homotopic to $f|_C$, that does not change variation on any arc $A_i$ in $C$ and has the properties listed below.

1. $f'(a_i) \in L$ for all $n + 1 \leq i \leq m + 1$. Hence $\text{var}(f', A_i, C)$ is defined for each $i > n$.
2. $\text{var}(f', A_i, C) = \text{var}(f, A_i, S)$ for all $n + 1 \leq i \leq m$.
3. $\text{var}(f', I, C) = \text{var}(f, I, S) = 0$.
4. $\text{ind}(f', C) = \text{ind}(f, C)$.

Having such a map, it then follows from Theorem 2.13, that

\[
\text{ind}(f', C) = \sum_{i=n+1}^{m} \text{var}(f', A_i, C) + \text{var}(f', I, C) + 1.
\]

By Theorem 2.5 $\text{ind}(f', C) = \text{ind}(f, C)$. By (2) and (3), $\sum_{i > n} \text{var}(f', A_i, C) + \text{var}(f', I, C) = \sum_{i > n} \text{var}(f, A_i, S)$ and the Theorem would follow.

It remains to define the map $f' : C \to \mathbb{C}$ with the above properties. For each $i$ such that $n + 1 \leq i \leq m + 1$, chose an arc $I_i$ joining $f(a_i)$ to $L$ as follows:

(a) If $f(a_i) \in L$, let $I_i$ be the degenerate arc $\{a_i\}$.
(b) If \( f(a_i) \in R \) and \( n + 1 < i < m + 1 \), let \( I_i \) be an arc in \( R \setminus \{a_0, a_{n+1}\} \) joining \( f(a_i) \) to \( I \).

(c) If \( f(a_0) \in R \), let \( I_0 \) be an arc joining \( f(a_0) \) to \( L \) such that \( I_0 \cap (L \cup J_{a_0}) \subset A_{n+1} \setminus \{a_{n+1}\} \).

Let \( x_{n+1} = y_{n+1} = a_{n+1} \), \( y_0 = y_{m+1} \in I \setminus \{a_0, a_{n+1}\} \) and \( x_0 = x_{m+1} \in A_m \setminus \{a_m, a_{m+1}\} \). For \( n + 1 < i < m + 1 \), let \( x_i \in A_{i-1} \) and \( y_i \in A_i \) such that \( y_{i-1} < x_i < a_i < y_i < x_{i+1} \). For \( n + 1 < i < m + 1 \) let \( f'(a_i) \) be the endpoint of \( I_i \) in \( L \), \( f'(x_i) = f'(y_i) = f(a_i) \) and extend \( f' \) continuously from \([x_i, a_i] \cup [a_i, y_i] \) onto \( I_i \) and define \( f' \) from \([y_i, x_{i+1}] \subset A_i \) onto \( f(A_i) \) by \( f'|_{[y_i, x_{i+1}]} = f \circ h_i \), where \( h_i : [y_i, x_{i+1}] \to A_i \) is a homeomorphism such that \( h_i(y_i) = a_i \) and \( h_i(x_{i+1}) = a_{i+1} \). Similarly, define \( f' \) on \([y_0, a_{n+1}] \subset I \) to...
Theorem 2.5. $f$, Extensions to variation for infinite partitions.

2.7. $T$ crosscuts, and with the domains cut off so close pointwise to

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Corollary 2.15. following Corollary follows.

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$\text{var}(f, A, C) = 0$ and the

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Moreover, if $f$ has no fixed points in $T(S)$, it follows from Theorems 2.3

and 2.5, that $\text{ind}(f, C) = \text{ind}(f, C)$.

Note that if $f$ is fixed point free on $T(S)$, then $\text{ind}(f, S) = 0$ and the

following Corollary follows.

Corollary 2.15. Assume the hypotheses of Theorem 2.14. Suppose, in

addition, $f$ is fixed point free on $T(S)$. Then if $f(a_{n+1}) \in R$

there exists $i \leq n$ such that $\text{var}(f, A, S) < 0$. If $f(a_{n+1}) \in L$

there exists $i > n$ such that $\text{var}(f, A, S) < 0$.

2.7. Extensions to variation for infinite partitions. Recall our Stand-

ing Hypotheses in 1.1: $f : \mathbb{C} \rightarrow \mathbb{C}$ takes continuum $X$

into $T(X)$ with no fixed points in $T(X)$, and $X$ is minimal with respect
to these properties.

Definition 2.16 (Bumping Simple Closed Curve). A simple closed curve $S$
in $\mathbb{C}$ which has the property that $S \cap X$ is nondegenerate and $T(X) \subset T(S)$
is said to be a bumping simple closed curve for $X$. A subarc $A$ of a bumping
simple closed curve, whose endpoints lie in $X$, is said to be a bumping
(sub)arc for $X$. Moreover, if $S'$ is any bumping simple closed curve for $X$
which contains $A$, then $S'$ is said to complete $A$.

A crosscut of $O_\infty = \mathbb{C}_\infty \setminus T(X)$ is an open arc $Q$
lying in $O_\infty$ such that $Q$ meets $\partial O_\infty$ in two endpoints $a \neq b \in T(X)$. (As
seems to be traditional, we use “crosscut of $T(X)$” interchangeably with “crosscut
of $O_\infty$.”) If $S \cap X$ is nondegenerate and proper in $S$, then each component of $S \setminus X$ is a crosscut
of $T(X)$. A similar statement holds for a bumping arc $A$.

Since $f$ has no fixed points in $T(X)$ and $X$ is compact, we can choose a
bumping simple closed curve $S$ so close pointwise to $T(X)$, with such small
crosscuts, and with the domains cut off so close pointwise to $T(X)$, that
$f$ has no fixed points in $T(S)$. Thus, we obtain the following corollary to
Theorem 2.5.
Corollary 2.17. There is a bumping simple closed curve $S$ for $X$ such that $f|_{T(S)}$ is fixed point free; hence, by 2.5, $\text{ind}(f, S) = 0$. Moreover, any bumping simple closed curve $S'$ such that $S' \subset T(S)$ has $\text{ind}(f, S') = 0$. Further, any crosscut $Q$ of $X$ for which $f$ has no fixed points in $T(X \cup Q)$ can be completed to a bumping simple closed curve $S$ for which $\text{ind}(f, S) = 0$.

Theorem 2.18. Suppose $S$ is a bumping simple closed curve for $X$. Then there is a $\delta > 0$ such that if $A \subset S$ is a bumping subarc for $X$ with $\text{diam}(A) \leq \delta$, then $\text{var}(f, A, S) = 0$.

Proof. Suppose not. Then, without loss of generality, there is a sequence $\{A_i\}_{i=1}^{\infty}$ of bumping subarcs converging to a point $a \in X \cap S$ such that $\text{var}(f, A_i, S) \neq 0$ for each $i$. Let $J_a$ be a junction based at $a$. Since $f(a) \in X$, there are connected neighborhoods $U$ of $a$ and $V$ of $f(a)$ such that $\overline{V} \cap J_a = \emptyset$ and $f(U) \subset V$. We may assume $U \cap S$ is connected. Since $A_i \to a$, there is a $k$ such that for all $i \geq k$, $\overline{A_i} \subset U$. We may adjust the junction $J_a$ to a junction $J_{a_i}$, keeping sufficiently close to $S$, so that for $i \geq k$, $a_i \in A_i$ and $f(\overline{A_i}) \cap J_{a_i} = \emptyset$. It follows that $\text{var}(f, A_i, S) = 0$. This contradiction completes the proof.

Corollary 2.19. Suppose $S$ is a bumping simple closed curve for $X$. Let $C$ be closed such that $S \setminus C = \bigcup_{i=1}^{\infty} A_i$, where the $A_i$ are disjoint bumping subarcs (or crosscuts) such that $\overline{f(A_i)} \cap \overline{A_i} = \emptyset$ for each $i$. Then for all but finitely many $A_i$, $\text{var}(f, A_i, S) = 0$.

The following Theorem follows from 2.19 and the remark following Definition 2.11.

Theorem 2.20. Suppose $S$ is a bumping simple closed curve with $A$ a bumping subarc in $S$ such that $f(A) \cap A = \emptyset$. Suppose $A = \bigcup_{i \in I} A_i$ is a partition of $A$ into possibly infinitely many bumping subarcs. Then $\text{var}(f, A, S) = \sum_{i \in I} \text{var}(f, A_i, S)$.

Remark 2.21. It follows from Corollary 2.19 and Theorem 2.20 that Theorems 2.13 and 2.14 hold for infinite partitions of bumping simple closed curves where the partition elements map off themselves.

2.7.1. Variation on a crosscut. We show that variation is local by defining it for a single bumping subarc (or single crosscut).

Proposition 2.22. Suppose $A$ is a bumping subarc on $X$. If $\text{var}(f, A, S)$ is defined for some bumping simple closed curve $S$ completing $A$, then for any bumping simple closed curve $S'$ completing $A$, $\text{var}(f, A, S) = \text{var}(f, A, S')$.

Proof. Let $A$ be a bumping subarc on $X$ for which $f(A) \cap A = \emptyset$. Let $S$ and $S'$ be two bumping simple closed curves completing $A$ for which variation is defined. Let $J_a$ and $J_{a'}$ be junctions whereby $\text{var}(f, A, S)$ and $\text{var}(f, A, S')$ are respectively computed. Suppose first that both junctions lie (except for $\{a, a'\}$) in $C \setminus (T(S) \cup T(S'))$. By the Junction Straightening
Proposition 2.9 and Remark 2.4, either junction can be used to compute either variation on $A$, so the result follows. Otherwise, at least one junction is not in $C \setminus (T(S) \cup T(S'))$. Hence, we can find another simple closed curve $S''$ such that $S''$ completes $Q$ and both junctions lie in $(C \setminus T(S'')) \cup \{a, a'\}$. Then by the Proposition 2.9 and Remark 2.4, $\var(f, A, S) = \var(f, A, S'') = \var(f, A, S')$.

It follows from Proposition 2.22 that variation on a crosscut of $X$ is independent of the simple closed curve surrounding $T(X)$ of which $Q$ is a subarc.

**Definition 2.23** (Variation on a crosscut). Suppose $Q$ is a crosscut of $X$ such that $f(Q) \cap Q = \emptyset$. Let $S$ be any bumping simple closed curve completing $Q$ for which variation is defined. Define the variation of $f$ on $Q$ with respect to $X$, denoted $\var(f, Q, X)$, by $\var(f, Q, X) = \var(f, Q, S)$.

We will need the following proposition in Section 3.3.

**Proposition 2.24.** Suppose $Q = [a, b]$ is a crosscut of $T(X)$ such that $f$ is fixed point free on $T(X \cup Q)$ and $f(Q) \cap Q = \emptyset$. Suppose $Q$ is replaced by a bumping subarc $A$ with the same endpoints such that $Q$ separates $A \setminus \{a, b\}$ from $\infty$ and each component $Q_i$ of $A \setminus X$ is a crosscut such that $f(Q_i) \cap Q_i = \emptyset$. Then

$$\var(f, Q, X) = \sum_i \var(f, Q_i, X).$$

**Proof.** Note that each of $Q$ and $A$ can be completed to a simple closed curve with the same bumping arc $B$ such that on both $T(Q \cup B)$ and $T(A \cup B)$, $f$ is fixed point free. By Corollary 2.17 and Remark 2.21 we have

$$\var(f, Q \cup B) + 1 = \text{ind}(f, Q \cup B) = \text{ind}(f, A \cup B) = \var(f, A \cup B) + 1.$$  

Thus,

$$\var(f, Q, Q \cup B) + \var(f, B, Q \cup B) = \var(f, Q \cup B) = \var(f, A \cup B) = \var(f, A \cup B) + \var(f, B, A \cup B).$$

Consequently, by Theorem 2.20 and Proposition 2.22,

$$\var(f, Q, X) = \var(f, Q, Q \cup B) = \var(f, A, A \cup B)$$

$$= \sum_i \var(f, Q_i, A \cup B) = \sum_i \var(f, Q_i, X).$$

\[\Box\]

**2.8. Prime Ends.** Prime ends provide a way of studying the approaches to the boundary of a simply-connected plane domain with non-degenerate boundary. See [7] or [9] for an analytic summary of the topic and [12] for a more topological approach. We will be interested in the prime ends of $O_\infty = \mathbb{C}_\infty \setminus T(X)$. Recall $Y = T(X)$. Let $\Delta_\infty = \{z \in \mathbb{C}_\infty \mid |z| > 1\}$ be the “unit disk about $\infty$.” The Riemann Mapping Theorem guarantees the existence of a conformal map $\phi : \Delta_\infty \to O_\infty$ taking $\infty \to \infty$, unique up
to the argument of the derivative at $\infty$. Fix such a map $\phi$. We identify $S^1 = \partial\Delta_\infty$ with $\mathbb{R}/\mathbb{Z}$ and identify points $e^{2\pi i t}$ in $\partial\Delta_\infty$ by their argument (mod $2\pi$). Crosscuts are defined in Section 2.7.

**Definition 2.25** (Prime End). A chain of crosscuts is a sequence $\{Q_i\}_{i=1}^{\infty}$ of crosscuts of $O_\infty$ such that for $i \neq j$, $Q_i \cap Q_j = \emptyset$, $\text{diam}(Q_i) \to 0$, and for all $j > i$, $Q_i$ separates $Q_j$ from $\infty$ in $O_\infty$. Two chains of crosscuts are said to be equivalent if it is possible to form a sequence of crosscuts by selecting alternately a crosscut from each chain so that the resulting sequence of crosscuts is again a chain. A prime end $E$ is an equivalence class of chains of crosscuts.

If $\{Q_i\}$ is a chain of crosscuts of $O_\infty$, it can be shown that $\{\phi^{-1}(Q_i)\}$ is a chain of crosscuts of $\Delta_\infty$ converging to a single point $t \in S^1 = \partial\Delta_\infty$, independent of the representative chain. Thus, we may name the prime end $E$ defined by $\{Q_i\}$, where $\phi^{-1}(Q_i) \to t \in S^1$, by $E_t$.

Let $E_t$ be a prime end with defining chain of crosscuts $\{Q_i\}$. Let $O_t$ denote the bounded complementary domain of $O_\infty \setminus Q_i$. We use $\{Q_i\}$ and $\{O_t\}$ to define two subcontinua of $\partial O_\infty$ associated with $E_t$.

**Definition 2.26** (Impression and Principal Continuum). The set

$$\text{Im}(E_t) = \bigcap_{i=1}^{\infty} \overline{O_i}$$

is a subcontinuum of $\partial O_\infty$ called the impression of $E_t$. The set

$$\text{Pr}(E_t) = \{z \in \partial O_\infty \mid \text{for some chain } \{Q_i\} \text{ defining } E_t, Q_i \to z\}$$

is a continuum called the principal continuum of $E_t$.

For a prime end $E_t$, $\text{Pr}(E_t) \subset \text{Im}(E_t)$, possibly properly. We will be interested in the existence of prime ends $E_t$ for which $\text{Pr}(E_t) = \text{Im}(E_t) = \partial O_\infty$.

**Definition 2.27** (External Rays). Let $t \in [0, 1)$ and define

$$R_t = \{z \in \mathbb{C} \mid z = \phi(re^{2\pi it}), 1 < r < \infty\}.$$  

We call $R_t$ the external ray at $t$. If $x \in R_t$ then the $(Y, x)$-end of $R_t$ is the component $K_x$ of $R_t \setminus \{x\}$ whose closure meets $Y$.

The external rays foliate $O_\infty$.

**Definition 2.28** (Essential crossing). An external ray $R_t$ is said to cross a crosscut $Q$ essentially if and only if an $(Y, x)$-end of $R_t$ is contained in the bounded complementary domain of $Y \cup Q$.

The properties below may readily be established.

**Proposition 2.29** ([7]). Let $E_t$ be a prime end of $O_\infty$. Then $\text{Pr}(E_t) = \overline{R_t} \setminus R_t$. Moreover, for each $1 < r < \infty$ there is a crosscut $Q_r$ at $\phi(re^{2\pi it})$ on $R_t$ with $\text{diam}(Q_r) \to 0$ as $r \to 1$ and such that $R_t$ crosses $Q_r$ essentially.
Definition 2.30 (Landing Points and Accessible Points). If \( \Pr(\mathcal{E}_t) = \{x\} \), then we say \( R_t \) lands at \( x \in T(X) \) and \( x \) is the landing point of \( R_t \). A point \( x \in \partial T(X) \) is said to be accessible (from \( O_\infty \)) iff there is a arc in \( O_\infty \cup \{x\} \) one endpoint of which is \( x \).

Proposition 2.31. A point \( x \in \partial T(X) \) is accessible iff \( x \) is the landing point of some external ray \( R_t \).

Definition 2.32 (Channels). A prime end \( \mathcal{E}_t \) of \( O_\infty \) for which \( \Pr(\mathcal{E}_t) \) is nondegenerate is said to be a channel in \( \partial O_\infty \) (or in \( T(X) \)). If moreover \( \Pr(\mathcal{E}_t) = \partial O_\infty = \partial T(X) \), we say \( \mathcal{E}_t \) is a dense channel. A crosscut \( Q \) with endpoints \( \{a,b\} \) is said to cross the channel \( \mathcal{E}_t \) iff \( R_t \) crosses \( Q \) essentially.

When \( X \) is locally connected, there are no channels, as the following classical theorem proves. In this case, every prime end has degenerate principal set and degenerate impression.

Theorem 2.33 (Caratheodory). \( X \) is locally connected iff the Riemann map \( \phi : \Delta_\infty \to O_\infty = \mathbb{C}_\infty \setminus T(X) \) taking \( \infty \to \infty \) extends continuously to \( S^1 = \partial \Delta_\infty \).

2.9. Index and Variation for Caratheodory Loops. We extend the definitions of index and variation and the theorem relating index to variation to Caratheodory loops.

Definition 2.34 (Caratheodory Loop). Let \( \phi : S^1 \to \mathbb{C} \) such that \( \phi \) is continuous and has an extension \( \psi : \mathbb{C} \setminus T(S^1) \to \mathbb{C} \setminus T(\phi(S^1)) \) such that \( \psi|_{\mathbb{C} \setminus T(S^1)} \) is an orientation preserving homeomorphism from \( \mathbb{C} \setminus T(S^1) \) onto \( \mathbb{C} \setminus T(\phi(S^1)) \). We call \( \phi \) (and loosely, \( S = \phi(S^1) \)), a Caratheodory loop.

In particular, if a Riemann map extends continuously to \( S^1 \), we have a Caratheodory loop. In order to define variation of \( f \) on a Caratheodory loop \( S = \phi(S^1) \), we do the partitioning in \( S^1 \) and transport it to the Caratheodory loop \( S = \phi(S^1) \). An allowable partition of \( S^1 \) is a set \( \{a_0 < a_1 < \cdots < a_n\} \) in \( S^1 \) ordered counterclockwise, where \( a_0 = a_n \) and \( A_i \) denotes the counterclockwise interval \([a_{i-1}, a_i]\), such that for each \( i \), \( f(\phi(a_i)) \in T(\phi(S^1)) \) and \( f(\phi(A_i)) \cap \phi(A_i) = \emptyset \). Variation on each path \( \phi(A_i) \) is then defined exactly as in Definition 2.7, except that the junction (see Definition 2.6) is chosen so that the vertex \( v \in \phi(A_i) \) and \( T(\phi(S^1)) \subset h(U) \cup \{v\} \), and the crossings of the junction by \( f(\phi(A_i)) \) are counted (see Definition 2.7). Variation on the whole loop, or an allowable subarc thereof, is defined just as in Definition 2.11, by adding the variations on the partition elements. At this point in the development, variation is defined only relative to the given allowable partition \( F \) of \( S^1 \) and the parameterization \( \phi \) of \( S \); \( \text{var}(f,F,\phi(S^1)) \).

Index on a Caratheodory loop \( S \) is defined exactly as in Section 2.1 with \( S = \phi(S^1) \) providing the parameterization of \( S \). Likewise, the definition of fractional index and Proposition 2.1 apply to Caratheodory loops.

Theorems 2.2, 2.3, Corollary 2.4, and Theorem 2.5 apply to Caratheodory loops. It follows that index on a Caratheodory loop \( S \) is independent of
the choice of parameterization $\phi$. It remains to extend Theorem 2.13 to Caratheodory loops. It then follows that variation on a Caratheodory loop $S$ is independent of choice of parameterization $\phi(S^1) = S$ and allowable partition of $S^1$. Thus, $\text{var}(f, S)$ is well-defined for any Caratheodory loop $S$ that has some parameterization and some allowable partition.

**Theorem 2.35.** Suppose $S = \phi(S^1)$ is a parameterized Caratheodory loop in $\mathbb{C}$ and $f$ has no fixed points on $S$. Suppose variation of $f$ on $S^1 = A_0 \cup \cdots \cup A_n$ with respect to $\phi$ is defined for some partition $A_0 \cup \cdots \cup A_n$ of $S^1$. Then

$$\text{ind}(f, \phi) = \sum_{i=0}^{n} \text{var}(f, A_i, \phi(S^1)) + 1.$$  

**Proof.** Let $\psi$ be the homeomorphic extension of $\phi$ carrying $\mathbb{C} \setminus T(S^1)$ onto $\mathbb{C} \setminus T(S)$. Let $S_i = \{1 + \frac{1}{r} e^{2\pi i \theta} \mid \theta \in [0, 1)\}$ be the concentric circles of radius $1 + \frac{1}{r}$ converging to $S^1$. For the given partition $A_0 \cup \cdots \cup A_n$ of $S^1$, let $A_j = [a_{j-1}, a_j]$ with $a_n = a_0$. Then $a_j = e^{2\pi i \theta_j}$ for some $\theta_j \in [0, 1)$ with $\theta_0 < \theta_1 < \cdots < \theta_n = \theta_0$. For each $j$, let $R_{i,j} = \{re^{2\pi i \theta} \mid 1 \leq r \leq 1 + \frac{1}{r}\}$ be the radial arc from $S_i$ to $S^1$ at $a_j$. Note $\text{diam}(R_{i,j}) \to 0$ as $i \to \infty$.

Let $C_{i,j} = \{(1 + \frac{1}{r}) e^{2\pi i \theta} \mid \theta_{j-1} \leq \theta \leq \theta_j\}$ be the subarc of $S_i$ between $c_{i,j-1} = S_i \cap R_{i,j-1}$ and $c_{i,j} = S_i \cap R_{i,j}$. Then $C_{i,j} = [c_{i,j-1}, c_{i,j}]$ approximates $A_j$ as $i \to \infty$. Moreover, $c_{i,j} \to a_i$ as $i \to \infty$. Since $\psi$ is a map, the same holds for the images.

For each $j$, choose a junction $J_v$, with vertex $v_j \in \psi(A_j)$ so that $T(\psi(S^1)) \subset U_j \cup \{v_j\}$, where $U_j$ is the usual complementary half-plane of the junction (see Definition 2.6).

Since $\psi(C_{i,j})$ approximates $\psi(A_j)$ with $\psi(R_{i,j})$ shrinking as $i \to \infty$, we may choose $i$ sufficiently large so that the following conditions are satisfied:

1. $\psi(C_{i,j}) \cap f(\psi(C_{i,j})) = \emptyset$.
2. $\text{var}(f, A_j, \phi(S^1)) = \text{var}(f, \psi(C_{i,j}), \psi(S_i))$.
3. There are no fixed points of $f$ in $T(\psi(S_i) \setminus T(S^1))$.
4. $\text{ind}(f, \psi(S_i)) = \text{ind}(f, \psi(S^1))$.

Condition (1) holds because of continuity of $\psi$ and the similar condition for $A_j$. To see condition (2), apply the observations about the stability of variation in Section 2.4. Condition (3) holds because there are no fixed points of $f$ on $S$, and $\psi(S_i)$ approximates $S$. Condition (4) then follows from the stability of index under fixed-point-free homotopy, noted in Section 2.2. (Use $\psi$ on $S_k$, $k \geq i$, to define the homotopy.)

By Theorem 2.13,

$$\text{ind}(f, \psi(S_i)) = \sum_{j=0}^{n} \text{var}(f, \psi(C_{i,j}), \psi(S_i)) + 1.$$
Hence, noting $\phi = \psi_{S_1}$, it follows from conditions (2) and (4) that
\[
\text{ind}(f, \phi) = \sum_{j=0}^{n} \text{var}(f, A_j, \phi(S_1)) + 1.
\]

\end{proof}

3. Geometric Prime Ends

For the proof of the principal new result in this paper, the uniqueness of the outchannel in the next section, we develop in this section Bell’s tools using a special collection of geometric crosscuts of $T(X)$. These results, and their connection to the standard conformal theory of prime-ends, are more fully developed in [1]. Recall our Standing Hypotheses in 1.1: $f : \mathbb{C} \to \mathbb{C}$ takes continuum $X$ into $Y = T(X)$ with no fixed points in $T(X)$, and $X$ is minimal with respect to these properties.

3.1. Geometric crosscuts. We are going to define a special class of geometric crosscuts (chords) of $O_\infty = \mathbb{C}_\infty \setminus Y$ and auxiliary nonseparating plane continua which contain $Y$ as subsets, and in some sense have some of the same channels as $Y$, but have a nicer boundary than $Y$.

**Definition 3.1** (Closest Points). Let $z \in O_\infty$. We define the set of closest points of $Y$ to $z$ by
\[
C(z) = \{ x \in Y \mid d(z, x) = d(z, Y) \}.
\]
The collection of all $z \in O_\infty$ with at least three closest points is denoted $C_3$; the collection of all $z \in O_\infty$ with exactly two closest points is denoted $C_2$.

**Definition 3.2.** For a given compact subset $Z \subset \mathbb{C}$, the convex hull of $Z$, denoted by Convexhull($Z$), consists of $Z$ together with all straight line segments whose endpoints are in $Z$. Then Convexhull($Z$) is a convex continuum. For $z \in \mathbb{C} \setminus Y$, $V(z) = \text{Convexhull}(C(z)) \setminus C(z)$ if $z \in C_2$ and $V(z) = \text{Int}($Convexhull($C(z)$)) if $z \in C_3$. If $z \in C_2$, we call $V(z)$ a chord. We call $V(z)$ for $z \in C_3$ a simplex.

Note that each simplex of $Y$ lies in the convex hull of $Y$. Note also that for $z \in C_2$, $V(z)$ is an open arc and for $z \in C_3$, $V(z)$ is an open topological disk with Jordan curve boundary composed of countably many straight line crosscuts plus a set contained in $C(z)$.

**Definition 3.3.** For $z \in O_\infty$, let $B(z, Y) = \{ w \in \mathbb{C} \mid d(z, w) < d(z, Y) \}$ and $S(z, Y) = \{ w \in \mathbb{C} \mid d(w, z) = d(z, Y) \}$. Then $B(z, Y)$ is the maximal open ball around $z$, disjoint from $Y$ and its boundary is $S(z, Y)$. Moreover, if $z \in Y$, let $S(z, Y) = \{ z \}$.

**Definition 3.4** (Geometric Crosscuts–Chords). Define the following collections of crosscuts and simplices:
\[
\mathcal{F}_s = \{ L \mid L \text{ is a component of } \partial V(z) \setminus C(z) \text{ for some } z \in C_3 \}\]
\[ \mathcal{F}_c = \{ K \mid K \text{ is a component of } \partial \text{Convexhull}(Y) \setminus Y \} \]

\[ \mathcal{F}_2 = \{ V(z) \mid z \in C_2 \} \]

\[ \mathcal{F}_3 = \{ V(z) \mid z \in C_3 \} \]

\[ \mathcal{F} = \mathcal{F}_n \cup \mathcal{F}_c \cup \mathcal{F}_2 \cup \mathcal{F}_3 \]

\[ \mathcal{G} = \mathcal{F}_2 \cup \mathcal{F}_c \cup \mathcal{F}_e \]

\[ \mathcal{G}_\delta = \{ Q \in \mathcal{G} \mid \text{diam}(Q) \leq \delta \}. \]

We call the elements of \( \mathcal{G} \) chords. Note that every element of \( \mathcal{G} \) and every element of \( \mathcal{G}_\delta \) is a straight line segment, while \( \mathcal{F} \) also contains simplices.

**Lemma 3.5.** Let \( z_1 \neq z_2 \in C_2 \cup C_3 \). Then either \( \overline{V(z_1)} \cap \overline{V(z_2)} \) is the empty set, or is the closure of an element \( Q \in \mathcal{G} \), or \( V(z_1) \cap V(z_2) \) is a single point in \( Y \). In particular, members of \( \mathcal{F} \) are either equal or disjoint.

**Proof.** Clearly \( V(z) \subset B(z, Y) \) for each \( z \in \mathbb{C} \). If \( S(z_1, Y) \cap S(z_2, Y) = y \) is a single point, then this point clearly is in \( Y \) and \( \overline{V(z_1)} \cap \overline{V(z_2)} = y \) is a single point in \( Y \). Hence, we may assume that \( z_1 \neq z_2 \in O_\infty \) such that \( B(z_1, Y) \cap B(z_2, Y) \neq \emptyset \). Then \( Y \cap [B(z_1, Y) \cup B(z_2, Y)] = \emptyset \) and, since \( \{z_1, z_2\} \subset C_2 \cup C_3 \), \( S(z_1, Y) \cap S(z_2, Y) = \{a, b\} \), with \( a \neq b \). Then it is easy to see that \( V(z_1) \subset K_1 = \overline{\text{Convexhull}(B(z_1, Y) \setminus B(z_2, Y))} \) and \( V(z_2) \subset K_2 = \overline{\text{Convexhull}(B(z_2, Y) \setminus B(z_1, Y))} \). Since \( K_1 \cap K_2 = [a, b] \), the desired result follows. \( \square \)

If \( \{H_i\} \) is a sequence in \( \mathcal{F} \) we say that the sequence \( \{H_i\} \) converges if the sequence \( \{\overline{H_i}\} \) converges in the space of compacta of \( \mathbb{C} \) with the Hausdorff metric topology. If the \( H_i \) are distinct and the limit exists, the limit is either a point of the continuum \( X \) or the closure of a chord \( Q \in \mathcal{G} \).

The proof of the following result is left to the reader. The last part follows from stability of variation (see Section 2.4).

**Proposition 3.6** (Compactness). If \( \{Q_i\} \) is a convergent sequence of chords in \( \mathcal{G}_\delta \) or of distinct elements of \( \mathcal{F}_3 \) of diameter less than or equal to \( \delta \), then either \( Q_i \) converges to a chord in \( \mathcal{G}_\delta \) or \( Q_i \) converges to a point of \( X \). Moreover, if \( \{Q_i\} \) converges to a chord \( Q \), then for sufficiently large \( i \),

\[ \text{var}(f, Q, Y) = \text{var}(f, Q_i, Y) \]

**Corollary 3.7.** For each \( \varepsilon > 0 \), there exist \( \delta > 0 \) such that for all \( Q \in \mathcal{G} \) with \( Q \subset B(Y, \delta) \), \( \text{diam}(Q) < \varepsilon \).

**Proof.** Suppose not, then there exist \( \varepsilon > 0 \) and a sequence \( Q_i \) in \( \mathcal{G} \) such that \( \lim Q_i \subseteq X \) and \( \text{diam}(Q_i) \geq \varepsilon \) a contradiction to Proposition 3.6. \( \square \)

The proof of the following proposition is left to the reader:

**Proposition 3.8.** For each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for each open arc \( A \) such that \( \overline{A} \cap Y = \{a, b\} \), with \( a \neq b \), and \( \text{diam}(A) < \delta \), \( T(Y \cup A) \subset B(Y, \varepsilon) \).
Proposition 3.9. Let $\varepsilon$, $\delta$ be as in Proposition 3.8 above with $\delta < \varepsilon/2$ and let $A$ be a crosscut of $Y$ such that $\text{diam}(A) < \delta$. Let $x \in T(A \cup Y) \cap V(z)$ for some $z \in C \setminus Y$ such that $d(x, A) \geq \varepsilon$, then $\text{diam}(V(z)) < 2\varepsilon$.

Proof. Suppose that $cl A$ is an arc from $a$ to $b$. If $z \in T(A \cup Y)$, then by Proposition 3.8, $d(z, Y) < \varepsilon$ and $\text{diam}(V(z)) < 2\varepsilon$. Hence we may assume that $z \notin T(A \cup Y)$. Then the straight line segment $B$ from $x$ to $z$ must cross $A$ at some point $w$. Hence $d(x, w) \geq \varepsilon$ and $d(x, z) = d(x, w) + d(w, z) > d(a, z)$, a contradiction since $B(z, d(z, Y)) \cap Y = \emptyset$. \hfill $\square$

Proposition 3.10. Let $C \subset \text{Convexhull}(Y)$ be a crosscut of $Y$ and let $A$ and $B$ be disjoint closed sets in $Y$ such that $\overline{C} \cap A \neq \emptyset \neq \overline{C} \cap B$ and for each $x \in C$ there exists $F_x \in \mathcal{F}$ such that $x \in F_x$ and $F_x \cap (A \cup B) \neq \emptyset$. Then there exists $F \in \mathcal{F}$ such that $F \cap \overline{C} \neq \emptyset$, $F \cap A \neq \emptyset$ and $F \cap B \neq \emptyset$.

Proof. The proof follows from Proposition 3.6, the connectedness of $C$ and the fact that $\overline{C}$ meets both $A$ and $B$. \hfill $\square$

Proposition 3.10 allows us to replace small crosscuts which cross the prime end $\mathcal{E}_t$ with non-trivial principal continuum essentially by small nearby chords which also cross $\mathcal{E}_t$ essentially. For if $C$ is a small crosscut in $\text{Convexhull}(Y)$ with endpoints $a$ and $b$ which crosses $R_t$ essentially, let $A$ and $B$ be the closures of the sets in $Y$ accessible from $a$ and $b$, respectively by small arcs missing $R_t$.

The following result of Bell’s is the linchpin of the theory of chords.

Theorem 3.11 (Geometric Foliation, Bell). The collection $\mathcal{F}$ foliates $\text{Convexhull}(Y) \setminus Y$.

Proof. Suppose that $K$ is a component of $\text{Convexhull}(Y) \setminus \bigcup \mathcal{F} \cup Y$. Then $K$ meets no $V(z)$.

Claim 1. $\partial K$ contains a geometric cross cut $Q = (a, b)$ of $Y$ and $K$ is open.

Proof of Claim 1. Since $Y$ is bounded, $\bigcup \mathcal{F} \cup Y$ is closed by 3.6 and $K$ is open. Hence $\partial K \subset C \cup Y$. Since $Y$ does not separate the plane, there exists $Q \in \mathcal{G}$ and a point $q \in Q \cap \partial K$. Without loss of generality $Q$ is a subset of the $x$-axis and there exists a sequence $q_i \in K$ converging to $q$ such that each $q_i$ is contained in the upper half plane $W$. We claim that there exists $r > 0$ such that $B(q, r) \cap W \subset K$. If not, then for each $i$ there exists $s_i \in W \cap B(q, 1/i) \cap \partial K$. Then, for $i$ large, $s_i \in C \setminus Y$ and, hence, there exists $Q_i = (a_i, b_i) \in \mathcal{G}$ with $s_i \in Q_i$. By 3.5, $(a, b) \cap (a_i, b_i) = \emptyset$. It follows from the definition of geometric cross cuts that $\lim Q_i = Q$ Hence there exist $i, j, k$ and an arc $M \subset C \setminus Y$ joining $q_i$ and $q_j$ which intersects $Q_k$ exactly once. It follows that $q_i$ and $q_j$ are in distinct components of $C \setminus Y \cup Q_k$, a contradiction. Thus $\{x \in (a, b) \mid x \in \partial K\}$ is open in $(a, b)$. However, it is also closed in $(a, b)$ and $(a, b) \subset \partial K$.

Claim 2. $K$ is convex.
Proof of Claim 2. Since $Y \cup \bigcup F$ is connected, it follows that $K$ is simply connected. Suppose that $K$ is not convex. Then there exist $p \neq q \in K$ and $x \in (p, q) \subset L$, where $L$ is the straight line through $p$ and $q$, such that $x \in \partial K$. Without loss of generality, $L$ is the $x$-axis. Let $A$ be an arc from $p$ to $q$ in $K$ and let $A'$ be an irreducible open subarc of $A$ which meets two components of $K \setminus \{x\}$. We may assume that $A = A'$ and $A$ is contained in the upper half plane. Let $W$ be the bounded complementary domain of the simple closed curve $A \cup (p, q)$. Note that $Y \cap W \neq \emptyset$. For if $Y \cap W = \emptyset$, then $x \in (c, d) \subset G$, where $(c, d) \subset \mathbb{C} \setminus L$ with one of the endpoints, say $c$, contained in the upper half plane. Since $(c, d) \cap A = \emptyset$, $c \in Y \cap W$.

Choose $y \in Y \cap W$ such that the second coordinate of $y$ is maximal. Note that $B(z, d(z, y) \cap Y = \emptyset$ for $z$ directly above and very close to $y$. Let $z$ be a point directly above $y$ such that $B(z, d(z, y) \cap Y = \emptyset$ and $d(z, y)$ is maximal for all such points. Since there exists points in $\partial K$ higher than $y$, such a maximal ball must exist. Then there exist $y' \in S((z, y)) \setminus \{y\} \neq \emptyset$, and $(y, y') \in G$. Since $y$ is the highest point of $Y$ in $W$, $y' \not\in W$. Hence $(y, y') \cap A \neq \emptyset$, a contradiction. This completes the proof of Claim 2.

Claim 3. If $Q = (a, b) \in \partial K$ and $Q \in G$, then $Q \in F_2 \cup F_s$.

Proof of Claim 3. Suppose not, then $Q \in F_c$. Please see figure 3 for a geometric picture of the notations defined below. Let $L$ be the straight line containing $Q$ and let $L_p$ be the perpendicular bisector of $Q$. Let $c = L \cap L_p$ and choose $d \in L_p \setminus L$. Since $K$ is convex, $(c, d) \subset K$. We may assume that $L_p$ is the $x$-axis and $L$ is the $y$-axis. Then $L$ divides the plane into a left half plane $H$ and a right half plane $H_K$ and we may assume $Y \subset \overline{H_K}$ since $Q \in F_c$. Let $H_a$ and $H_b$ be the components of $\mathbb{C} \setminus L_p$ which contain $a$ and $b$, respectively. Let $L_a \subset H$ and $L_b \subset \overline{H}$ be the half lines parallel to $L_p$ and containing $a$ and $b$, respectively. Then $C(x) = \{a\}$ for each $x \in L_a$ and $C(x) = \{b\}$ for each $x \in L_b$. Let $E = \{x \in \overline{H} \mid d(x, \overline{T_a \cap Y}) = d(x, \overline{T_b \cap Y})\}$. Then $E$ is closed and $E$ separates $L_a$ from $L_b$. Hence $E$ contains an unbounded sequence $e_i$ which is located between the half lines $L_a$ and $L_b$. For each $i$ choose $a_i \in S(e_i, Y) \cap \overline{T_a}$ and $b_i \in S(e_i, Y) \cap \overline{T_b}$. Then $Q_i = (a_i, b_i) \subset G$ and $\lim a_i = a$ and $\lim b_i = b$. Note that since $Q \notin F_2 \cup F_s$, $\{a, b\} \setminus S((z, Y)) \neq \emptyset$ for each $z \in \mathbb{C} \setminus Y$. In particular, $\{a_i, b_i\} \setminus \{a, b\} \neq \emptyset$ for each $i$. Hence $Q_i \subset H_K$ for each $i$ and $Q_i \cap (c, d) \neq \emptyset$ for $i$ large, a contradiction. This completes the proof of Claim 3.

We will continue to use the notation used in the proof of Claim 3 and illustrated in figure 3 with the exception that it is no longer necessarily true that $Y \subset \overline{H_K}$. Since $K$ is convex, we may assume that $K \subset H_K$. By Claim 3, there exists $Q = (a, b) \in F_2 \cup F_s$ such that $Q \subset \partial K$. Then there exists $e \in L_p$ such that $\{a, b\} \subset S(e, Y)$. Suppose first that there exists $e \in (L_p \cap H_K) \cup \{c\}$ such that $\{a, b\} \subset S(e, Y)$. If $S(e, Y)$ meets $H_K$, it follows that $K \supset V(e)$, a contradiction. Hence $S(e, Y) \cap H_K = \emptyset$. As in the proof of Claim 2, we may assume that the first coordinate of $e$ is maximal.

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Let \( T \) be the (possibly degenerate) triangle with vertices \( a, b \) and \( e \). Let \( \varepsilon \) be sufficiently small such that any arc which is not contained in the \( y \)-axis and joins a point of \( B(a, \varepsilon) \cap H_K \) to any point of \( B(b, \varepsilon) \cap H_K \), meets \((c, d) \subset K\). Since \( S(e, Y) \cap H_K = \emptyset \), there exists \( \delta > 0 \) such that for each \( w \in B(e, \delta), S(w, Y) \cap H_K \subset B(a, \varepsilon) \cup B(b, \varepsilon) \). Clearly, for all \( p \) in the intersection of the straight line segment \((e, a)\) and \( B(e, \delta) \) which separates \( \{a, b\} \subset S(e, Y) \) and the first coordinate of \( e \) was maximal with respect to the property that \( \{a, b\} \subset S(e, Y) \), either \( a_x \neq a \) or \( b_x \neq b \). Hence \( Q_x = (a_x, b_x) \in \mathcal{G} \) is not contained in the \( y \)-axis. By the choice of \( \varepsilon, Q_x \cap (c, d) \neq \emptyset \) and \( Q_x \) meets \( K \), a contradiction.

Suppose next that for all \( e \in L_p \cap H_K, \{a, b\} \setminus S(e, Y) \neq \emptyset \). Then there exists \( e \in L_p \cap H \) such that \( \{a, b\} \subset S(e, Y) \) and the first coordinate of
proof. Suppose first that \( Q \) must land on a point of \( \partial V \). Hence

\[
\text{of generality } \lim_{y \to \infty} = \emptyset
\]

is not defined by \( Q \). By proposition 3.6, \( \lim_{x} Q \) is the least element of \( Q \) is non-degenerate and choose \( x, y \) must land on \( t \in S^1 = \partial \Delta_\infty \). Then for sufficiently large \( i \), \( \var(f, Q_i, Y) = 0 \). 

Proof. Since \( f \) is fixed point free on \( Y \) and \( f(x) \in Y \), we may choose a connected neighborhood \( W \) of \( x \) such that \( f(W) \cap (W \cup R_t) = \emptyset \). For sufficiently large \( i \), \( Q_i \subset W \). For each such \( i \), let \( J_i \) be a junction starting from a point in \( Q_i \), staying in \( W \) until it reaches \( R_t \), then following \( R_t \) to \( \infty \). By our choice of \( W \), \( \var(f, Q_i, Y) = 0 \). 

Proposition 3.13. If \( R_t \) is an external ray of \( Y \). Then one of the following must hold:

1. \( R_t \) lands on a point of \( Y \cap \partial \text{Convexhull}(Y) \),
2. There exist \( z \in C_2 \cup C_3 \) such that \( R_t \) lands on a point of \( S(z, Y) \),
3. There is a defining sequence \( Q_i \) of chords for \( R_t \).

Proof. Let \( R_t \) be an external ray and let

\[
Q = \{ Q \in G \mid R_t \text{ crosses } Q \text{ essentially} \}.
\]

Suppose first that \( Q = \emptyset \). If there is a \( (Y, x) \)-end of \( R_t \) in \( Y \backslash \text{Convexhull}(Y) \), then there exists \( y \in \overline{R_t} \cap Y \cap \partial \text{Convexhull}(Y) \). Since \( y \) is accessible, \( R_t \) must land on \( y \) and \( (1) \) holds. If not, there is a cofinal sequence \( \{ x_i \} \) in \( R_t \) such that \( x_i \in \partial \text{Convexhull}(Y) \). Let \( y \in \limsup x_i \). As above, \( R_t \) lands on \( y \) and \( (1) \) holds.

Suppose, therefore, that \( Q \neq \emptyset \). Then there exist a partial order on \( Q \) defined by \( Q_1 \prec Q_2 \) if \( Q_1 \) is contained in the bounded complementary domain of \( Y \cup Q_2 \). Let \( Q_i \) be a cofinal decreasing sequence in \( Q \). Without loss of generality \( \lim_{x} Q_i = Q \) in the hyperspace of subcontinua of \( \text{Convexhull}(Y) \). By proposition 3.6, \( Q \) is either a point \( y \) of \( Y \), or \( Q = [a, b] \), where \( (a, b) \in G \) is a chord. If \( Q \) is a singleton \( y \), then \( (3) \) holds.

Suppose \( Q \) is non-degenerate and choose \( x_i \in R_t \cap Q_i \) such that the \( (Y, x_i) \)-end of \( R_t \) is contained in the bounded complementary domain of \( Y \cup Q_i \). If \( Q \notin Q_i \), then it follows that \( e \in \overline{R_t} \) for some \( e \in \{a, b\} \). In this case, as above, the ray \( R_t \) must land on \( e \) and \( (2) \) holds. We suppose, therefore, that \( Q \in Q \) is the least element of \( Q \). Let \( x \in R_t \cap Q \) such that the \( (Y, x) \)-end \( K_x \) of \( R_t \) is contained in the bounded complementary domain of \( Y \cup Q \). If \( \{Q'_i\} \) were a sequence in \( G \) converging to \( Q \) such that \( Q'_i < Q \) for each \( i \), \( R_t \) would cross \( Q'_i \) essentially for large \( i \), contradicting the minimality of \( Q \). Hence \( Q \subset \partial V(z) \) for some \( z \in C_3 \). As in the case \( Q = \emptyset \), it follows that \( R_t \) must land on a point of \( \partial V(z) \cap Y \) and \( (2) \) holds. 

\( \square \)
Corollary 3.14. Let $E_t$ be a channel in $Y$ such that $Pr(E_t)$ is non-degenerate. Then for each $x \in Pr(E_t)$, for every $\delta > 0$, there is a chain $\{Q_i\}_{i=1}^{\infty}$ of chords defining $E_t$ selected from $\mathcal{G}_\delta$ with $Q_i \to x \in \partial Y$.

Proof. Let $x \in Pr(E_t)$ and let $\{C_i\}$ be a defining chain of crosscuts for $Pr(E_t)$ with $\{x\} = \lim C_i$. By the remark following Proposition 3.10, there is a sequence $\{Q_i\}$ of chords such that $Q_i \cap C_i \neq \emptyset$ and $Pr(E_t)$ crosses each $Q_i$ essentially. By Proposition 3.9, the sequence $Q_i$ converges to $\{x\}$. \hfill \square

Lemma 3.15. Suppose an external ray $R_t$ lands on $a \in Y$ with $\{a\} = Pr(E_t) \neq Im(E_t)$. Suppose $\{x_i\}_{i=1}^{\infty}$ is a collection of points in $O_\infty$ with $x_i \to x \in Im(E_t) \setminus \{a\}$ and $\phi^{-1}(x_i) \to t$. Then for sufficiently large $i$, there is a sequence of chords $\{Q_i\}_{i=1}^{\infty}$ such that $Q_i$ separates $x_i$ from $\infty$, $Q_i \to a$ and $\phi^{-1}(Q_i) \to t$.

Proof. The existence of the chords $Q_i$ follows from the remark following Proposition 3.10. It is easy to see that $\lim \phi^{-1}(Q_i) = t$. \hfill \square

3.2. Auxiliary Continua.

Definition 3.16. Fix $\delta > 0$. Define the following collections of chords:

$\mathcal{G}_\delta^+ = \{Q \in \mathcal{G}_\delta | \text{var}(f, Q, Y) \geq 0\}$

$\mathcal{G}_\delta^- = \{Q \in \mathcal{G}_\delta | \text{var}(f, Q, Y) \leq 0\}$

To each collection of chords above, there corresponds an auxiliary continuum defined as follows:

$Y_\delta = T(Y \cup (\cup \mathcal{G}_\delta))$

$Y_\delta^+ = T(Y \cup (\cup \mathcal{G}_\delta^+))$

$Y_\delta^- = T(Y \cup (\cup \mathcal{G}_\delta^-))$

Proposition 3.17. Let $Z \in \{Y_\delta, Y_\delta^+, Y_\delta^-, Y_\delta^{-}\}$, and correspondingly $\mathcal{W} \in \{\mathcal{G}_\delta, \mathcal{G}_\delta^+, \mathcal{G}_\delta^-\}$. Then the following hold:

1. $Z$ is a nonseparating plane continuum.
2. $\partial Z \subset Y \cup (\cup \mathcal{W})$.
3. Every accessible point $p$ in $\partial Z$ is either a point of $Y$ or a point interior to a chord $A \in \mathcal{W}$.
4. If $p$ is an accessible point of $\partial Z$ and in the interior of the chord $A \in \mathcal{W}$, then every point of $A$ is accessible in $\partial Z$.

Proof. By Proposition 3.6, $Y \cup (\cup \mathcal{W})$ is compact. Moreover, $Y$ is connected and each crosscut $A \in \mathcal{W}$ has endpoints in $Y$. Hence, the topological hull $T(Y \cup (\cup \mathcal{W}))$ is a nonseparating plane continuum, establishing (1).

Since $Z$ is the topological hull of $Y \cup (\cup \mathcal{W})$, no boundary points can be in complementary domains of $Y \cup (\cup \mathcal{W})$. Hence, $\partial Z \subset Y \cup (\cup \mathcal{W})$, establishing (2). Conclusion (3) follows immediately.

To prove (4), suppose $p \in A$ is an accessible point of $\partial Z$ in the interior of a chord $A \in \mathcal{W}$. By Proposition 3.11, $\mathcal{F}$ foliates ConvexHull$(Y) \setminus Y$. Since $\mathcal{W} \subset \mathcal{G}$, if $p$ is accessible, then no other crosscut $B \in \mathcal{W}$ separates $p$ from $\infty$. 
By Proposition 3.6, every point of \( A \) is accessible (by moving along parallel to \( A \) left and right of \( p \)). □

**Proposition 3.18.** \( Y_\delta \) is locally connected; hence, \( \partial Y_\delta \) is a Carathéodory loop.

**Proof.** Suppose that \( Y_\delta \) is not locally connected. Then there exists \( k \in Y_\delta \) and \( \varepsilon, 0 < \varepsilon < \delta / 3 \), and distinct components \( K_i \) of \( B(k, 2\varepsilon) \) \( \cap Y_\delta \) with \( k \in \lim K_i \subset K_0 \), where \( K_0 \) is the component containing \( k \). We also suppose that \( K_i \cap \partial B(k, \varepsilon) \) is not connected and \( K_{i+1} \) separates \( K_i \) from \( K_0 \) for each \( i > 0 \). Since \( Y_\delta \) is a non-separating continuum, there exist \( x_i \in [\mathbb{C} \setminus Y_\delta] \cap \partial B(k, \varepsilon) \) and if \( R_i \) is a ray joining \( x_i \) to infinity in \( \mathbb{C} \setminus Y_\delta \), then \( R_i \) separates \( K_i \) from \( K_0 \cup K_{i+1} \cup \ldots \) in \( B(k, \varepsilon) \). We may assume that each \( K_i \) meets \( B(k, \varepsilon / 3) \). Let \( A_i \subset B(k, \varepsilon / 3) \) be an irreducible arc from \( R_i \) to \( R_{i+1} \) and let \( U_i \) be the component of \( \mathbb{C} \setminus [R_i \cup A_i \cup R_{i+1}] \) which does not contain \( k \). Then \( U_i \cap K_{i+1} \cap \partial B(k, \varepsilon) \neq \emptyset \) for each \( i \). Since \( K_i \subset Y_\delta \) is \( T = (Y \cup \partial Y_\delta) \) it follows that \( U_i \cap Y \setminus B(k, \varepsilon) = \emptyset \). Since \( Y \) is a continuum, there exists continua \( L_i \subset Y \), such that \( L_i \) separates \( R_i \) from \( R_{i+1} \) in \( B(k, \varepsilon) \).

Let \( B(k, \varepsilon) \setminus R_i = B_i \cup C_i \) be separated sets such that \( K_i+1 \subset B_i \) and \( K_i \subset C_i \). Let \( M_i = Y \cap B_i \) and \( N_i = Y \cap C_i \), then \( M_i \) and \( N_i \) are disjoint compact sets. Let \( M = \{ x \in \mathbb{C} \mid d(x, M_i) = d(x, N_i) \} \). Then separates \( N_i \) from \( M_i \) in \( C \). Since \( C \) is unicoherent \([13]\), a component \( L \) of \( N \) separates. Choose \( z \in L \cap B(k, \varepsilon / 3) \), then \( x_i \notin B(z, Y) \). Note that \( B(z, Y) \cup \partial Y \) separates \( x_i \) from infinity. Hence, \( \partial B(z, Y) \setminus S(z, Y) \) separates \( x_i \) from infinity in \( \mathbb{C} \setminus Y \). Since \( \mathbb{C} \setminus Y \) is unicoherent, a component \( P \) of \( \partial B(z, Y) \setminus S(z, Y) \) separates \( x_i \) from \( \infty \). Let \( Q \) be the straight line segment joining the two end points of \( P \), then \( Q \in \mathcal{G}_\delta \) for large \( i \) and \( Q \) also separates \( x_i \) from infinity, a contradiction. □

### 3.3. Outchannels.

**Definition 3.19 (Outchannel).** An outchannel of the nonseparating plane continuum \( Y \) is a prime end \( \mathcal{E}_i \) of \( O_\infty = C_\infty \setminus Y \) such that for some chain \( \{ Q_i \} \) of crosscuts defining \( \mathcal{E}_i \), \( \var(f, Q_i, Y) = \var(f, Y) = \emptyset \) for every \( i \). We call an outchannel \( \mathcal{E}_i \) of \( Y \) a geometric outchannel iff for sufficiently small \( \delta \), every chord in \( \mathcal{G}_\delta \), which crosses \( \mathcal{E}_i \) essentially, has nonzero variation. We call a geometric outchannel negative (respectively, positive) iff every chord in \( \mathcal{G}_\delta \), which crosses \( \mathcal{E}_i \) essentially, has negative (respectively, positive) variation.

**Lemma 3.20.** Let \( Z \in \{ Y_\delta, Y_\delta^+, Y_\delta^- \} \). Fix a Riemann map \( \phi : \Delta_\infty \to C_\infty \setminus Z \) such that \( \phi(\infty) = \infty \). Suppose \( R_t \) lands at \( x \in \partial Z \). Then there is an open interval \( M \subset \partial \Delta_\infty \) containing \( t \) such that \( \phi \) can be extended continuously over \( M \).

**Proof.** Let \( t_i \) converge to \( t \) in \( S^1 \) such that \( R_{t_i} \) lands on \( x_i \) in \( Z \) and \( x_i \) converges to \( x \). Without loss of generality, \( \{ t_i \} \) is an increasing sequence. Let \( t_i < s_i < t_{i+1} \) in \( S^1 \). Note that \( \lim \sup R_{s_i} \supset R_{t_i} \). If \( \lim \sup R_{s_i} \) is not equal to \( R_{t_i} \), then by Lemma 3.15 there exist chords \( Q_i \) (by passing to a subsequence if
necessary) such that $R_{s_i}$ crosses $Q_i$ essentially. By Proposition 3.9, $Q_i \in G_\delta$ for large $i$. But by Proposition 3.12, $\text{var}(f, Q_i, X) = 0$ so $Q_i \subset Z$ for large $i$, a contradiction. So $\limsup R_{s_i} = \overline{R}$. In particular, $R_{s_i}$ must land on some point $y_i$ and $\lim y_i = x$. So on an interval $M$ about $t$ in $S^1$ each $R_s$ lands $y_s \in Z$ for each $s \in M$ and if $\{s_i\}$ converges to $s$, then $\lim y_{s_i} = y_s$. Hence the map taking $s \in M$ to the landing point of $R_s$ in $Z$ is continuous. □

Lemma 3.21. If there is a chord $Q$ of $Y$ of negative (respectively, positive) variation, such that there is no fixed point in $T(Y \cup Q)$, then there is a negative (respectively, positive) geometric outchannel $E_t$ of $Y$ for which a defining chain begins with $Q$.

Proof. Without loss of generality, assume $\text{var}(f, Q, Y) < 0$. Choose $\delta > 0$ so small that $Q \notin G_\delta$, no chord in $G_\delta$ separates $Q$ from $\infty$, and, since there are no fixed points in $T(Y \cup Q)$, every chord in $G_\delta$ separated from $\infty$ by $Q$ moves off itself under $f$ (so variation on it is defined). Let $\varphi : \Delta_\infty \to C \setminus Y^+_{\delta}$ be the Riemann map. Assume $t' < t \in S^1$ such that $\varphi(t') = a$ and $\varphi(t) = b$. By Lemma 3.20, $\varphi$ extends to a continuous function $\hat{\varphi} : \Delta_\infty \to [C \setminus Y^+_{\delta}] \cup \{\text{accessible points of } Y^+_{\delta}\}$. Consider the arc-component $M$ of $[\Delta_\infty \setminus \Delta_\infty] \cap [t', t]$ containing $t'$. If $t \in M$, then $\hat{\varphi}([t', t])$ is an arc in $Y^+_{\delta}$ joining $a$ to $b$ which is separated by $Q$ from $\infty$. But

$$\text{var}(f, Q, X) = \sum_{C \in G_\delta, C \subset \hat{\varphi}([t', t])} \text{var}(f, C, X).$$

This is a contradiction since $\text{var}(f, Q, X) < 0$ and all $\text{var}(f, C, X) \geq 0$. Hence $b$ is not in $M$. Let $s = \sup M$, then $E_s$ is a negative geometric outchannel. For if some chord $C \in G_\delta$ crosses $R_s$ essentially, then $s \in \text{Int}(M)$, a contradiction. □

3.4. Invariant Channel in $X$. We are now in a position to prove Bell’s principal result on any possible counter-example to the fixed point property, under our standing hypothesis.

Lemma 3.22. Suppose $E_t$ is a geometric outchannel of $Y = T(X)$ under $f$. Then the principal continuum $\text{Pr}(E_t)$ of $E_t$ is invariant under $f$. So $\text{Pr}(E_t) = X$.

Proof. To see that $\text{Pr}(E_t)$ is invariant under $f$, let $E_t$ be a geometric outchannel of $T(X)$ under $f$. Let $x \in \text{Pr}(E_t)$. Then for some chain $\{Q_i\}_{i=1}^\infty$ of cross-cuts defining $E_t$ selected from $G_\delta$, we may suppose $Q_i \to x \in \partial T(X)$ and $\text{var}(f, Q_i, X) \neq 0$ for each $i$. Let $R_t$ be the image under the conformal map $\phi$ of the radial ray to $t \in S^1$. Since $Q_i$ meets $R_t$ and $\text{var}(f, Q_i, X) \neq 0$, we have $f(Q_i)$ meeting $R_t$ (since any junction from $Q_i$ “parallels” $R_t$). Since $\text{diam}(f(Q_i)) \to 0$, we have $f(Q_i) \to f(x)$ and $f(x) \in \text{Pr}(E_t)$. □

Theorem 3.23 (Dense channel, Bell). Under our standing Hypothesis, $Y = T(X)$ contains a negative geometric outchannel; hence, $\partial O_\infty = \partial T(X) = X = f(X)$ is an indecomposable continuum.
Proof. Recall that the map \( f : \mathbb{C} \to \mathbb{C} \) taking \( X \) into \( T(X) \) has no fixed points in \( T(X) \), and \( X \) is minimal with respect to these properties. Choose \( \delta > 0 \) so that each crosscut \( Q \in \mathcal{G}_\delta \) is sufficiently close to \( Y = T(X) \) so that \( f \) has no fixed points in \( T(Y \cup Q) \), and so that for any geometric crosscut \( Q \in \mathcal{G}_\delta \), \( f(Q) \cap Q = \emptyset \). Let \( C = \partial Y_\delta \). Since \( Y_\delta \) is locally connected, \( C \) is a Caratheodory loop. Since \( f \) is fixed point free on \( C \), \( \text{ind}(f, C) = 0 \). Consequently, by Theorem 2.13 for Caratheodory loops, \( \text{var}(f, C) = -1 \).

By the summability of variation on \( C \), it follows that on some chord \( Q \subset C \), \( \text{var}(f, Q, Y) < 0 \). By Lemma 3.21, there is a negative geometric outchannel \( \mathcal{E}_t \) under the crosscut \( Q \).

Since \( \Pr(\mathcal{E}_t) \) is invariant under \( f \) by Lemma 3.22, it follows that \( \Pr(\mathcal{E}_t) \) is an invariant subcontinuum of \( \partial O_\infty \subset \partial T(X) \subset X \). By the minimality condition in our Standing Hypothesis, \( \Pr(\mathcal{E}_t) \) is dense in \( \partial O_\infty \). Hence, \( \partial O_\infty = \partial T(X) = X \) and \( \Pr(\mathcal{E}_t) \) is dense in \( X \). It then follows from a theorem of Rutt [10] that \( X \) is an indecomposable continuum. \( \square \)

Theorem 3.24. The boundary of \( Y_\delta \) is a simple closed curve. The set of accessible points in the boundary of each of \( Y_\delta^+ \) and \( Y_\delta^- \) is a countable union of continuous one-to-one images of \( \mathbb{R} \).

Proof. Since \( X \) is indecomposable by Theorem 3.23, it has no cut points. By Proposition 3.18 \( \partial Y_\delta \) is a Caratheodory loop. Since \( X \) has no cut points, neither does \( Y_\delta \). A Caratheodory loop with no cut points is a simple closed curve.

Let \( Z \in \{Y_\delta^+, Y_\delta^-\} \). Fix a Riemann map \( \phi : \Delta_\infty \to \mathbb{C}_\infty \setminus Z \) such that \( \phi(\infty) = \infty \). Corresponding to the choice of \( Z \), let \( W \in \{\mathcal{G}_\delta^+, \mathcal{G}_\delta^-\} \). Apply Lemma 3.20 and find the maximal collection \( J \) of open subarcs of \( \partial \Delta_\infty \) over which \( \phi \) can be extended continuously. Since \( X \) has no cutpoints this extension is one-to-one. Since angles corresponding to accessible points are dense in \( \partial \Delta_\infty \), \( C = \partial \Delta_\infty \setminus \cup J \) contains no open arc. If \( Z = Y_\delta^+ \), then it is possible that \( \cup J \) is all of \( \partial \Delta_\infty \) except one point, but it cannot be all of \( \partial \Delta_\infty \) since there is at least one (negative) geometric outchannel by Theorem 3.23. Since it is a collection of open arcs in \( \partial \Delta_\infty \), \( J \) is countable. \( \square \)

Theorem 3.24 still leaves open the possibility that \( Z \in \{Y_\delta^+, Y_\delta^-\} \) has a very complicated boundary. Note that \( \phi \) is not continuous at points of the closed zero-dimensional set \( C \). We may call \( C \) the set of outchannels of \( Z \). In principle, there could be an uncountable set of outchannels, each dense in \( X \). The one-to-one continuous images of \( \mathbb{R} \) lying in \( \partial Z \) are the “sides” of the outchannels. If two elements \( J_1 \) and \( J_2 \) of the collection \( J \) happen to share a common endpoint \( t \), then the prime end \( \mathcal{E}_t \) is an outchannel in \( Z \), dense in \( X \), with \( \phi(J_1) \) and \( \phi(J_2) \) as its sides. It seems possible that an endpoint \( t \) of \( J \in J \) might have a sequence of elements \( J_i \) from \( J \) converging to it. Then the outchannel \( \mathcal{E}_t \) would have only one (continuous) “side.” This possibility, and even having more than one outchannel in \( Z \), is eliminated in the next section.
In the lemma below we show that for $Y^\_\delta$ those pieces of the boundary, which correspond to arc components in the set of accessible points, are well behaved and do not contain large unnecessary “wiggles.”

**Lemma 3.25.** Assume that $\partial Y^\_\delta$ is not a simple closed curve. Let $K$ be an arc component of the set of accessible points of $Y^\_\delta$. Then for each $\varepsilon$, $0 < \varepsilon < \delta/2$, there exists $\xi > 0$ such that for any two points $x, y \in K \cap X$, if $Q$ is any crosscut of $Y^\_\delta$ joining $x$ to $y$, and $\text{diam}(Q) < \xi$, then there exists an arc $B \subset K$, joining $x$ to $y$ such that $\text{diam}(B) < 8\varepsilon$.

**Proof.** Let $\varepsilon > 0$ be fixed and choose $\xi$ as in Proposition 3.9. Let $B$ be the unique arc in $K$ joining $x$ to $y$. By Theorem 3.24, $K$ is a one-to-one continuous image of $\mathbb{R}$. We will denote the unique subarc of $K$ which joins two points $p, q \in K$ by $\langle p, q \rangle$. Hence $B = \langle x, y \rangle$. Suppose there exist $z \in B$ such that $d(z, Q) > 8\varepsilon$. Let $b \in T(Q \cup Y^\_\delta) \setminus Y^\_\delta$ such that $d(b, z) \leq \varepsilon/2$. Let $P = \langle x, z \rangle \setminus B(b, \varepsilon)$ and $M = \langle z, y \rangle \setminus B(b, \varepsilon)$. Then $P$ and $Q$ are disjoint closed sets in $Y^\_\delta$. Let $N$ be a component of $B(Q, 3\varepsilon) \setminus Y^\_\delta$ which separates $b$ from $\infty$ in $\mathbb{C} \setminus Y^\_\delta$ and such that $N$ is contained in the bounded component of $T(Q \cup Y^\_\delta)$. By Proposition 3.9, each point of $N$ lies in an element $F_x \in \mathcal{F}$ with diameter at most $2\varepsilon$. Since $F_x$ is small and meets $Y$, $F_x \cap (P \cup M) \neq \emptyset$ for each $x \in N$. It follows from Proposition 3.10 that there exists $x \in N$ such that $F_x = F$ meets both $P$ and $M$ and, hence, $F$ separates $b$ from $\infty$ in $\mathbb{C} \setminus Y^\_\delta$. We may assume that $F \in \mathcal{G}$ since, if $F = V(z)$ is a simplex for some $z$, we can replace $F$ by one of the components of $V(z) \setminus (Y \cup V(z))$.

Now, $\text{var}(f, F, X) \leq 0$. For if $\text{var}(f, F, X) > 0$, there exists a positive geometric outchannel $E_\alpha$ for which a defining chain starts with $F$. But, if the end points of $F$ are $x', y'$, then $R_\alpha$ would cross some chord $G \subset \langle x', y' \rangle \subset K$ essentially. This is a contradiction since $K$ contains no crosscuts of positive variation. So $\text{var}(f, F, X) \leq 0$.

It follows that $F \subset Y^\_\delta$ and $T(Y \cup F) \subset Y^\_\delta$. This contradicts that $b \in \mathbb{C} \setminus Y^\_\delta$. \qed

4. **Uniqueness of the Outchannel**

Theorem 3.23 asserts the existence of at least one negative geometric outchannel which is dense in $X$. We show below that there is exactly one geometric outchannel, and that its variation is $-1$. Of course, $X$ could have other dense channels, but they are “neutral” as far as variation is concerned.

**Theorem 4.1 (Unique Outchannel).** Assume the standing hypothesis 1.1. Then there exists a unique geometric outchannel $E_t$ for $X$, which is dense in $X = \partial Y$. Moreover, for any sufficiently small chord $Q$ in any chain defining $E_t$, $\text{var}(f, Q, X) = -1$, and for any sufficiently small chord $Q'$ not crossing $R_t$ essentially, $\text{var}(f, Q', X) = 0$.

**Proof.** Suppose by way of contradiction that $X$ has a positive outchannel. Let $\delta > 0$ such that $T(B(Y, 2\delta))$ contains no fixed points of $f$ and such that,
if $M \subset B(Y, 2\delta)$ with $\text{diam}(M) < 2\delta$, then $f(M) \cap M = \emptyset$. Since $X$ has a positive outchannel, $\partial Y_{\delta}^-$ is not a simple closed curve. By Theorem 3.24 $\partial Y_{\delta}^-$ contains an arc component $K$ which is the one-to-one continuous image of $\mathbb{R}$. Note that each point of $K$ is accessible.

Let $\varphi : \Delta_{\infty} \to U_{\infty} = \mathbb{C} \setminus Y_{\delta}^-$ a conformal map. By Theorem 3.24, and its proof, $\varphi$ extends continuously and injectively to a map $\tilde{\varphi} : \tilde{\Delta}_{\infty} \to \tilde{U}_{\infty}$, where $\tilde{\Delta}_{\infty} \setminus \Delta_{\infty}$ is a dense and open subset of $S^1$ which contains $K$ in its image. Then $\tilde{\varphi}^{-1}(K) = (t', t) \subset S^1$ is an open arc with $t' < t$ in the counterclockwise order on $S^1$. Let $<$ denote the order in $K$ induced by $\tilde{\varphi}$ and for $x < y$ in $K$, denote the arc in $K$ with endpoints $x$ and $y$ by $\langle x, y \rangle$. Let $\langle x, \infty \rangle = \cup_{y > x} \langle x, y \rangle$.

Let $\mathcal{E}_t$ be the prime-end corresponding to $t$. Then $\text{Pr}(\mathcal{E}_t)$ is a positive geometric outchannel and, hence, by Lemma 3.22, $\text{Pr}(\mathcal{E}_t) = X$. Let $R_t = \varphi(re^{it})$, $r > 1$, be the external conformal ray corresponding to the prime-end
$\mathcal{E}_t$. Since $R_t \setminus R_t = X$ and the small chords which define $\Pr(\mathcal{E}_t)$ have one end point in $K$ (c.f., Proposition 3.9), $(x, \infty) \cap Y^-_\delta = X$.

Let $\varepsilon > 0$ such that $T(B(Y, \varepsilon)) \subset B(Y, \delta)$ (by Proposition 3.8). It follows from Propositions 3.8, 3.10 and 3.13, there exists $x \in K$ such that in each arc $M \subset (x, \infty)$ with $\text{diam}(M) > \varepsilon/4$, there exists $y \in M$ and a chord $G \in G_\delta$ with end point $y$ which crosses $R_t$ essentially.

Let $a_0 \in K \cap X$ so that $a_0 > x$ and $J_{a_0}$ a junction of $Y^-_\delta$. Let $W$ be a topological disk about $a_0$ with simple closed curve boundary of diameter less than $\varepsilon$ so that the component of $K \cap W$ containing $a_0$ has closure $\langle a, b \rangle$, $a < a_0 < b$ in $K$ and $f(W) \cap (W \cup J_{a_0}) = \emptyset$. We may suppose that $(K \cap W) \setminus \langle a, b \rangle$ is contained in one component of $W \setminus \langle a, b \rangle$ since one side of $K$ is accessible from $\infty$ in $C \setminus Y^-_\delta$. Since $X \subset (a_0, \infty)$, there are components of $W \cap \langle b, \infty \rangle$ which pass arbitrarily close to $a_0$. Choose $\langle c, d \rangle$ to be the closure of a component of $W \cap \langle b, \infty \rangle$ such that:

1. $a$ and $d$ lie in the same component of $\partial W \setminus \{b, c\}$,
2. there exists $y \in \langle c, d \rangle \cap X \cap W$ and an arc $I \subset (W \setminus \langle a, d \rangle) \cup \{a_0, y\}$ joining $a_0$ to $y$, and
3. there is a chord $Q \subset W$ with $y$ and $z$ as endpoints which crosses $R_t$ essentially.

To see the above, note that there are small chords or simplexes which cross $R_t$ essentially through each point of $R_t$. By Proposition 3.10 given an arc $A$ that crosses $R_t$ essentially and is sufficiently close to $X$, there is a small diameter chord that essentially crosses $R_t$ and meets $A$. By Lemma 3.25, it follows that if two small chords both cross $R_t$ essentially and both meet a small diameter arc in $R_t$, then they both meet a small diameter arc in $K$. Thus we can satisfy (3) on any arc in $W \cap K$ which gets close to $a_0$. Note that (2) holds for any point of $\langle c, d \rangle$ for which (3) holds.

Let $B$ be the arc in $\partial W \setminus \{b\}$ with end points $a$ and $d$. Let $A$ be a bumping arc in $(C \setminus [J_{a_0} \cup T((a, d) \cup B)]) \cup \{a, d\}$ with end-points $a$ and $d$ such that $Y \setminus T((a, d) \cup B) \subset T(A \cup B)$. Hence, $S = \langle a, d \rangle \cup A$ is a simple closed curve and $Y \subset T(S)$. We may suppose that $A \subset B(Y, \varepsilon)$ so that $f$ is fixed point free on $T(S)$ and each component of $A \setminus X$ has diameter less than $\delta$ so that variation is defined on each such component.

Since $\overline{Q} \cap Y = \{y, z\}$ we may suppose that $A \cap \overline{Q} = \{z\}$. Note that $I$ is an arc in $T(S)$ which meets $S$ only at its end points $a_0$ and $y$. Since $I \subset W$, $f(I) \cap J_{a_0} = \emptyset$. Let $R = T(\langle a_0, y \rangle \cup I)$ and let $L = T(\langle y, d \rangle \cup I \cup A \cup \langle a, a_0 \rangle)$. Let $J_y$ be a junction for $S$ such that $J_y \cap \overline{Q} = \{y\}$, $J_{a_0} \setminus W \subset J_y$ so that $R^*_y \setminus W \subset R^*_y$ for each $* \in \{+, i, -\}$ and $J_y$ runs very close to $(a_0, y) \cup J_{a_0}$.

Note that the order $<$ on $K$ coincides with the counterclockwise order on $S$. It follows that $W \cup R^*_y$ separates $L \cup R^-_t \setminus J_{a_0}$ from $R \cup R^+_t \setminus J_{a_0}$. Since $Q$ crosses $R_t$ essentially, we know that $\text{var}(f, Q, Y) > 0$. We will use this information to show that $f(y) \in R$. To compute $\text{var}(f, Q, Y) = \text{var}(f, Q, S)$ we will use the fact that the variation is invariant under a homotopy which keeps $y$ and $z$ in $h(U_y)$ (see Proposition 2.10 and the remark following that
we know that \( \text{var}(f', Q, S) = \text{var}(f, Q, S) \). Moreover, we can choose \( f' \) such that the number of components of \( f'^{-1}(\mathbb{C} \setminus T(S)) \) is minimal (the set of components of \( \overline{Q} \cap f'^{-1}(\mathbb{C} \setminus T(S)) \) whose closures meet both \( f^{-1}(T(L)) \) and \( f^{-1}(T(R)) \) is finite since \( f(\overline{Q}) \cap W = \emptyset \). Then to compute \( \text{var}(f', Q, S) \) we use the following recipe: As we go along \( Q \) from \( y \) to \( z \), each time the image of \( f' \) goes from \( R \) to \( L \) count +1. Each time the image goes from \( L \) to \( R \) count −1. Make no other counts. Then it follows that if \( f'(y) \in R \) and \( f(z) \in L \), then \( \text{var}(f', Q, S) = +1 \), if \( f'(y) \in L \) and \( f'(z) \in R \), then \( \text{var}(f', Q, S) = −1 \). Otherwise \( \text{var}(f', Q, S) = 0 \). Since \( \text{var}(f', Q, S) > 0 \) if \( f(y) \in R \).

The Lollipop Lemma, Theorem 2.14, applies to \( S \) and the arc \( I \). Also, since \( Y \subset T(S) \), \( \text{var}(f, C, S) = \text{var}(f, C, Y) \) for each chord \( C \) contained in \( S \).

Hence, there exists a chord \( Q_1 \subset \langle a_0, y \rangle \) such that \( \text{var}(f, Q_1, Y) < 0 \). Since there are no chords of positive variation on \( \langle a_0, y \rangle \) and

\[
0 = \text{ind}(f, I \cup \langle a_0, y \rangle) = \sum_{C \in G, C \subseteq (a_0, y)} \text{var}(f, C, Y) + 1,
\]

we know that \( \text{var}(f, Q_1, Y) = −1 \).

We repeat the above argument starting with \( y \in K \) in place of \( a_0 \) and \( J_y \) in place of \( J_{a_0} \) and an open disk \( V \subset W \) about \( y \) to find a second chord \( Q_2 \subset \langle y, \infty \rangle \) with \( \text{var}(f, Q_2, Y) = −1 \).

We will now show that the existence of chords \( Q_1 \) and \( Q_2 \) in \( K \) with variation −1 on each leads to a contradiction. Choose \( c' < d' \in K \) such that \( \langle c', d' \rangle \) is the closure of a component of \( K \cap W \) satisfying the following conditions.

1. \( Q_1, Q_2 \subset \langle a_0, c' \rangle \subset K \),
2. \( \{a, d'\} \) is contained in one component of \( \partial W \setminus \{b, c'\} \),
3. there exist \( y' \in \langle c', d' \rangle \cap Y \cap W \) and an arc \( I' \) from \( a_0 \) to \( y' \) in \( \{a_0, y'\} \cup (W \setminus \langle a, d'\rangle) \), and
4. there exists a chord \( Q' \subset W \) with endpoints \( y' \) and \( z' \) such that \( Q' \) crosses \( R_t \) essentially.

Let \( B' \) be the arc in \( \partial W \setminus \{b\} \) with endpoints \( \{a\} \) and \( \{d'\} \). Let \( A' \) be an arc in \( \{a, d'\} \cup (Y \setminus \langle a, d'\rangle \cup B') \) such that \( Y \setminus T(\langle a, d'\rangle \cup B') \subset T(A' \cup B') \) and such that the components of \( A' \setminus X \) have diameter less than \( \delta \). We may suppose that \( \overline{Q} \cap A' = \{z'\} \). We can prove, as above, that \( f(y') \in R = T(\langle a_0, y' \rangle \cup I') \) and, hence all conditions of the Lollipop Lemma 2.14 are again satisfied for \( S' \) and \( I' \) and

\[
\text{ind}(f, \langle a_0, y' \rangle \cup I') = \sum_{C \in G, C \subseteq (a_0, y')} \text{var}(f, C, S') + 1.
\]

Since \( \langle a_0, y' \rangle \) contains \( Q_1 \) and \( Q_2 \), \( \text{var}(f, Q_1, S') = \text{var}(f, Q_1, Y) < 0 \) and contains no chords of positive variation, \( \sum \text{var}(f, C, S') + 1 \leq −1 \).
Since $f$ is fixed point free on $R'$, $\text{ind}(f, (a_0, y') \cup I') = 0$ by Theorem 2.5. This contradiction shows that $X$ has no positive outchannels.

By Theorems 3.23 and 2.13, $X$ has exactly one negative outchannel and its variation is $-1$. □

REFERENCES


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