VARIATION AND UNIQUENESS OF OUTCHANNELS

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ABSTRACT. In this paper we try to present in a coherent fashion proofs of basic results developed so far (primarily by H. Bell) for the plane fixed point problem. Some of these results have been announced much earlier but without formal proof. We define the concept of the variation of a map on a simple closed curve and relate it to the index of the map on that curve: Index = Variation + 1. We define the concept of an *outchannel* for a fixed point free map which carries a nonseparating plane continuum into itself. We then prove Bell's *Lollipop Lemma* and use it to show that such a map has a unique outchannel, and that outchannel must have variation $= -1$. We also define a special class of straight line crosscuts and show that these suffice for a satisfactory treatment of prime-ends of a non-separating plane continuum.

1. INTRODUCTION

By $\mathbb{C}$ we denote the plane and by $\mathbb{C}_\infty$ the Riemann sphere. Let X be a plane continuum. By $T(X)$ we denote the *topological hull* of X consisting of X union all of its bounded complementary domains. Thus, $\mathbb{C}_\infty \setminus T(X)$ is a simply-connected domain containing ∞. The following is a long-standing question in topology.

Fixed Point Question: *"Does a continuous function taking a non-separating plane continuum into itself always have a fixed point?"*

We study the slightly more more general question, "Is there a plane continuum Z and a continuous function $f : \mathbb{C} \to \mathbb{C}$ taking Z into $T(Z)$ with no fixed points in $T(Z)$?" A Zorn's Lemma argument shows that if one assumes the answer is "yes," then there is a subcontinuum $X \subset Z$ minimal with respect to these properties. Therefore, we will assume the following throughout this paper:

1.1. **Standing Hypotheses.** *We assume that $f : \mathbb{C} \to \mathbb{C}$ is a map and X is a plane continuum such that $f(X) \subset T(X) = Y$, f has no fixed points in Y, and X is minimal with respect to these properties.*

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It will follow from Theorem 3.23 that for such a minimal continuum, $f(X) = X = \partial Y$ (though it may not be the case that $f(Y) \subset Y$).

By results of Cartwright-Littlewood and Bell, if one replaces "map" by "homeomorphism of the plane" in the fixed point question, the answer is yes. By results of Bell [3] (see also Sieklucki [11], and Iliadis [8]), the only unsolved general case (with no special assumptions on the map) is where the boundary of X is indecomposable (with a dense channel, explained later). In this paper we use tools first developed by Bell to elucidate the action of a fixed point free map (should one exist). Theorem 4.1 (Unique Outchannel) is new and complete proofs of Theorems 2.13 and 2.14 appear in print for the first time. In subsequent papers [6], we will apply the tools developed here to prove some additional special cases (with restrictions on the map) of a "fixed point theorem."

2. Tools

Let S^1 denote the unit circle in the complex plane and let $p : \mathbb{R} \to S^1$ denote the covering map $p(x) = e^{2\pi ix}$. Let $g : S^1 \to S^1$ be a map. By the *degree* of the map g, denoted by degree(g), we mean the number $\hat{g}(1) - \hat{g}(0)$, where $\hat{g} : \mathbb{R} \to \mathbb{R}$ is a lift of the map g to the universal covering space $\mathbb{R}$ of S^1 (i.e., $p \circ \hat{g} = g \circ p$). It is well-known that degree(g) is independent of the choice of the lift.

2.1. **Index.** Let $g : S^1 \to \mathbb{C}$ be a map and let $S = g(S^1)$. Suppose $f : S \to \mathbb{C}$ has no fixed points on S. Then for all $z \in S$, the vector $z - f(z) \neq 0$. Hence the unit vector $v(z) = \frac{z-f(z)}{|z-f(z)|}$ always exists. Define the map $\overline{v} = v \circ g : S^1 \to S^1$ by

$$\overline{v}(t) = v(g(t)) = \frac{g(t) - f(g(t))}{|g(t) - f(g(t))|}.$$

Then the map $\overline{v} : S^1 \to S^1$ *lifts* to a map $\widehat{v} : \mathbb{R} \to \mathbb{R}$ such that $p \circ \widehat{v} = \overline{v} \circ p$. Define the *index of f with respect to g*, denoted ind(f, g) by

$$\text{ind}(f, g) = \widehat{v}(1) - \widehat{v}(0) = \text{degree}(\overline{v}).$$

More generally, for any parameters $0 \leq a < b \leq 1$ in $S^1 = \mathbb{R}/\mathbb{Z}$, define the *fractional index* of f on the path $g|_{[a,b]}$ in S by

$$\text{ind}(f, g|_{[a,b]}) = \widehat{v}(b) - \widehat{v}(a).$$

While necessarily, the index of f with respect to g is an integer, the fractional index of f on $g|_{[a,b]}$ need not be. We shall have occasion to use fractional index in the proof of Theorem 2.13. Note that (fractional) index is the net change in argument of the vector $g(t) - f(g(t))$ as t runs along S^1 from a to b.

Proposition 2.1. *Let $g : S^1 \to \mathbb{C}$ be a map with $g(S^1) = S$, and suppose $f : S \to \mathbb{C}$ has no fixed points on S. Let $a \neq b \in S^1$ with $[a, b]$ denoting* *the counterclockwise subarc on* S^1 *from* a *to* b *(so* $S^1 = [a,b] \cup [b,a]$*). Then* $\operatorname{ind}(f,g) = \operatorname{ind}(f, g|_{[a,b]}) + \operatorname{ind}(f, g|_{[b,a]})$*.*

2.2. **Stability of Index.** The following standard theorems and observations about the stability of index under fixed-point-free homotopy are consequences of the fact that index is continuous and integer-valued.

Theorem 2.2. *Suppose* $f : \mathbb{C} \to \mathbb{C}$ *is a map and* $g_1 : S^1 \to \mathbb{C}$ *and* $g_2 : S^1 \to \mathbb{C}$ *are homotopic maps in* $\mathbb{C}$ *such that the homotopy misses the fixed point set of* f*. Then* $\operatorname{ind}(f, g_1) = \operatorname{ind}(f, g_2)$*.*

An embedding $g : S^1 \to S \subset \mathbb{C}$ is *orientation preserving* if g is isotopic to the indentity map $id|_{S^1}$. In particular, the index of f on a simple closed curve S missing the fixed point set of f is independent of choice of parameterizations of S with the same orientation. If g_1 and g_2 are orientation-preserving embeddings of S^1 with the same image set $g_1(S^1) = S = g_2(S^1)$, then we have a well-defined index of f on S, namely $\operatorname{ind}(f,S) = \operatorname{ind}(f,g_1) = \operatorname{ind}(f,g_2)$.

Theorem 2.3. *Suppose* $g : S^1 \to \mathbb{C}$ *is a map with* $g(S^1) = S$*, and* $f_1, f_2 : S \to \mathbb{C}$ *are homotopic maps such that each level of the homotopy is fixed-point-free on* S*. Then* $\operatorname{ind}(f_1, g) = \operatorname{ind}(f_2, g)$*.*

In particular, if S is a simple closed curve and $f_1, f_2 : S \to \mathbb{C}$ are maps such that there is a homotopy $h_t : S \to \mathbb{C}$ from f_1 to f_2 with h_t fixed-point free on S for each $t \in [0,1]$, then $\operatorname{ind}(f_1, S) = \operatorname{ind}(f_2, S)$.

Corollary 2.4. *Suppose* $g : S^1 \to \mathbb{C}$ *is an an orientation preserving embedding with* $g(S^1) = S$*, and* $f : T(S) \to \mathbb{C}$ *is a map such that* f *has no fixed points on* S *and* $f(S) \subset T(S)$*. Then* $\operatorname{ind}(f,g) = 1$*.*

Proof. Since $f(S) \subset T(S)$ which is a disk with boundary S and f has no fixed point on S, there is a fixed point free homotopy of $f|_S$ to a constant map $c : S \to \mathbb{C}$ taking S to a point in $T(S) \setminus S$. By Theorem 2.3, $\operatorname{ind}(f,g) = \operatorname{ind}(c,g) = 1$. □

Theorem 2.5. *Suppose* $g : S^1 \to \mathbb{C}$ *is a map with* $g(S^1) = S$*, and* $f : T(S) \to \mathbb{C}$ *is a map such that* $\operatorname{ind}(f,g) \neq 0$*, then* f *has a fixed point in* $T(S)$*.*

Proof. Notice that $T(S)$ is a locally connected non-separating plane continuum and, hence, contractible. Suppose f has no fixed point in $T(S)$. Choose point $q \in T(S)$. Let $c : S^1 \to \mathbb{C}$ be the constant map to q. Let H be a homotopy from g to c with image in $T(S)$. Since H misses the fixed point set of f, Theorem 2.2 implies $\operatorname{ind}(f,g) = \operatorname{ind}(f,c) = 0$. □

2.3. **Variation.** In this section we introduce the notion of variation of a map on an arc and relate it to winding number.

Definition 2.6 (Junctions)**.** *The* standard junction J_0 *is the union of the three rays* $R_i = \{z \in \mathbb{C} \mid z = re^{i\pi/2},\ r \in [0,\infty)\}$*,* $R_+ = \{z \in \mathbb{C} \mid z =$

re^0, $r \in [0,\infty)\}$, $R_- = \{z \in \mathbb{C} \mid z = re^{i\pi},\ r \in [0,\infty)\}$, having the origin 0 in common. By U we denote the lower half-plane $\{z \in \mathbb{C} \mid z = x + iy,\ x \in \mathbb{R},\ y < 0\}$. A junction *$J_v$ is the image of J_0 under any orientation-preserving homeomorphism $h : \mathbb{C} \to \mathbb{C}$ where $v = h(0)$.*

We will often suppress h and refer to $h(R_i)$ as R_i, and similarly for the remaining rays and the region $h(U)$. When needed we will write R_{v^+} etc. when we want to refer to a particular $h(R_+)$ of a junction J_v based at $v = h(0)$.

Suppose S is a simple closed curve and $A \subset S$ is a subarc of S with endpoints a and b, with $a < b$ in the counter-clockwise orientation on S. We will usually denote such a subarc by $A = [a, b]$ and by (a, b) its interior in S^1.

Definition 2.7 (Variation on an arc)**.** *Let S be a simple closed curve and $A = [a, b]$ a subarc of S such that $f(a), f(b) \in T(S)$ and $f(A) \cap A = \emptyset$. We define the* variation of f on A with respect to S, *denoted* $\mathrm{var}(f, A, S)$, *by the following algorithm:*

1. *Choose an orientation-preserving homeomorphism h of $\mathbb{C}$ such that $h(0) = v \in A$ and $T(S) \subset h(U) \cup \{v\}$.*
2. *As always we assume that $a < b$ in the counterclockwise order.*
3. Counting crossings: *Consider the set $M = f^{-1}(J_v) \cap [a, b]$. Each time a point of $f^{-1}(h(R^+)) \cap [a, b]$ is immediately followed in M, in the natural order on $[a, b]$, by a point of $f^{-1}(h(R^i))$ count $+1$ and each time a point of $f^{-1}(h(R^i)) \cap [a, b]$ is immediately followed in M, in the natural order on $[a, b]$, by a point of $f^{-1}(h(R^+))$ count -1. Count no other crossings.*
4. *The sum of the crossings found above is the variation, denoted* $\mathrm{var}(f, A, S)$.

Note that $f^{-1}(h(R^+)) \cap [a, b]$ and $f^{-1}(h(R^i)) \cap [a, b]$ are disjoint closed sets in $[a, b]$. Hence, in (3) in the above definition, we count only a finite number of crossings and $\mathrm{var}(f, A, S)$ is a finite integer.

Let $g : S^1 \to \mathbb{C}$ be a map and $w \in \mathbb{C} \setminus g(S^1)$ be a point. By the *winding number of g about the point w*, denoted by $\mathrm{win}(g, w)$, we mean the number $\mathrm{ind}(c, g)$, where $c : \mathbb{C} \to \mathbb{C}$ is the constant map $c(z) = w$. It is well-known that the winding number is invariant under homotopies of g in $\mathbb{C} \setminus w$ and independent of the choice of the point w in a particular component of $\mathbb{C}\setminus g(S^1)$. Note that if B is the closure of $S\setminus A$ and $\alpha : S \to \mathbb{C}$ is any map such that $\alpha|_A = f|_A$ and $\alpha(B) \subset T(S) \setminus \{v\} \subset U$, then $\mathrm{var}(f, A, S) = \mathrm{win}(\alpha, v)$.

In case A is an open arc $(a, b) \subset S$ such that $\mathrm{var}(f, \overline{A}, S)$ is defined, it will be convenient to denote $\mathrm{var}(f, \overline{A}, S)$ by $\mathrm{var}(f, A, S)$

The following Lemma follows immediately from the definition.

Lemma 2.8. *Let S be a simple closed curve. Suppose that $a < c < b$ are three points in S such that $\{f(a), f(b), f(c)\} \subset T(S)$ and $f([a, b]) \cap [a, b] = \emptyset$. Then* $\mathrm{var}(f, [a, b], S) = \mathrm{var}(f, [a, c], S) + \mathrm{var}(f, [c, b], S)$.

2.4. **Stability of Variation.** By the above remark that variation is a winding number, the invariance of winding number under suitable homotopies implies that the variation $\mathrm{var}(f, A, S)$ also remains invariant under such homotopies. That is, even though the specific crossings in (3) in the algorithm may change, the sum remains invariant. We will state the required results about variation below without proof. Proofs can also be obtained directly by using the fact that $\mathrm{var}(f, A, S)$ is integer valued and continuous under suitable homotopies.

Proposition 2.9 (Junction Straightening)**.** *Any two junctions h_1, h_2 with $v_1 = h_1(0) \in A$, $v_2 = h_2(0) \in A$, $T(S) \subset h_1(U) \cup \{v_1\}$, and $T(S) \subset h_2(U) \cup \{v_2\}$ give the same variation.*

Proposition 2.10. *Variation $\mathrm{var}(f, A, S)$ is an integer, well-defined, and independent of h.*

Since U is open for a given junction J_v for $A = [a, b]$ with $T(S) \subset U \cup \{v\}$, the computation of $\mathrm{var}(f, A, S)$ depends only upon the crossings of the junction coming from a proper compact subarc of the open arc (a, b). Consequently, $\mathrm{var}(f, A, S)$ remains invariant under homotopies h_t of $f|_{[a,b]}$ such that $h_t(a)$ and $h_t(b)$ remain in U and $v \notin h_t([a, b])$ for all t. Moreover, the computation is stable under an isoptopy of the plane that moves the entire junction J_v (even off A), provided in the the isotopy v never crosses the image $f(A)$ and, $f(a)$ and $f(b)$ remain in the corresponding domain U_t.

Definition 2.11 (Variation on a finite union of arcs)**.** *Let S be a simple closed curve and $A = [a, b]$ a subcontinuum of S with partition a finite set $F = \{a = a_0 < a_1, \dots, a_n = b\}$. For each i let $A_i = [a_i, a_{i+1}]$. Suppose that f satisfies $f(a_i) \in T(S)$ and $f(A_i) \cap A_i = \emptyset$ for each i. We define the* variation of f on A with respect to S, *denoted* $\mathrm{var}(f, A, S)$, *by*

$$\mathrm{var}(f, A, S) = \sum_{i=0}^{n-1} \mathrm{var}(f, [a_i, a_{i+1}], S).$$

In particular, we include the possibility that $a_n = a_0$ in which case $A = S$.

By considering a common refinement of two partitions F_1 and F_2 of an arc $A \subset S$ such that $f(F_1) \cup f(F_2) \subset T(S)$ and satisfying the conditions in Definition 2.11, it follows from Lemma 2.8 that we get the same value for $\mathrm{var}(f, A, S)$ whether we use the partition F_1 or the partition F_2. Hence, $\mathrm{var}(f, A, S)$ is well-defined. If $A = S$ we denote $\mathrm{var}(f, S, S)$ simply by $\mathrm{var}(f, S)$.

2.5. **Index and variation for finite partitions.** What links Theorem 2.5 with variation is Theorem 2.13 below, first obtained by Bell and announced in the mid 1980's, and later by Akis [2]. Our proof is a modification of Bell's unpublished proof. We first need a variant of Proposition 2.9. Let $r : \mathbb{C} \to \mathbb{D}$ be radial retraction: $r(z) = \frac{z}{|z|}$ when $|z| \geq 1$ and $r|_{\mathbb{D}} = id|_{\mathbb{D}}$.

Lemma 2.12 (Curve Straightening). *Suppose $f : S^1 \to \mathbb{C}$ is a map with no fixed points on S^1. If $[a,b] \subset S^1$ is a proper subarc with $f([a,b]) \cap [a,b] = \emptyset$, $f((a,b)) \subset \mathbb{C} \setminus T(S^1)$ and $f(\{a,b\}) \subset S^1$, then there exists a map $h : S^1 \to \mathbb{C}$ homotopic to f in $\mathbb{C} \setminus T(S)$ relative to $\{a,b\}$, with each level of the homotopy fixed-point-free, such that $r \circ h : [a,b] \to S^1$ is locally one-to-one. Moreover,* $\operatorname{var}(f,[a,b],S^1) = \operatorname{var}(h,[a,b],S^1)$.

Note that if $\operatorname{var}(f,[a,b],S^1) = 0$, then $r \circ h$ carries $[a,b]$ one-to-one onto the arc in $S^1 \setminus [a,b]$ from $f(a)$ to $f(b)$. If the $\operatorname{var}(f,[a,b],S^1) = m > 0$, then $r \circ h$ wraps the arc $[a,b]$ counterclockwise about S^1 so that $h([a,b])$ meets each ray in J_v m times. A similar statement holds for negative variation.

Theorem 2.13 (Index = Variation + 1, Bell). *Suppose $g : S^1 \to \mathbb{C}$ is an orientation preserving embedding onto a simple closed curve S and f has no fixed points on S. If $F = \{a_0 < a_1 < \cdots < a_n\}$ is a partition of S and $A_i = [a_i, a_{i+1}]$ for $i = 1, \ldots, n$ with $a_{n+1} = a_0$ such that $f(F) \subset T(S)$ and $f(A_i) \cap A_i = \emptyset$ for each i, then*

$$\operatorname{ind}(f,g) = \sum_{i=0}^{n} \operatorname{var}(f, A_i, S) + 1 = \operatorname{var}(f,S) + 1.$$

Note that it is possible for index to be defined yet variation not to be defined on a simple closed curve S. For example, consider the map $z \to 2z$ with S the unit circle.

Proof. By an appropriate conjugation of f and g, we may assume without loss of generality that $S = S^1$ and $g = id$. Let F and $A_i = [a_i, a_{i+1}]$ be as in the hypothesis. Consider the collection of arcs

$$\mathcal{K} = \{K \subset S \mid K \text{ is the closure of a component of } S \cap f^{-1}(f(S) \setminus T(S))\}.$$

For each $K \in \mathcal{K}$, there is an i such that $K \subset A_i$. Since $f(A_i) \cap A_i = \emptyset$, it follows from the remark after Proposition 2.10 that $\operatorname{var}(f, A_i, S) = \sum_{K \subset A_i, K \in \mathcal{K}} \operatorname{var}(f, K, S)$. In particular, we can compute $\operatorname{var}(f, K, S)$ using one fixed junction for A_i and it is now clear that there are at most finitely many such K with $\operatorname{var}(f, K, S) \neq 0$. Moreover, the images of the endpoints of K lie on S.

Let m be the cardinality of the set $\mathcal{K}_f = \{K \in \mathcal{K} \mid \operatorname{var}(f, K, S) \neq 0\}$. By the above remarks, $m < \infty$ and $\mathcal{K}_f$ is independent of F. We prove the theorem by induction on m.

Suppose for a given f we have $m = 0$. Observe that from the definition of variation and the fact that the computation of variation is independent of the choice of an appropriate partition, it follows that,

$$\operatorname{var}(f, S) = \sum_{K \in \mathcal{K}} \operatorname{var}(f, K, S) = 0.$$

We claim that there is a map $f_1 : S \to \mathbb{C}$ with $f_1(S) \subset T(S)$ and a homotopy H from $f|_S$ to f_1 such that each level H_t of the homotopy is fixed-point-free and $\operatorname{ind}(f_1, id|_S) = 1$.

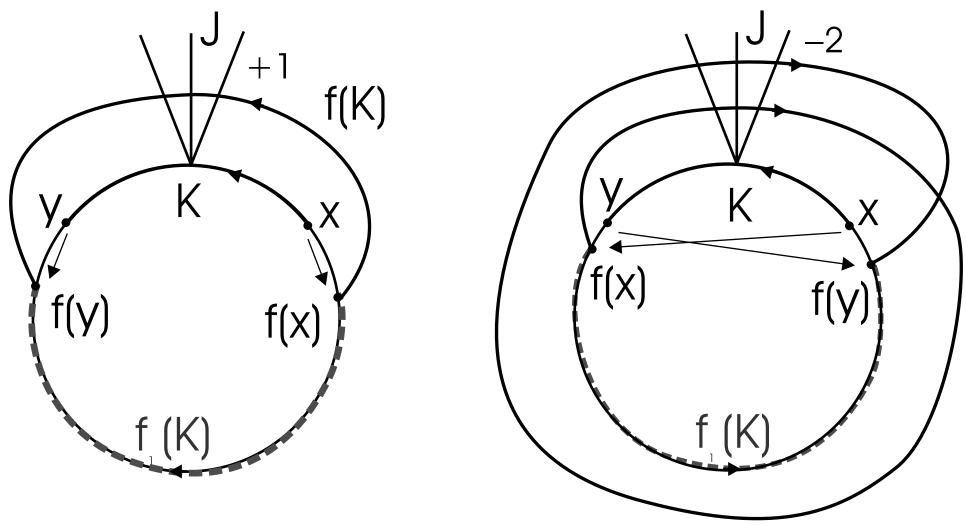

FIGURE 1. Replacing $f : S \to \mathbb{C}$ by $g : S \to \mathbb{C}$ with one less subarc of nonzero variation.

To see the claim, first apply the Curve Straightening Lemma 2.12 to each $K \in \mathcal{K}$ (if there are infinitely many, they form a null sequence) to obtain a fixed-point-free homotopy of $f|_S$ to a map $h : S \to \mathbb{C}$ such that $r \circ h$ is locally one-to-one on each $K \in \mathcal{K}$, where r is radial retraction of $\mathbb{C}$ to $\mathbb{D}$, and $\text{var}(h, K, S) = 0$ for each $K \in \mathcal{K}$. Let K be in $\mathcal{K}$ with endpoints x, y. Since $h(K) \cap K = \emptyset$, $r \circ h$ is one-to-one, and $\text{var}(h, K, S) = 0$. Since $\mathcal{K}$ is a null family, we can do this for each $K \in \mathcal{K}$ so that we obtain the desired $f_1 : S \to \mathbb{C}$ as the end map of a fixed-point-free homotopy from f to f_1. Since f_1 carries S into $T(S)$, Corollary 2.4 implies $\text{ind}(f_1, id|_S) = 1$.

Since the homotopy $f \simeq f_1$ is fixed-point-free, it follows from Theorem 2.3 that $\text{ind}(f, id|_S) = 1$. Hence, the theorem holds if $m = 0$ for any f and any appropriate partition F.

By way of contradiction, consider the collection of all maps f on S^1 which satisfy the hypotheses of the theorem, but not the conclusion. By the above $0 < |\mathcal{K}_f| < \infty$ for each. Let f and partition F be a counterexample for which $m = |\mathcal{K}_f|$ is minimal. By modifying f, we will show there exists another counterexample f' with $|\mathcal{K}_{f'}| < m$, a contradiction.

Choose $K \in \mathcal{K}$ such that $\text{var}(f, K, S) \neq 0$. Then $K = [x, y] \subset A_i = [a_i, a_{i+1}]$ for some i. By the Curve Straightening Lemma 2.12 and Theorem 2.3, we may suppose $r \circ f$ is locally one-to-one on K. Define a new map $f_1 : S \to \mathbb{C}$ by setting $f_1|_{\overline{S \setminus K}} = f|_{\overline{S \setminus K}}$ and setting $f_1|_K$ equal to the linear map taking $[x, y]$ to the subarc $f(x)$ to $f(y)$ on S missing $[x, y]$. Figure 1 (left) shows an example of a (straightened) f and the corresponding f_1 for a case where $\text{var}(f, K, S) = 1$, while Figure 1 (right) shows a case where $\text{var}(f, K, S) = -2$.

Since on $\overline{S \setminus K}$, f and f_1 are the same map, we have

$$\operatorname{var}(f, S \setminus K, S) = \operatorname{var}(f_1, S \setminus K, S).$$

Likewise for the fractional index,

$$\operatorname{ind}(f, S \setminus K) = \operatorname{ind}(f_1, S \setminus K).$$

By definition (refer to the observation we made in the case $m = 0$),

$$\operatorname{var}(f, S) = \operatorname{var}(f, S \setminus K, S) + \operatorname{var}(f, K, S)$$

$$\operatorname{var}(f_1, S) = \operatorname{var}(f_1, S \setminus K, S) + \operatorname{var}(f_1, K, S)$$

and by Proposition 2.1,

$$\operatorname{ind}(f, S) = \operatorname{ind}(f, S \setminus K) + \operatorname{ind}(f, K)$$

$$\operatorname{ind}(f_1, S) = \operatorname{ind}(f_1, S \setminus K) + \operatorname{ind}(f_1, K).$$

Consequently,

$$\operatorname{var}(f, S) - \operatorname{var}(f_1, S) = \operatorname{var}(f, K, S) - \operatorname{var}(f_1, K, S)$$

and

$$\operatorname{ind}(f, S) - \operatorname{ind}(f_1, S) = \operatorname{ind}(f, K) - \operatorname{ind}(f_1, K).$$

We will now show that the changes in index and variation, going from f to f_1 are the same (i.e., we will show that $\operatorname{var}(f, K, S) - \operatorname{var}(f_1, K, S) = \operatorname{ind}(f, K) - \operatorname{ind}(f_1, K)$). We suppose first that $\operatorname{ind}(f, K) = n + \alpha$ for some nonnegative $n \in \mathbb{Z}$ and $0 < \alpha < 1$. That is, the vector $z - f(z)$ turns through n full revolutions counterclockwise and α part of a revolution counterclockwise as z varies from x to y in K. (See Figure 1 (left) for a case $n = 0$ and α about $\frac{2}{3}$.) Then as z varies from x to y, $f_1(z)$ goes along S from $f(x)$ to $f(y)$ in the clockwise direction, so $z - f_1(z)$ turns through the angle $-(1-\alpha) = \alpha - 1$. Hence, $\operatorname{ind}(f_1, K) = -(1-\alpha)$. It is easy to see that $\operatorname{var}(f, K, S) = n + 1$ and $\operatorname{var}(f_1, K, S) = 0$. Consequently,

$$\operatorname{var}(f, K, S) - \operatorname{var}(f_1, K, S) = n + 1 - 0 = n + 1$$

and

$$\operatorname{ind}(f, K) - \operatorname{ind}(f_1, K) = n + \alpha - (\alpha - 1) = n + 1.$$

In Figure 1 on the left we assumed that $f(x) < x < y < f(y)$. The cases where $f(y) < x < y < f(x)$ and $f(x) = f(y)$ are treated similarly.

Thus when $n \geq 0$, in going from f to f_1, the change in variation and the change in index are the same. However, in obtaining f_1 we have removed one $K \in \mathcal{K}$, reducing the minimal m for f_1 by one, producing a counterexample for smaller m, a contradiction.

The cases where $\operatorname{ind}(f, K) = n + \alpha$ for negative n and $0 < \alpha < 1$ are handled similarly, and illustrated for $n = -2$ and α about $\frac{1}{2}$ in Figure 1 (right).

□

2.6. **Locating arcs of negative variation.** The principal tool in proving the main theorem in the next section is the following theorem first obtained by Bell. It provides a method for locating arcs of negative variation on a curve of index zero.

Theorem 2.14 (Lollipop Lemma, Bell)**.** *Let $S \subset \mathbb{C}$ be a simple closed curve such that f has no fixed points on S. Suppose $F = \{a_0 < \cdots < a_n < a_{n+1} < \cdots < a_m\}$ is a partition of S, $a_{m+1} = a_0$ and $A_i = [a_i, a_{i+1}]$ such that $f(F) \subset T(S)$ and $f(A_i) \cap A_i = \emptyset$ for $i = 0, \ldots, m$. Suppose I is an arc in $T(S)$ meeting S only at its endpoints a_0 and a_{n+1}. Let J_{a_0} be a junction in $(\mathbb{C} \setminus T(S)) \cup \{a_0\}$ and suppose that $f(I) \cap (I \cup J_{a_0}) = \emptyset$. Let $R = T([a_0, a_{n+1}] \cup I)$ and $L = T([a_{n+1}, a_{m+1}] \cup I)$. Then one of the following holds*

(1) *If $f(a_{n+1}) \in R$, then*

$$\sum_{i \le n} \mathrm{var}(f, A_i, S) + 1 = \mathrm{ind}(f, I \cup [a_0, a_{n+1}]).$$

(2) *If $f(a_{n+1}) \in L$, then*

$$\sum_{i > n} \mathrm{var}(f, A_i, S) + 1 = \mathrm{ind}(f, I \cup [a_{n+1}, a_{m+1}].$$

(Note that in (1) in effect we compute $\mathrm{var}(f, \partial R)$ but technically, we have not defined $\mathrm{var}(f, A_i, \partial R)$ since the endpoints of A_i do not have to map inside R but they do map into $T(S)$. Similarly in Case (2).)

Proof. Without loss of generality, suppose $f(a_{n+1}) \in L$. Let $C = [a_{n+1}, a_{m+1}] \cup I$ (so $T(C) = L$). We want to construct a map $f' : C \to \mathbb{C}$, fixed-point-free homotopic to $f|_C$, that does not change variation on any arc A_i in C and has the properties listed below.

(1) $f'(a_i) \in L$ for all $n + 1 \le i \le m + 1$. Hence $\mathrm{var}(f', A_i, C)$ is defined for each $i > n$.
(2) $\mathrm{var}(f', A_i, C) = \mathrm{var}(f, A_i, S)$ for all $n + 1 \le i \le m$.
(3) $\mathrm{var}(f', I, C) = \mathrm{var}(f, I, S) = 0$.
(4) $\mathrm{ind}(f', C) = \mathrm{ind}(f, C)$.

Having such a map, it then follows from Theorem 2.13, that

$$\mathrm{ind}(f', C) = \sum_{i=n+1}^{m} \mathrm{var}(f', A_i, C) + \mathrm{var}(f', I, C) + 1.$$

By Theorem 2.5 $\mathrm{ind}(f', C) = \mathrm{ind}(f, C)$. By (2) and (3), $\sum_{i>n} \mathrm{var}(f', A_i, C) + \mathrm{var}(f', I, C) = \sum_{i>n} \mathrm{var}(f, A_i, S)$ and the Theorem would follow.

It remains to define the map $f' : C \to \mathbb{C}$ with the above properties. For each i such that $n + 1 \le i \le m + 1$, chose an arc I_i joining $f(a_i)$ to L as follows:

(a) If $f(a_i) \in L$, let I_i be the degenerate arc $\{a_i\}$.

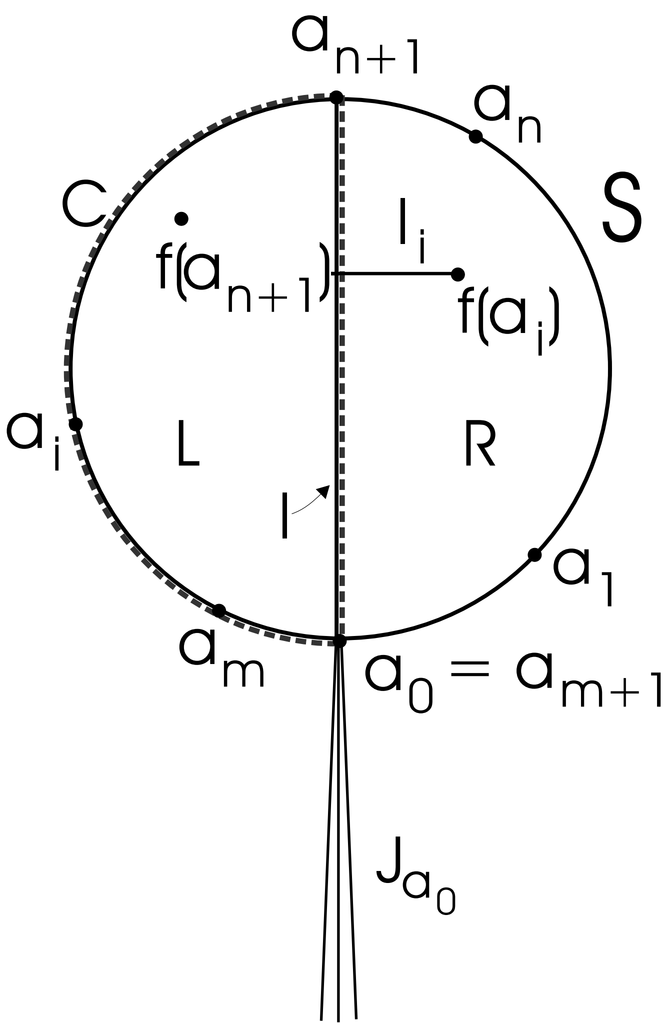

FIGURE 2. Bell's Lollipop.

(b) If $f(a_i) \in R$ and $n+1 < i < m+1$, let I_i be an arc in $R \setminus \{a_0, a_{n+1}\}$ joining $f(a_i)$ to I.
(c) If $f(a_0) \in R$, let I_0 be an arc joining $f(a_0)$ to L such that $I_0 \cap (L \cup J_{a_0}) \subset A_{n+1} \setminus \{a_{n+1}\}$.

Let $x_{n+1} = y_{n+1} = a_{n+1}$, $y_0 = y_{m+1} \in I \setminus \{a_0, a_{n+1}\}$ and $x_0 = x_{m+1} \in A_m \setminus \{a_m, a_{m+1}\}$. For $n+1 < i < m+1$, let $x_i \in A_{i-1}$ and $y_i \in A_i$ such that $y_{i-1} < x_i < a_i < y_i < x_{i+1}$. For $n+1 < i < m+1$ let $f'(a_i)$ be the endpoint of I_i in L, $f'(x_i) = f'(y_i) = f(a_i)$ and extend f' continuously from $[x_i, a_i] \cup [a_i, y_i]$ onto I_i and define f' from $[y_i, x_{i+1}] \subset A_i$ onto $f(A_i)$ by $f'|_{[y_i, x_{i+1}]} = f \circ h_i$, where $h_i : [y_i, x_{i+1}] \to A_i$ is a homeomorphism such that $h_i(y_i) = a_i$ and $h_i(x_{i+1}) = a_{i+1}$. Similarly, define f' on $[y_0, a_{n+1}] \subset I$ to

$f(I)$ by $f|_{[y_0,a_{n+1}]} = f \circ h_0$, where $h_0 : [y_0, a_{n+1}] \to I$ is a homeomorphism such that $h(a_{n+1}) = a_{n+1}$ and extend extend f' from $[x_{m+1}, a_0] \subset A_m$ and $[a_o, y_0] \subset I$ onto I_0 such that $f'(x_{m+1}) = f'(y_0) = f(a_0)$ and $f'(a_0)$ is the endpoint of I_0 in L.

Note that $f'(A_i) \cap A_i = \emptyset$ for $i = n+1, \ldots, m$ and $f'(I) \cap [I \cup J_{a_0}] = \emptyset$. To compute the variation of f' on each A_m and I we can use the junction J_{a_0} Hence $\mathrm{var}(f', I, C) = 0$ and, by the definition of f' on A_m, $\mathrm{var}(f', A_m, C) = \mathrm{var}(f(A_m, S)$. For $i = n+1, \ldots, m-1$ we can use the same junction J_{v_i} to compute $\mathrm{var}(f', A_i, C)$ as we did to compute $\mathrm{var}(f, A_i, S)$. Since $I_i \cup I_{i+1} \subset T(S)$ we have that $f'([a_i, y_i]) \cup f'([x_{i+1}, a_{i+1}]) \subset I_i \cup I_{i+1}$ misses that junction and, hence, make no contribution to variation $\mathrm{var}(f', A_i, C)$. Since $f'^{-1}(J_{v_i}) \cap [y_i, x_{i+1}]$ is isomorphic to $f^{-1}(J_{v_i}) \cap A_i$, $\mathrm{var}(f', A_i, C) = \mathrm{var}(f(A_i, S)$ for $i = n+1, \ldots, m$.

To see that f' is fixed-point-free homotopic to $f|_C$, note that we can pull the image of A_i back along the arcs I_i and I_{i+1} in R without fixing a point of A_i at any level of the homotopy. Since f' and $f|_C$ are fixed-point-free homotopic and f has no fixed points in $T(S)$, it follows from Theorems 2.3 and 2.5, that $\mathrm{ind}(f', C) = \mathrm{ind}(f, C)$. □

Note that if f is fixed point free on $T(S)$, then $\mathrm{ind}(f, S) = 0$ and the following Corollary follows.

Corollary 2.15. *Assume the hypotheses of Theorem 2.14. Suppose, in addition, f is fixed point free on $T(S)$. Then if $f(a_{n+1}) \in R$ there exists $i \leq n$ such that* $\mathrm{var}(f, A_i, S) < 0$. *If $f(a_{n+1}) \in L$ there exists $i > n$ such that* $\mathrm{var}(f, A_i, S) < 0$.

2.7. **Extensions to variation for infinite partitions.** Recall our Standing Hypotheses in 1.1: $f : \mathbb{C} \to \mathbb{C}$ takes continuum X into $T(X)$ with no fixed points in $T(X)$, and X is minimal with respect to these properties.

Definition 2.16 (Bumping Simple Closed Curve)**.** *A simple closed curve S in $\mathbb{C}$ which has the property that $S \cap X$ is nondegenerate and $T(X) \subset T(S)$ is said to be a* bumping simple closed curve for X. *A subarc A of a bumping simple closed curve, whose endpoints lie in X, is said to be a* bumping (sub)arc for X. *Moreover, if S' is any bumping simple closed curve for X which contains A, then S' is said to* complete A.

A *crosscut* of $O_\infty = \mathbb{C}_\infty \setminus T(X)$ is an open arc Q lying in O_∞ such that $\overline{Q}$ meets ∂O_∞ in two endpoints $a \neq b \in T(X)$. (As seems to be traditional, we use "crosscut of $T(X)$" interchangeably with "crosscut of O_∞.") If $S \cap X$ is nondegenerate and proper in S, then each component of $S \setminus X$ is a crosscut of $T(X)$. A similar statement holds for a bumping arc A.

Since f has no fixed points in $T(X)$ and X is compact, we can choose a bumping simple closed curve S so close pointwise to $T(X)$, with such small crosscuts, and with the domains cut off so close pointwise to $T(X)$, that f has no fixed points in $T(S)$. Thus, we obtain the following corollary to Theorem 2.5.

Corollary 2.17. *There is a bumping simple closed curve S for X such that $f|_{T(S)}$ is fixed point free; hence, by 2.5,* $\text{ind}(f, S) = 0$*. Moreover, any bumping simple closed curve S' such that $S' \subset T(S)$ has* $\text{ind}(f, S') = 0$*. Furthermore, any crosscut Q of X for which f has no fixed points in $T(X \cup Q)$ can be completed to a bumping simple closed curve S for which* $\text{ind}(f, S) = 0$*.*

Theorem 2.18. *Suppose S is a bumping simple closed curve for X. Then there is a $\delta > 0$ such that if $A \subset S$ is a bumping subarc for X with* $\text{diam}(A) \leq \delta$*, then* $\text{var}(f, A, S) = 0$*.*

Proof. Suppose not. Then, without loss of generality, there is a sequence $\{A_i\}_{i=1}^{\infty}$ of bumping subarcs converging to a point $a \in X \cap S$ such that $\text{var}(f, A_i, S) \neq 0$ for each i. Let J_a be a junction based at a. Since $f(a) \in X$, there are connected neighborhoods U of a and V of $f(a)$ such that $\overline{V} \cap J_a = \emptyset$ and $f(U) \subset V$. We may assume $U \cap S$ is connected. Since $A_i \to a$, there is a k such that for all $i \geq k$, $\overline{A_i} \subset U$. We may adjust the junction J_a to a junction J_{a_i}, keeping sufficiently close to S, so that for $i \geq k$, $a_i \in A_i$ and $f(\overline{A_i}) \cap J_{a_i} = \emptyset$. It follows that $\text{var}(f, A_i, S) = 0$. This contradiction completes the proof. □

Corollary 2.19. *Suppose S is a bumping simple closed curve for X. Let C be closed such that $S \setminus C = \bigcup_{i=1}^{\infty} A_i$, where the A_i are disjoint bumping subarcs (or crosscuts) such that $\overline{f(A_i)} \cap \overline{A_i} = \emptyset$ for each i. Then for all but finitely many A_i,* $\text{var}(f, A_i, S) = 0$*.*

The following Theorem follows from 2.19 and the remark following Definition 2.11.

Theorem 2.20. *Suppose S is a bumping simple closed curve with A a bumping subarc in S such that $f(A) \cap A = \emptyset$. Suppose $A = \bigcup_{i \in I} A_i$ is a partition of A into possibly infinitely many bumping subarcs. Then* $\text{var}(f, A, S) = \sum_{i \in I} \text{var}(f, A_i, S)$*.*

Remark 2.21. *It follows from Corollary 2.19 and Theorem 2.20 that Theorems 2.13 and 2.14 hold for infinite partitions of bumping simple closed curves where the partition elements map off themselves.*

2.7.1. *Variation on a crosscut.* We show that variation is local by defining it for a single bumping subarc (or single crosscut).

Proposition 2.22. *Suppose A is a bumping subarc on X. If* $\text{var}(f, A, S)$ *is defined for some bumping simple closed curve S completing A, then for any bumping simple closed curve S' completing A,* $\text{var}(f, A, S) = \text{var}(f, A, S')$*.*

Proof. Let A be a bumping subarc on X for which $f(A) \cap A = \emptyset$. Let S and S' be two bumping simple closed curves completing A for which variation is defined. Let J_a and $J_{a'}$ be junctions whereby $\text{var}(f, A.S)$ and $\text{var}(f, A, S')$ are respectively computed. Suppose first that both junctions lie (except for $\{a, a'\}$) in $\mathbb{C} \setminus (T(S) \cup T(S'))$. By the Junction Straightening

Proposition 2.9 and Remark 2.4, either junction can be used to compute either variation on A, so the result follows. Otherwise, at least one junction is not in $\mathbb{C} \setminus (T(S) \cup T(S'))$. But both junctions are in $\mathbb{C} \setminus T(X \cup A)$. Hence, we can find another simple closed curve S'' such that S'' completes Q and both junctions lie in $(\mathbb{C} \setminus T(S'')) \cup \{a, a'\}$. Then by the Proposition 2.9 and Remark 2.4, $\text{var}(f, A, S) = \text{var}(f, A, S'') = \text{var}(f, A, S')$. □

It follows from Proposition 2.22 that variation on a crosscut of X is independent of the simple closed curve surrounding $T(X)$ of which Q is a subarc.

Definition 2.23 (Variation on a crosscut)**.** *Suppose Q is a crosscut of X such that $f(Q) \cap Q = \emptyset$. Let S be any bumping simple closed curve completing Q for which variation is defined. Define the* variation of f on Q with respect to X, *denoted* $\text{var}(f, Q, X)$, *by* $\text{var}(f, Q, X) = \text{var}(f, Q, S)$.

We will need the following proposition in Section 3.3.

Proposition 2.24. *Suppose $Q = [a, b]$ is a crosscut of $T(X)$ such that f is fixed point free on $T(X \cup Q)$ and $f(Q) \cap Q = \emptyset$. Suppose Q is replaced by a bumping subarc A with the same endpoints such that Q separates $A \setminus \{a, b\}$ from ∞ and each component Q_i of $A \setminus X$ is a crosscut such that $f(Q_i) \cap Q_i = \emptyset$. Then*

$$\text{var}(f, Q, X) = \sum_i \text{var}(f, Q_i, X).$$

Proof. Note that each of Q and A can be completed to a simple closed curve with the same bumping arc B such that on both $T(Q \cup B)$ and $T(A \cup B)$, f is fixed point free. By Corollary 2.17 and Remark 2.21 we have

$$\text{var}(f, Q \cup B) + 1 = \text{ind}(f, Q \cup B) = \text{ind}(f, A \cup B) = \text{var}(f, A \cup B) + 1.$$

Thus,

$$\begin{aligned} \text{var}(f, Q, Q \cup B) + \text{var}(f, B, Q \cup B) &= \text{var}(f, Q \cup B) = \text{var}(f, A \cup B) \\ &= \text{var}(f, A, A \cup B) + \text{var}(f, B, A \cup B). \end{aligned}$$

Consequently, by Theorem 2.20 and Proposition 2.22,

$$\begin{aligned} \text{var}(f, Q, X) &= \text{var}(f, Q, Q \cup B) = \text{var}(f, A, A \cup B) \\ &= \sum_i \text{var}(f, Q_i, A \cup B) = \sum_i \text{var}(f, Q_i, X). \end{aligned}$$

□

2.8. **Prime Ends.** Prime ends provide a way of studying the approaches to the boundary of a simply-connected plane domain with non-degenerate boundary. See [7] or [9] for an analytic summary of the topic and [12] for a more topological approach. We will be interested in the prime ends of $O_\infty = \mathbb{C}_\infty \setminus T(X)$. Recall $Y = T(X)$. Let $\Delta_\infty = \{z \in \mathbb{C}_\infty \mid |z| > 1\}$ be the "unit disk about ∞." The Riemann Mapping Theorem guarantees the existence of a conformal map $\phi : \Delta_\infty \to O_\infty$ taking $\infty \to \infty$, unique up to the argument of the derivative at ∞. Fix such a map ϕ. We identify $S^1 = \partial\Delta_\infty$ with $\mathbb{R}/\mathbb{Z}$ and identify points $e^{2\pi it}$ in $\partial\Delta_\infty$ by their argument (mod 2π). Crosscuts are defined in Section 2.7.

Definition 2.25 (Prime End). *A* chain of crosscuts *is a sequence* $\{Q_i\}_{i=1}^{\infty}$ *of crosscuts of* O_∞ *such that for* $i \neq j$, $Q_i \cap Q_j = \emptyset$, $\operatorname{diam}(Q_i) \to 0$, *and for all* $j > i$, Q_i *separates* Q_j *from* ∞ *in* O_∞. *Two chains of crosscuts are said to be* equivalent *iff it is possible to form a sequence of crosscuts by selecting alternately a crosscut from each chain so that the resulting sequence of crosscuts is again a chain. A* prime end $\mathcal{E}$ *is an equivalence class of chains of crosscuts.*

If $\{Q_i\}$ is a chain of crosscuts of O_∞, it can be shown that $\{\phi^{-1}(Q_i)\}$ is a chain of crosscuts of Δ_∞ converging to a single point $t \in S^1 = \partial\Delta_\infty$, independent of the representative chain. Thus, we may name the prime end $\mathcal{E}$ defined by $\{Q_i\}$, where $\phi^{-1}(Q_i) \to t \in S^1$, by $\mathcal{E}_t$.

Let $\mathcal{E}_t$ be a prime end with defining chain of crosscuts $\{Q_i\}$. Let O_i denote the bounded complementary domain of $O_\infty \setminus Q_i$. We use $\{Q_i\}$ and $\{O_i\}$ to define two subcontinua of ∂O_∞ associated with $\mathcal{E}_t$.

Definition 2.26 (Impression and Principal Continuum). *The set*

$$\operatorname{Im}(\mathcal{E}_t) = \bigcap_{i=1}^{\infty} \overline{O_i}$$

is a subcontinuum of ∂O_∞ *called the* impression *of* $\mathcal{E}_t$. *The set*

$$\operatorname{Pr}(\mathcal{E}_t) = \{z \in \partial O_\infty \mid \textit{for some chain } \{Q_i\} \textit{ defining } \mathcal{E}_t,\ Q_i \to z\}$$

is a continuum called the principal continuum *of* $\mathcal{E}_t$.

For a prime end $\mathcal{E}_t$, $\operatorname{Pr}(\mathcal{E}_t) \subset \operatorname{Im}(\mathcal{E}_t)$, possibly properly. We will be interested in the existence of prime ends $\mathcal{E}_t$ for which $\operatorname{Pr}(\mathcal{E}_t) = \operatorname{Im}(\mathcal{E}_t) = \partial O_\infty$.

Definition 2.27 (External Rays). *Let* $t \in [0, 1)$ *and define*

$$R_t = \{z \in \mathbb{C} \mid z = \phi(re^{2\pi it}), 1 < r < \infty\}.$$

We call R_t *the* external ray *at* t. *If* $x \in R_t$ *then the* (Y, x)-end of R_t *is the component* K_x *of* $R_t \setminus \{x\}$ *whose closure meets* Y.

The external rays foliate O_∞.

Definition 2.28 (Essential crossing). *An external ray* R_t *is said to* cross *a crosscut* Q essentially *if and only if an* (Y, x)*-end of* R_t *is contained in the bounded complementary domain of* $Y \cup Q$.

The properties below may readily be established.

Proposition 2.29 ([7]). *Let* $\mathcal{E}_t$ *be a prime end of* O_∞. *Then* $\operatorname{Pr}(\mathcal{E}_t) = \overline{R_t} \setminus R_t$. *Moreover, for each* $1 < r < \infty$ *there is a crosscut* Q_r *at* $\phi(re^{2\pi it})$ *on* R_t *with* $\operatorname{diam}(Q_r) \to 0$ *as* $r \to 1$ *and such that* R_t *crosses* Q_r *essentially.*

Definition 2.30 (Landing Points and Accessible Points). *If* $\Pr(\mathcal{E}_t) = \{x\}$, *then we say* R_t lands *at* $x \in T(X)$ *and* x *is the* landing point *of* R_t. *A point* $x \in \partial T(X)$ *is said to be* accessible *(from* O_∞*) iff there is a arc in* $O_\infty \cup \{x\}$ *one endpoint of which is* x.

Proposition 2.31. *A point* $x \in \partial T(X)$ *is accessible iff* x *is the landing point of some external ray* R_t.

Definition 2.32 (Channels). *A prime end* $\mathcal{E}_t$ *of* O_∞ *for which* $\Pr(\mathcal{E}_t)$ *is nondegenerate is said to be a* channel in ∂O_∞ *(or in* $T(X)$*). If moreover* $\Pr(\mathcal{E}_t) = \partial O_\infty = \partial T(X)$, *we say* $\mathcal{E}_t$ *is a* dense *channel. A crosscut* Q *with endpoints* $\{a, b\}$ *is said to* cross *the channel* $\mathcal{E}_t$ *iff* R_t *crosses* Q *essentially.*

When X is locally connected, there are no channels, as the following classical theorem proves. In this case, every prime end has degenerate principal set and degenerate impression.

Theorem 2.33 (Caratheodory). X *is locally connected iff the Riemann map* $\phi : \Delta_\infty \to O_\infty = \mathbb{C}_\infty \setminus T(X)$ *taking* $\infty \to \infty$ *extends continuously to* $S^1 = \partial \Delta_\infty$.

2.9. **Index and Variation for Caratheodory Loops.** We extend the definitions of index and variation and the theorem relating index to variation to *Caratheodory loops*.

Definition 2.34 (Caratheodory Loop). *Let* $\phi : S^1 \to \mathbb{C}$ *such that* ϕ *is continuous and has an extension* $\psi : \mathbb{C} \setminus T(S^1) \to \mathbb{C} \setminus T(\phi(S^1))$ *such that* $\psi|_{\mathbb{C}\setminus T(S^1}$ *is an orientation preserving homeomorphism from* $\mathbb{C} \setminus T(S^1)$ *onto* $\mathbb{C} \setminus T(\phi(S^1))$. *We call* ϕ *(and loosely,* $S = \phi(S^1)$*), a* Caratheodory loop.

In particular, if a Riemann map extends continuously to S^1, we have a Caratheodory loop. In order to define variation of f on a Caratheodory loop $S = \phi(S^1)$, we do the partitioning in S^1 and transport it to the Caratheodory loop $S = \phi(S^1)$. An *allowable* partition of S^1 is a set $\{a_0 < a_1 < \cdots < a_n\}$ in S^1 ordered counterclockwise, where $a_0 = a_n$ and A_i denotes the counterclockwise interval $[a_{i-1}, a_i]$, such that for each i, $f(\phi(a_i)) \in T(\phi(S^1))$ and $f(\phi(A_i)) \cap \phi(A_i) = \emptyset$. Variation on each path $\phi(A_i)$ is then defined exactly as in Definition 2.7, except that the junction (see Definition 2.6) is chosen so that the vertex $v \in \phi(A_i)$ and $T(\phi(S^1)) \subset h(U) \cup \{v\}$, and the crossings of the junction by $f(\phi(A_i))$ are counted (see Definition 2.7). Variation on the whole loop, or an allowable subarc thereof, is defined just as in Definition 2.11, by adding the variations on the partition elements. At this point in the development, variation is defined only relative to the given allowable partition F of S^1 and the parameterization ϕ of S: $\mathrm{var}(f, F, \phi(S^1))$.

Index on a Caratheodory loop S is defined exactly as in Section 2.1 with $S = \phi(S^1)$ providing the parameterization of S. Likewise, the definition of fractional index and Proposition 2.1 apply to Caratheodory loops.

Theorems 2.2, 2.3, Corollary 2.4, and Theorem 2.5 apply to Caratheodory loops. It follows that index on a Caratheodory loop S is independent of the choice of parameterization ϕ. It remains to extend Theorem 2.13 to Caratheodory loops. It then follows that variation on a Caratheodory loop S is independent of choice of parameterization $\phi(S^1) = S$ and allowable partition of S^1. Thus, $\mathrm{var}(f, S)$ is well-defined for any Caratheodory loop S that has some parameterization and some allowable partition.

Theorem 2.35. *Suppose $S = \phi(S^1)$ is a parameterized Caratheodory loop in $\mathbb{C}$ and f has no fixed points on S. Suppose variation of f on $S^1 = A_0 \cup \cdots \cup A_n$ with respect to ϕ is defined for some partition $A_0 \cup \cdots \cup A_n$ of S^1. Then*

$$\mathrm{ind}(f, \phi) = \sum_{i=0}^{n} \mathrm{var}(f, A_i, \phi(S^1)) + 1.$$

Proof. Let ψ be the homeomorphic extension of ϕ carrying $\mathbb{C} \setminus T(S^1)$ onto $\mathbb{C} \setminus T(S)$. Let $S_i = \{1 + \frac{1}{i})e^{2\pi i\theta} \mid \theta \in [0, 1)\}$ be the concentric circles of radius $1 + \frac{1}{i}$ converging to S^1. For the given partition $A_0 \cup \cdots \cup A_n$ of S^1, let $A_j = [a_{j-1}, a_j]$ with $a_n = a_0$, Then $a_j = e^{2\pi i\theta_j}$ for some $\theta_j \in [0, 1)$ with $\theta_0 < \theta_1 < \cdots < \theta_n = \theta_0$. For each j, let $R_{i,j} = \{re^{2\pi i\theta_j} \mid 1 \le r \le 1 + \frac{1}{i}\}$ be the radial arc from S_i to S^1 at a_j. Note $\mathrm{diam}(R_{i,j}) \to 0$ as $i \to \infty$. Let $C_{i,j} = \{(1 + \frac{1}{i})e^{2\pi i\theta} \mid \theta_{j-1} \le \theta \le \theta_j\}$ be the subarc of S_i between $c_{i,j-1} = S_i \cap R_{i,j-1}$ and $c_{i,j} = S_i \cap R_{i,j}$. Then $C_{i,j} = [c_{i,j-1}, c_{i,j}]$ approximates A_j as $i \to \infty$. Moreover, $c_{i,j} \to a_{i,j}$ as $i \to \infty$. Since ψ is a map, the same holds for the images.

For each j, choose a junction J_{v_j} with vertex $v_j \in \psi(A_j)$ so that $T(\psi(S^1)) \subset U_j \cup \{v_j\}$, where U_j is the usual complementary half-plane of the junction (see Definition 2.6).

Since $\psi(C_{i,j})$ approximates $\psi(A_j)$ with $\psi(R_{i,j})$ shrinking as $i \to \infty$, we may choose i sufficiently large so that the following conditions are satisfied:

(1) $\psi(C_{i,j}) \cap f(\psi(C_{i,j})) = \emptyset$.
(2) $\mathrm{var}(f, A_j, \psi(S^1)) = \mathrm{var}(f, \psi(C_{i,j}), \psi(S_i))$.
(3) There are no fixed points of f in $T(\psi(S_i)) \setminus T(\psi(S^1))$.
(4) $\mathrm{ind}(f, \psi(S_i)) = \mathrm{ind}(f, \psi(S^1))$.

Condition (1) holds because of continuity of ψ and the similar condition for A_j. To see condition (2), apply the observations about the stability of variation in Section 2.4. Condition (3) holds because there are no fixed points of f on S, and $\psi(S_i)$ approximates S. Condition (4) then follows from the stability of index under fixed-point-free homotopy, noted in Section 2.2. (Use ψ on S_k, $k \ge i$, to define the homotopy.)

By Theorem 2.13,

$$\mathrm{ind}(f, \psi(S_i)) = \sum_{j=0}^{n} \mathrm{var}(f, \psi(C_{i,j}), \psi(S_i)) + 1.$$

Hence, noting $\phi = \psi|_{S^1}$, it follows from conditions (2) and (4) that

$$\mathrm{ind}(f, \phi) = \sum_{j=0}^{n} \mathrm{var}(f, A_j, \phi(S^1)) + 1.$$

□

3. Geometric Prime Ends

For the proof of the principal new result in this paper, the uniqueness of the outchannel in the next section, we develop in this section Bell's tools using a special collection of geometric crosscuts of $T(X)$. These results, and their connection to the standard conformal theory of prime-ends, are more fully developed in [1]. Recall our Standing Hypotheses in 1.1: $f : \mathbb{C} \to \mathbb{C}$ takes continuum X into $Y = T(X)$ with no fixed points in $T(X)$, and X is minimal with respect to these properties.

3.1. **Geometric crosscuts.** We are going to define a special class of *geometric crosscuts (chords)* of $O_\infty = \mathbb{C}_\infty \setminus Y$ and auxiliary nonseparating plane continua which contain Y as subsets, and in some sense have some of the same channels as Y, but have a nicer boundary than Y.

Definition 3.1 (Closest Points)**.** *Let $z \in O_\infty$. We define the set of* closest points *of Y to z by*

$$C(z) = \{x \in Y \mid \mathrm{d}(z, x) = \mathrm{d}(z, Y)\}.$$

The collection of all $z \in O_\infty$ with at least three closest points is denoted $\mathcal{C}_3$; the collection of all $z \in O_\infty$ with exactly two closest points is denoted $\mathcal{C}_2$.

Definition 3.2. *For a given compact subset $Z \subset \mathbb{C}$, the* convex hull *of Z, denoted by* Convexhull(Z)*, consists of Z together with all straight line segments whose endpoints are in Z. Then* Convexhull(Z) *is a convex continuum. For $z \in \mathbb{C} \setminus Y$, $V(z) =$* Convexhull$(C(z)) \setminus C(z)$ *if $z \in \mathcal{C}_2$ and $V(z) = \mathrm{Int}(\mathrm{Convexhull}(C(z)))$ if $z \in \mathcal{C}_3$. If $z \in \mathcal{C}_2$, we call $V(z)$ a* chord*. We call $V(z)$ for $z \in \mathcal{C}_3$ a* simplex.

Note that each simplex of Y lies in the convex hull of Y. Note also that for $z \in \mathcal{C}_2$, $V(z)$ is an open arc and for $z \in \mathcal{C}_3$, $V(z)$ is an open topological disk with Jordan curve boundary composed of countably many straight line crosscuts plus a set contained in $C(z)$.

Definition 3.3. *For $z \in O_\infty$, let $B(z, Y) = \{w \in \mathbb{C} \mid d(z, w) < d(z, Y)\}$ and $S(z, Y) = \{w \in \mathbb{C} \mid d(w, z) = d(z, Y)\}$. Then $B(z, Y)$ is the maximal open ball around z, disjoint from Y and its boundary is $S(z, Y)$. Moreover, if $z \in Y$, let $S(z, Y) = \{z\}$.*

Definition 3.4 (Geometric Crosscuts–Chords)**.** *Define the following collections of crosscuts and simplices:*

$$\mathcal{F}_s = \{L \mid L \text{ is a component of } \partial V(z) \setminus C(z) \text{ for some } z \in \mathcal{C}_3\}$$

$$\mathcal{F}_c = \{K \mid K \text{ is a component of } \partial\text{Convexhull}(Y) \setminus Y\}$$
$$\mathcal{F}_2 = \{V(z) \mid z \in \mathcal{C}_2\}$$
$$\mathcal{F}_3 = \{V(z) \mid z \in \mathcal{C}_3\}$$
$$\mathcal{F} = \mathcal{F}_s \cup \mathcal{F}_c \cup \mathcal{F}_2 \cup \mathcal{F}_3$$
$$\mathcal{G} = \mathcal{F}_2 \cup \mathcal{F}_s \cup \mathcal{F}_c$$
$$\mathcal{G}_\delta = \{Q \in \mathcal{G} \mid \text{diam}(Q) \le \delta\}.$$

We call the elements of $\mathcal{G}$ *chords*. Note that every element of $\mathcal{G}$ and every element of $\mathcal{G}_\delta$ is a straight line segment, while $\mathcal{F}$ also contains simplexes.

Lemma 3.5. *Let $z_1 \neq z_2 \in \mathcal{C}_2 \cup \mathcal{C}_3$. Then either $\overline{V(z_1)} \cap \overline{V(z_2)}$ is the empty set, or is the closure of an element $Q \in \mathcal{G}$, or $\overline{V(z_1)} \cap \overline{V(z_2)}$ is a single point in Y. In particular, members of $\mathcal{F}$ are either equal or disjoint.*

Proof. Clearly $V(z) \subset B(z, Y)$ for each $z \in \mathbb{C}$. If $S(z_1, Y) \cap S(z_2, Y) = y$ is a single point, then this point clearly is in Y and $\overline{V(z_1)} \cap \overline{V(z_2)} = y$ is a single point in Y. Hence, we may assume that $z_1 \neq z_2 \in O_\infty$ such that $B(z_1, Y) \cap B(z_2, Y) \neq \emptyset$. Then $Y \cap [B(z_1, Y) \cup B(z_2, Y)] = \emptyset$ and, since $\{z_1, z_2\} \subset \mathcal{C}_2 \cup \mathcal{C}_3$, $S(z_1, Y) \cap S(z_2, Y) = \{a, b\}$, with $a \neq b$. Then it is easy to see that $V(z_1) \subset K_1 = \overline{\text{Convexhull}(B(z_1, Y) \setminus B(z_2, Y))}$ and $V(z_2) \subset K_2 = \overline{\text{Convexhull}(B(z_2, Y) \setminus B(z_1, Y))}$. Since $K_1 \cap K_2 = [a, b]$, the desired result follows. □

If $\{H_i\}$ is a sequence in $\mathcal{F}$ we say that the sequence $\{H_i\}$ converges if the sequence $\{\overline{H}_i\}$ converges in the space of compacta of $\mathbb{C}$ with the Hausdorff metric topology. If the H_i are distinct and the limit exists, the limit is either a point of the continuum X or the closure of a chord $Q \in \mathcal{G}$.

The proof of the following result is left to the reader. The last part follows from stability of variation (see Section 2.4).

Proposition 3.6 (Compactness). *If $\{Q_i\}$ is a convergent sequence of chords in $\mathcal{G}_\delta$ or of distinct elements of $\mathcal{F}_3$ of diameter less than or equal to δ, then either Q_i converges to a chord in $\mathcal{G}_\delta$ or Q_i converges to a point of X. Moreover, if $\{Q_i\}$ converges to a chord Q, then for sufficiently large i,* $\text{var}(f, Q, Y) = \text{var}(f, Q_i, Y)$

Corollary 3.7. *For each $\varepsilon > 0$, there exist $\delta > 0$ such that for all $Q \in \mathcal{G}$ with $Q \subset B(Y, \delta)$,* $\text{diam}(Q) < \varepsilon$.

Proof. Suppose not, then there exist $\varepsilon > 0$ and a sequence Q_i in $\mathcal{G}$ such that $\lim Q_i \subset X$ and $\text{diam}(Q_i) \ge \varepsilon$ a contradiction to Proposition 3.6. □

The proof of the following proposition is left to the reader:

Proposition 3.8. *For each $\varepsilon > 0$ there exists $\delta > 0$ such that for each open arc A such that $\overline{A} \cap Y = \{a, b\}$, with $a \neq b$, and* $\text{diam}(A) < \delta$, *$T(Y \cup A) \subset B(Y, \varepsilon)$.*

Proposition 3.9. *Let ε, δ be as in Proposition 3.8 above with $\delta < \varepsilon/2$ and let A be a crosscut of Y such that* $\text{diam}(A) < \delta$. *Let $x \in T(A \cup Y) \cap V(z)$ for some $z \in \mathbb{C} \setminus Y$ such that $d(x, A) \geq \varepsilon$, then* $\text{diam}(V(z)) < 2\varepsilon$.

Proof. Suppose that clA is an arc from a to b. If $z \in T(A \cup Y)$, then by Proposition 3.8, $d(z, Y) < \varepsilon$ and $\text{diam}(V(z) < 2\varepsilon$. Hence we may assume that $z \notin T(A \cup Y)$. Then the straight line segment B from x to z must cross A at some point w. Hence $d(x, w) \geq \varepsilon$ and $d(x, z) = d(x, w) + d(w, z) > d(a, z)$, a contradiction since $B(z, d(z, Y)) \cap Y = \emptyset$. □

Proposition 3.10. *Let $C \subset$ Convexhull(Y) be a crosscut of Y and let A and B be disjoint closed sets in Y such that $\overline{C} \cap A \neq \emptyset \neq \overline{C} \cap B$ and for each $x \in C$ there exists $F_x \in \mathcal{F}$ such that $x \in F_x$ and $\overline{F_x} \cap (A \cup B) \neq \emptyset$. Then there exists $F \in \mathcal{F}$ such that $\overline{F} \cap \overline{C} \neq \emptyset$, $\overline{F} \cap A \neq \emptyset$ and $\overline{F} \cap B \neq \emptyset$.*

Proof. The proof follows from Proposition 3.6, the connectedness of C and the fact that $\overline{C}$ meets both A and B. □

Proposition 3.10 allows us to replace small crosscuts which cross the prime end $\mathcal{E}_t$ with non-trivial principal continuum essentially by small nearby chords which also cross $\mathcal{E}_t$ essentially. For if C is a small crosscut in Convexhull(Y) with endpoints a and b which crosses R_t essentially, let A and B be the closures of the sets in Y accessible from a and b, respectively by small arcs missing R_t.

The following result of Bell's is the linchpin of the theory of chords.

Theorem 3.11 (Geometric Foliation, Bell)**.** *The collection $\mathcal{F}$ foliates* Convexhull(Y) $\setminus Y$.

Proof. Suppose that K is a component of Convexhull(Y) $\setminus \bigcup \mathcal{F} \cup Y$. Then K meets no $V(z)$.

Claim 1. ∂K contains a geometric cross cut $Q = (a, b)$ of Y and K is open.

Proof of Claim 1. Since Y is bounded, $\bigcup \mathcal{F} \cup Y$ is closed by 3.6 and K is open. Hence $\partial K \subset \mathcal{G} \cup Y$. Since Y does not separate the plane, there exists $Q \in \mathcal{G}$ and a point $q \in Q \cap \partial K$. Without loss of generality Q is a subset of the x-axis and there exists a sequence $q_i \in K$ converging to q such that each q_i is contained in the upper half plane W. We claim that there exists $r > 0$ such that $B(q, r) \cap W \subset K$. If not, then for each i there exists $s_i \in W \cap B(q, 1/i) \cap \partial K$. Then, for i large, $s_i \in \mathbb{C} \setminus Y$ and, hence, there exists $Q_i = (a_i, b_i) \in \mathcal{G}$ with $s_i \in Q_i$. By 3.5, $(a, b) \cap (a_i, b_i) = \emptyset$. It follows from the definition of geometric cross cuts that $\lim Q_i = Q$ Hence there exist i, j, k and an arc $M \subset \mathbb{C} \setminus Y$ joining q_i and q_j which intersects Q_k exactly once. It follows that q_i and q_j are in distinct components of $\mathbb{C} \setminus Y \cup Q_k$, a contradiction. Thus $\{x \in (a, b) \mid x \in \partial K\}$ is open in (a, b). However, it is also closed in (a, b) and $(a, b) \subset \partial K$.

Claim 2. K is convex.

Proof of Claim 2. Since $Y \cup \bigcup \mathcal{F}$ is connected, it follows that K is simply connected. Suppose that K is not convex. Then there exist $p \neq q \in K$ and $x \in (p,q) \subset L$, where L is the straight line through p and q, such that $x \in \partial K$. Without loss of generality, L is the x-axis. Let A be an arc from p to q in K and let A' be an irreducible open subarc of A which meets two components of $K \setminus \{x\}$. We may assume that $A = A'$ and A is contained in the upper half plane. Let W be the bounded complementary domain of the simple closed curve $A \cup (p,q)$. Note that $Y \cap \overline{W} \neq \emptyset$. For if $Y \cap \overline{W} \neq \emptyset$, then $x \in (c,d) \in \mathcal{G}$, where $\{c,d\} \subset \mathbb{C} \setminus L$ with one of the endpoints, say c, contained in the upper half plane. Since $(c,d) \cap A = \emptyset$, $c \in Y \cap W$.

Choose $y \in Y \cap \overline{W}$ such that the second coordinate of y is maximal. Note that $B(z, d(z,y) \cap Y = \emptyset$ for z directly above and very close to y. Let z be a point directly above y such that $B(z, d(z,y) \cap Y = \emptyset$ and $d(z,y)$ is maximal for all such points. Since there exists points in ∂K higher than y, such a maximal ball must exist. Then there exist $y' \in S(z,Y) \setminus \{y\} \neq \emptyset$. and $(y,y') \in \mathcal{G}$. Since y is the highest point of Y in $\overline{W}$, $y' \notin \overline{W}$. Hence $(y,y') \cap A \neq \emptyset$, a contradiction. This completes the proof of Claim 2.

Claim 3. If $Q = (a,b) \subset \partial K$ and $Q \in \mathcal{G}$, then $Q \in \mathcal{F}_2 \cup \mathcal{F}_s$.

Proof of Claim 3. Suppose not, then $Q \in \mathcal{F}_c$. Please see figure 3 for a geometric picture of the notations defined below. Let L be the straight line containing Q and let L_p be the perpendicular bisector of Q. Let $c = L \cap L_p$ and choose $d \in L_p \cap K$. Since K is convex, $(c,d] \subset K$. We may assume that L_p is the x-axis and L is the y-axis. Then L divides the plane into a left half plane H and a right half plane H_K and we may assume $Y \subset \overline{H_K}$ since $Q \in \mathcal{F}_c$. Let H_a and H_b be the components of $\mathbb{C} \setminus L_p$ which contain a and b, respectively. Let $L_a \subset \overline{H}$ and $L_b \subset \overline{H}$ be the half lines parallel to L_p and containing a and b, respectively. Then $C(x) = \{a\}$ for each $x \in L_a$ and $C(x) = \{b\}$ for each $x \in L_b$. Let $E = \{x \in \overline{H} \mid d(x, \overline{H_a} \cap Y) = d(x, \overline{H_b} \cap Y)\}$. Then E is closed and E separates L_a from L_b. Hence E contains an unbounded sequence e_i which is located between the half lines L_a and L_b. For each i choose $a_i \in S(e_i, Y) \cap \overline{H_a}$ and $b_i \in S(e_i, Y) \cap \overline{H_b}$. Then $Q_i = (a_i, b_i) \in \mathcal{G}$ and $\lim a_i = a$ and $\lim b_i = b$. Note that since $Q \notin \mathcal{F}_2 \cup \mathcal{F}_s$, $\{a,b\} \setminus S(z,Y) \neq \emptyset$ for each $z \in \mathbb{C} \setminus Y$. In particular, $\{a_i, b_i\} \setminus \{a,b\} \neq \emptyset$ for each i. Hence $Q_i \subset H_K$ for each i and $Q_i \cap (c,d] \neq \emptyset$ for i large, a contradiction. This completes the proof of Claim 3.

We will continue to use the notation used in the proof of Claim 3 and illustrated in figure 3 with the exception that it is no longer necessarily true that $Y \subset \overline{H_K}$. Since K is convex, we may assume that $K \subset H_K$. By Claim 3, there exists $Q = (a,b) \in \mathcal{F}_2 \cup \mathcal{F}_s$ such that $Q \subset \partial K$. Then there exists $e \in L_p$ such that $\{a,b\} \subset S(e,Y)$. Suppose first that there exists $e \in (L_p \cap H_K) \cup \{c\}$ such that $\{a,b\} \subset S(e,Y)$. If $S(e,Y)$ meets H_K, it follows that $K \supset V(e)$, a contradiction. Hence $S(e,Y) \cap H_k = \emptyset$. As in the proof of Claim 2, we may assume that the first coordinate of e is maximal.

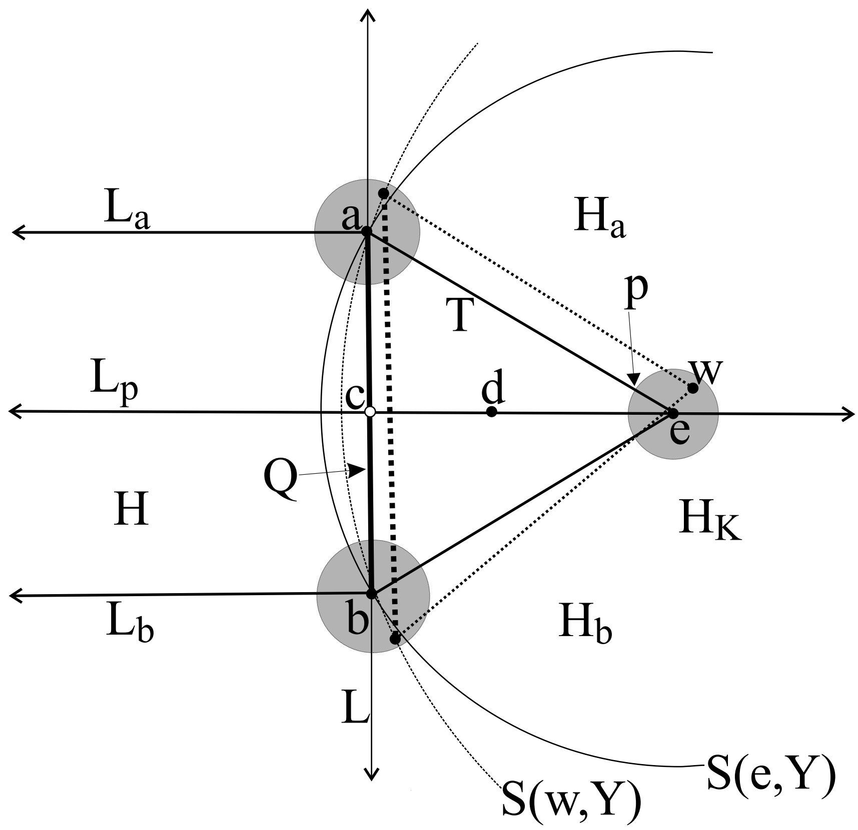

FIGURE 3. Foliation of the convex hull of Y by geometric crosscuts and simplexes.

Let T be the (possibly degenerate) triangle with vertices a, b and e. Let ε be sufficiently small such that any arc which is not contained in the y-axis and joins a point of $B(a,\varepsilon)\cap\overline{H_K}$ to any point of $B(b,\varepsilon)\cap\overline{H_K}$, meets $(c,d]\subset K$. Since $S(e,Y)\cap H_K=\emptyset$, there exists $\delta>0$ such that for each $w\in B(e,\delta)$, $S(w,Y)\cap\overline{H_K}\subset B(a,\varepsilon)\cup B(b,\varepsilon)$. Clearly, for all p in the intersection of the straight line segment (e,a) and $B(e,\delta)$, $S(p,Y)=\{a\}$. Similarly, for all $q\in(e,b)\cap B(e,\delta)$, $S(q,Y)=\{b\}$. It follows that there exists a set D in $B(e,\delta)\setminus T$ which separates $\{x\in B(e,\delta)\setminus T\mid d(x,\overline{B(a,\varepsilon)}\cap Y)<d(x,\overline{B(b,\varepsilon)}\cap Y)\}$ from $\{x\in B(e,\delta)\setminus T\mid d(x,\overline{B(a,\varepsilon)}\cap Y)>d(x,\overline{B(b,\varepsilon)}\cap Y)\}$. If $x\in D$, then there exist $a_x\in\overline{B(a,\varepsilon)}\cap Y\cap\overline{H_k}$ and $b_x\in\overline{B(b,\varepsilon)}\cap Y\cap\overline{H_k}$ such that $\{a_x,b_x\}\subset S(x,Y)$. Choose a point $x\in D$ whose first coordinate is larger than the first coordinate of e. Since the first coordinate of e was maximal with respect to the property that $\{a,b\}\subset S(e,Y)$, either $a_x\neq a$ or $b_x\neq b$. Hence $Q_x=(a_x,b_x)\in\mathcal{G}$ is not contained in the y-axis. By the choice of ε, $Q_x\cap(c,d]\neq\emptyset$ and Q_x meets K, a contradiction.

Suppose next that for all $e\in L_p\cap\overline{H_K}$, $\{a,b\}\setminus S(e,Y)\neq\emptyset$. Then there exists $e\in L_p\cap H$ such that $\{a,b\}\subset S(e,Y)$ and the first coordinate of

e is minimal. Let T be the triangle with vertices a, b and e. A similar argument, this time using points in $B(e,\delta)\cap T$, shows that there exists a point $x \in D\cap T\cap B(e,\delta)$ such that there exist $\{a_x,b_x\}\subset S(x,Y)$ with $Q_x=(a_x,b_x)\in\mathcal{G}$ and $Q_x\cap K\neq\emptyset$. This completes the proof of the theorem. □

Fix a Riemann map $\phi:\Delta_\infty\to O_\infty=\mathbb{C}_\infty\setminus Y$ taking $\infty\to\infty$.

Proposition 3.12. *Suppose the external ray R_t lands on $x\in Y$, and $\{Q_i\}_{i=1}^{\infty}$ is a sequence of crosscuts converging to x with $\phi^{-1}(Q_i)\to t\in S^1=\partial\Delta_\infty$. Then for sufficiently large i,* $\operatorname{var}(f,Q_i,Y)=0$.

Proof. Since f is fixed point free on Y and $f(x)\in Y$, we may choose a connected neighborhood W of x such that $f(\overline{W})\cap(\overline{W}\cup R_t)=\emptyset$. For sufficiently large i, $Q_i\subset W$. For each such i, let J_i be a junction starting from a point in Q_i, staying in W until it reaches R_t, then following R_t to ∞. By our choice of W, $\operatorname{var}(f,Q_i,Y)=0$. □

Proposition 3.13. *If R_t is an external ray of Y. Then one of the following must hold:*

1. *R_t lands on a point of $Y\cap\partial$Convexhull(Y),*
2. *There exist $z\in\mathcal{C}_2\cup\mathcal{C}_3$ such that R_t lands on a point of $S(z,Y)$,*
3. *There is a defining sequence Q_i of chords for R_t.*

Proof. Let R_t be an external ray and let

$$\mathcal{Q}=\{Q\in\mathcal{G}\mid\ R_t \text{ crosses } Q \text{ essentially}\}.$$

Suppose first that $\mathcal{Q}=\emptyset$. If there is a (Y,x)-end of R_t in $\mathbb{C}\setminus$Convexhull(Y), then there exists $y\in\overline{R_t}\cap Y\cap\partial$Convexhull$(Y)$. Since y is accessible, R_t must land on y and (1) holds. If not, there is a cofinal sequence $\{x_i\}$ in R_t such that $x_i\in\partial$Convexhull(Y). Let $y\in\limsup x_i$. As above, R_t lands on y and (1) holds.

Suppose, therefore, that $\mathcal{Q}\neq\emptyset$. Then there exist a partial order on $\mathcal{Q}$ defined by $Q_1<Q_2$ if Q_1 is contained in the bounded complementary domain of $Y\cup Q_2$. Let Q_i be a cofinal decreasing sequence in $\mathcal{Q}$. Without loss of generality $\lim\overline{Q_i}=Q$ in the hyperspace of subcontinua of Convexhull(Y). By proposition 3.6, Q is either a point y of Y, or $Q=[a,b]$, where $(a,b)\in\mathcal{G}$ is a chord. If Q is a singleton y, then (3) holds.

Suppose Q is non-degenerate and choose $x_i\in R_t\cap Q_i$ such that the (Y,x_i)-end of R_t is contained in the bounded complementary domain of $Y\cup Q_i$. If $Q\notin\mathcal{Q}$, then it follows that $e\in\overline{R_t}$ for some $e\in\{a,b\}$. In this case, as above, the ray R_t must land on e and (2) holds. We suppose, therefore, that $Q\in\mathcal{Q}$ is the least element of $\mathcal{Q}$. Let $x\in R_t\cap Q$ such that the (Y,x)-end K_x of R_t is contained in the bounded complementary domain of $Y\cup Q$. If $\{Q_i'\}$ were a sequence in $\mathcal{G}$ converging to Q such that $Q_i'<Q$ for each i, R_t would cross Q_i' essentially for large i, contradicting the minimality of Q. Hence $Q\subset\partial V(z)$ for some $z\in\mathcal{C}_3$. As in the case $\mathcal{Q}=\emptyset$, it follows that R_t must land on a point of $\partial V(z)\cap Y$ and (2) holds. □

Corollary 3.14. *Let $\mathcal{E}_t$ be a channel in Y such that $\Pr(\mathcal{E}_t)$ is non-degenerate. Then for each $x \in \Pr(\mathcal{E}_t)$, for every $\delta > 0$, there is a chain $\{Q_i\}_{i=1}^{\infty}$ of chords defining $\mathcal{E}_t$ selected from $\mathcal{G}_\delta$ with $Q_i \to x \in \partial Y$.*

Proof. Let $x \in \Pr(\mathcal{E}_t)$ and let $\{C_i\}$ be a defining chain of crosscuts for $\Pr(\mathcal{E}_t)$ with $\{x\} = \lim C_i$. By the remark following Proposition 3.10, there is a sequence $\{Q_i\}$ of chords such that $Q_i \cap C_i \neq \emptyset$ and $\Pr(\mathcal{E}_t)$ crosses each Q_i essentially. By Proposition 3.9, the sequence Q_i converges to $\{x\}$. □

Lemma 3.15. *Suppose an external ray R_t lands on $a \in Y$ with $\{a\} = \Pr(\mathcal{E}_t) \neq \mathrm{Im}(\mathcal{E}_t)$. Suppose $\{x_i\}_{i=1}^{\infty}$ is a collection of points in O_∞ with $x_i \to x \in \mathrm{Im}(\mathcal{E}_t) \setminus \{a\}$ and $\phi^{-1}(x_i) \to t$. Then for sufficiently large i, there is a sequence of chords $\{Q_i\}_{i=1}^{\infty}$ such that Q_i separates x_i from ∞, $Q_i \to a$ and $\phi^{-1}(Q_i) \to t$.*

Proof. The existence of the chords Q_i follows from the remark following proposition 3.10. It is easy to see that $\lim \varphi^{-1}(Q_i) = t$. □

3.2. **Auxiliary Continua.**

Definition 3.16. *Fix $\delta > 0$. Define the following collections of chords:*

$$\mathcal{G}_\delta^+ = \{Q \in \mathcal{G}_\delta \mid \mathrm{var}(f, Q, Y) \geq 0\}$$
$$\mathcal{G}_\delta^- = \{Q \in \mathcal{G}_\delta \mid \mathrm{var}(f, Q, Y) \leq 0\}$$

To each collection of chords above, there corresponds an auxiliary continuum defined as follows:

$$Y_\delta = T(Y \cup (\cup \mathcal{G}_\delta))$$
$$Y_\delta^+ = T(Y \cup (\cup \mathcal{G}_\delta^+))$$
$$Y_\delta^- = T(Y \cup (\cup \mathcal{G}_\delta^-))$$

Proposition 3.17. *Let $Z \in \{Y_\delta, Y_\delta^+, Y_\delta^-\}$, and correspondingly $\mathcal{W} \in \{\mathcal{G}_\delta, \mathcal{G}_\delta^+, \mathcal{G}_\delta^-\}$. Then the following hold:*

1. *Z is a nonseparating plane continuum.*
2. *$\partial Z \subset Y \cup (\cup \mathcal{W})$.*
3. *Every accessible point p in ∂Z is either a point of Y or a point interior to a chord $A \in \mathcal{W}$.*
4. *If p is an accessible point of ∂Z and in the interior of the chord $A \in \mathcal{W}$, then every point of A is accessible in ∂Z.*

Proof. By Proposition 3.6, $Y \cup (\cup \mathcal{W})$ is compact. Moreover, Y is connected and each crosscut $A \in \mathcal{W}$ has endpoints in Y. Hence, the topological hull $T(Y \cup (\cup \mathcal{W}))$ is a nonseparating plane continuum, establishing (1).

Since Z is the topological hull of $Y \cup (\cup \mathcal{W})$, no boundary points can be in complementary domains of $Y \cup (\cup \mathcal{W})$. Hence, $\partial Z \subset Y \cup (\cup \mathcal{W})$, establishing (2). Conclusion (3) follows immediately.

To prove (4), suppose $p \in A$ is an accessible point of ∂Z in the interior of a chord $A \in \mathcal{W}$. By Proposition 3.11, $\mathcal{F}$ foliates $\mathrm{ConvexHull}(Y) \setminus Y$. Since $\mathcal{W} \subset \mathcal{G}$, if p is accessible, then no other crosscut $B \in \mathcal{W}$ separates p from ∞.

By Proposition 3.6, every point of A is accessible (by moving along parallel to A left and right of p). □

Proposition 3.18. *Y_δ is locally connected; hence, ∂Y_δ is a Caratheodory loop.*

Proof. Suppose that Y_δ is not locally connected. Then there exists $k \in Y_\delta$ and ε, $0 < \varepsilon < \delta/3$, and distinct components K_i of $B(k, 2\varepsilon)) \cap Y_\delta$ with $k \in \lim K_i \subset K_0$, where K_0 is the component containing k. We also suppose that $K_i \cap \partial B(k, \varepsilon)$ is not connected and K_{i+1} separates K_i from K_0 for each $i > 0$. Since Y_δ is a non-separating continuum, there exist $x_i \in [\mathbb{C} \setminus Y_\delta] \cap \partial B(k, \varepsilon)$ and if R_i is a ray joining x_i to infinity in $\mathbb{C} \setminus Y_\delta$, then R_i separates K_i from $K_0 \cup K_{i+1} \cup \dots$ in $B(k, \varepsilon)$. We may assume that each K_i meets $B(k, \varepsilon/3)$. Let $A_i \subset B(k, \varepsilon/3)$ be an irreducible arc from R_i to R_{i+1} and let U_i be the component of $\mathbb{C} \setminus [R_i \cup A_i \cup R_{i+1}]$ which does not contain k. Then $U_i \cap K_{+1} i \cap \partial B(k, \varepsilon) \neq \emptyset$ for each i. Since $K_i \subset Y_\delta = T(Y \cup \mathcal{G}_\delta)$ it follows that $U_i \cap Y \setminus B(k, \varepsilon) \neq \emptyset$. Since Y is a continuum, there exists continua $L_i \subset Y$, such that L_i separates R_i from R_{i+1} in $\overline{B(k, \varepsilon)}$.

Let $\overline{B(k, \varepsilon)} \setminus R_i = B_i \cup C_i$ be separated sets such that $K_{i+1} \subset B_i$ and $K_i \subset C_i$. Let $M_i = Y \cap B_i$ and $N_i = Y \cap C_i$, then M_i and N_i are disjoint compact sets. Let $M = \{x \in \mathbb{C} \mid d(x, M_i) = d(x, N_i)\}$. Then separates N_i from M_i in $\mathbb{C}$. Since $\mathbb{C}$ is unicoherent [13], a component L of N separates. Choose $z \in L \cap B(k, \varepsilon/3)$, then $x_i \notin \overline{B(z, Y)}$. Note that $B(z, Y) \cup Y$ separates x_i from infinity. Hence, $\partial B(z, Y) \setminus S(z, Y)$ separates x_i from infinity in $\mathbb{C} \setminus Y$. Since $\mathbb{C} \setminus Y$ is unicoherent, a component P of $\partial B(z, Y) \setminus S(z, Y)$ separates x_i from ∞. Let Q be the straight line segment joining the two end points of P, then $Q \in \mathcal{G}_\delta$ for large i and Q also separates x_i from infinity, a contradiction. □

3.3. **Outchannels.**

Definition 3.19 (Outchannel)**.** *An* outchannel *of the nonseparating plane continuum Y is a prime end $\mathcal{E}_t$ of $O_\infty = \mathbb{C}_\infty \setminus Y$ such that for some chain $\{Q_i\}$ of crosscuts defining $\mathcal{E}_t$,* $\text{var}(f, Q_i, Y) \neq 0$ *for every i. We call an outchannel $\mathcal{E}_t$ of Y* a geometric outchannel *iff for sufficiently small δ, every chord in $\mathcal{G}_\delta$, which crosses $\mathcal{E}_t$ essentially, has nonzero variation. We call a geometric outchannel* negative *(respectively,* positive*) iff every chord in $\mathcal{G}_\delta$, which crosses $\mathcal{E}_t$ essentially, has negative (respectively, positive) variation.*

Lemma 3.20. *Let $Z \in \{Y_\delta, Y_\delta^+, Y_\delta^-\}$. Fix a Riemann map $\phi : \Delta_\infty \to \mathbb{C}_\infty \setminus Z$ such that $\phi(\infty) = \infty$. Suppose R_t lands at $x \in \partial Z$. Then there is an open interval $M \subset \partial \Delta_\infty$ containing t such that ϕ can be extended continuously over M.*

Proof. Let t_i converge to t in S^1 such that R_{t_i} lands on x_i in Z and x_i converges to x. Without loss of generality, $\{t_i\}$ is an increasing sequence. Let $t_i < s_i < t_{i+1}$ in S^1. Note that $\limsup \overline{R_{s_i}} \supset \overline{R_t}$. If $\limsup \overline{R_{s_i}}$ is not equal to $\overline{R_t}$, then by Lemma 3.15 there exist chords Q_i (by passing to a subsequence if necessary) such that R_{s_i} crosses Q_i essentially. By Proposition 3.9, $Q_i \in \mathcal{G}_\delta$ for large i. But by Proposition 3.12, $\mathrm{var}(f, Q_i, X) = 0$ so $Q_i \subset Z$ for large i, a contradiction. So $\limsup R_{s_i} = \overline{R_t}$. In particular, R_{s_i} must land on some point y_i and $\lim y_i = x$. So on an interval M about t in S^1 each R_s lands $y_s \in Z$ for each $s \in M$ and if $\{s_i\}$ converges to s, then $\lim y_{s_i} = y_s$. Hence the map taking $s \in M$ to the landing point of R_s in Z is continuous. □

Lemma 3.21. *If there is a chord Q of Y of negative (respectively, positive) variation, such that there is no fixed point in $T(Y \cup Q)$, then there is a negative (respectively, positive) geometric outchannel $\mathcal{E}_t$ of Y for which a defining chain begins with Q.*

Proof. Without loss of generality, assume $\mathrm{var}(f, Q, Y) < 0$. Choose $\delta > 0$ so small that $Q \notin \mathcal{G}_\delta$, no chord in $\mathcal{G}_\delta$ separates Q from ∞, and, since there are no fixed points in $T(Y \cup Q)$, every chord in $\mathcal{G}_\delta$ separated from ∞ by Q moves off itself under f (so variation on it is defined). Let $\varphi : \Delta_\infty \to \mathbb{C} \setminus Y^+_\delta$ be the Riemann map. Assume $t' < t \in S^1$ such that $\varphi(t') = a$ and $\varphi(t) = b$. By Lemma 3.20, φ extends to a continuous function $\tilde{\varphi} : \tilde{\Delta}_\infty \to [\mathbb{C} \setminus Y^+_\delta] \cup \{\text{accessible points of } Y^+_\delta\}$. Consider the arc-component M of $[\tilde{\Delta}_\infty \setminus \Delta_\infty] \cap [t', t]$ containing t'. If $t \in M$, then $\tilde{\varphi}([t', t])$ is an arc in Y^+_δ joining a to b which is separated by Q from ∞. But

$$\mathrm{var}(f, Q, X) = \sum_{C \in \mathcal{G}_\delta,\ C \subset \tilde{\varphi}([t',t])} \mathrm{var}(f, C, X).$$

This is a contradiction since $\mathrm{var}(f, Q, X) < 0$ and all $\mathrm{var}(f, C, X) \geq 0$. Hence b is not in M. Let $s = \sup M$, then $\mathcal{E}_s$ is a negative geometric outchannel. For if some chord $C \in \mathcal{G}_\delta$ crosses R_s essentially, then $s \in \mathrm{Int}(M)$, a contradiction. □

3.4. **Invariant Channel in X.** We are now in a position to prove Bell's principal result on any possible counter-example to the fixed point property, under our standing hypothesis.

Lemma 3.22. *Suppose $\mathcal{E}_t$ is a geometric outchannel of $Y = T(X)$ under f. Then the principal continuum* $\mathrm{Pr}(\mathcal{E}_t)$ *of $\mathcal{E}_t$ is invariant under f. So* $\mathrm{Pr}(\mathcal{E}_t) = X$.

Proof. To see that $\mathrm{Pr}(\mathcal{E}_t)$ is invariant under f, let $\mathcal{E}_t$ be a geometric outchannel of $T(X)$ under f. Let $x \in \mathrm{Pr}(\mathcal{E}_t)$. Then for some chain $\{Q_i\}_{i=1}^\infty$ of crosscuts defining $\mathcal{E}_t$ selected from $\mathcal{G}_\delta$, we may suppose $Q_i \to x \in \partial T(X)$ and $\mathrm{var}(f, Q_i, X) \neq 0$ for each i. Let R_t be the image under the conformal map ϕ of the radial ray to $t \in S^1$. Since Q_i meets R_t and $\mathrm{var}(f, Q_i, X) \neq 0$, we have $f(Q_i)$ meeting R_t (since any junction from Q_i "parallels" R_t). Since $\mathrm{diam}(f(Q_i)) \to 0$, we have $f(Q_i) \to f(x)$ and $f(x) \in \mathrm{Pr}(\mathcal{E}_t)$. □

Theorem 3.23 (Dense channel, Bell). *Under our standing Hypothesis, $Y = T(X)$ contains a negative geometric outchannel; hence, $\partial O_\infty = \partial T(X) = X = f(X)$ is an indecomposable continuum.*

Proof. Recall that the map $f : \mathbb{C} \to \mathbb{C}$ taking X into $T(X)$ has no fixed points in $T(X)$, and X is minimal with respect to these properties. Choose $\delta > 0$ so that each crosscut $Q \in \mathcal{G}_\delta$ is sufficiently close to $Y = T(X)$ so that f has no fixed points in $T(Y \cup Q)$, and so that for any geometric crosscut $Q \in \mathcal{G}_\delta$, $f(Q) \cap Q = \emptyset$. Let $C = \partial Y_\delta$. Since Y_δ is locally connected, C is a Caratheodory loop. Since f is fixed point free on C, $\text{ind}(f, C) = 0$. Consequently, by Theorem 2.13 for Caratheodory loops, $\text{var}(f, C) = -1$. By the summability of variation on C, it follows that on some chord $Q \subset C$, $\text{var}(f, Q, Y) < 0$. By Lemma 3.21, there is a negative geometric outchannel $\mathcal{E}_t$ under the crosscut Q.

Since $\text{Pr}(\mathcal{E}_t)$ is invariant under f by Lemma 3.22, it follows that $\text{Pr}(\mathcal{E}_t)$ is an invariant subcontinuum of $\partial O_\infty \subset \partial T(X) \subset X$. So by the minimality condition in our Standing Hypothesis, $\text{Pr}(\mathcal{E}_t)$ is dense in ∂O_∞. Hence, $\partial O_\infty = \partial T(X) = X$ and $\text{Pr}(\mathcal{E}_t)$ is dense in X. It then follows from a theorem of Rutt [10] that X is an indecomposable continuum. □

Theorem 3.24. *The boundary of Y_δ is a simple closed curve. The set of accessible points in the boundary of each of Y_δ^+ and Y_δ^- is a countable union of continuous one-to-one images of $\mathbb{R}$.*

Proof. Since X is indecomposable by Theorem 3.23, it has no cut points. By Proposition 3.18 ∂Y_δ is a Caratheodory loop. Since X has no cut points, neither does Y_δ. A Caratheodory loop with no cut points is a simple closed curve.

Let $Z \in \{Y_\delta^+, Y_\delta^-\}$. Fix a Riemann map $\phi : \Delta_\infty \to \mathbb{C}_\infty \setminus Z$ such that $\phi(\infty) = \infty$. Corresponding to the choice of Z, let $\mathcal{W} \in \{\mathcal{G}_\delta^+, \mathcal{G}_\delta^-\}$. Apply Lemma 3.20 and find the maximal collection $\mathcal{J}$ of open subarcs of $\partial\Delta_\infty$ over which ϕ can be extended continuously. Since X has no cutpoints this extension is one-to-one. Since angles corresponding to accessible points are dense in $\partial\Delta_\infty$, $C = \partial\Delta_\infty \setminus \cup\mathcal{J}$ contains no open arc. If $Z = Y_\delta^+$, then it is possible that $\cup\mathcal{J}$ is all of $\partial\Delta_\infty$ except one point, but it cannot be all of $\partial\Delta_\infty$ since there is at least one (negative) geometric outchannel by Theorem 3.23. Since it is a collection of open arcs in $\partial\Delta_\infty$, $\mathcal{J}$ is countable. □

Theorem 3.24 still leaves open the possibility that $Z \in \{Y_\delta^+, Y_\delta^-\}$ has a very complicated boundary. Note that ϕ is not continuous at points of the closed zero-dimensional set C. We may call C the set of outchannels of Z. In principle, there could be an uncountable set of outchannels, each dense in X. The one-to-one continuous images of $\mathbb{R}$ lying in ∂Z are the "sides" of the outchannels. If two elements J_1 and J_2 of the collection $\mathcal{J}$ happen to share a common endpoint t, then the prime end $\mathcal{E}_t$ is an outchannel in Z, dense in X, with $\phi(J_1)$ and $\phi(J_2)$ as its sides. It seems possible that an endpoint t of $J \in \mathcal{J}$ might have a sequence of elements J_i from $\mathcal{J}$ converging to it. Then the outchannel $\mathcal{E}_t$ would have only one (continuous) "side." This possibility, and even having more than one outchannel in Z, is eliminated in the next section.

In the lemma below we show that for Y_δ^- those pieces of the boundary, which correspond to arc components in the set of accessible points, are well behaved and do not contain large unnecessary "wiggles."

Lemma 3.25. *Assume that ∂Y_δ^- is not a simple closed curve. Let K be an arc component of the set of accessible points of Y_δ^-. Then for each ε, $0 < \varepsilon < \delta/2$, there exists $\xi > 0$ such that for any two points $x, y \in K \cap X$, if Q is any crosscut of Y_δ^- joining x to y, and* $\operatorname{diam}(Q) < \xi$*, Then there exists an arc $B \subset K$, joining x to y such that* $\operatorname{diam}(B) < 8\varepsilon$.

Proof. Let $\varepsilon > 0$ be fixed and choose ξ as in Proposition 3.9. Let B be the unique arc in K joining x to y. By Theorem 3.24, K is a one-to-one continuous image of $\mathbb{R}$. We will denote the unique subarc of K which joins two points $p, q \in K$ by $\langle p, q\rangle$. Hence $B = \langle x, y\rangle$. Suppose there exist $z \in B$ such that $d(z, Q) \geq 8\varepsilon$. Let $b \in T(Q \cup Y_\delta^-) \setminus Y_\delta^-$ such that $d(b, z) \leq \varepsilon/2$. Let $P = \langle x, z\rangle \setminus B(b, \varepsilon)$ and $M = \langle z, y\rangle \setminus B(b, \varepsilon)$. Then P and Q are disjoint closed sets in Y_δ^-. Let N be a component of $B(Q, 3\varepsilon) \setminus Y_\delta^-$ which separates b from ∞ in $\mathbb{C} \setminus Y_\delta^-$ and such that N is contained in the bounded component of $T(Q \cup Y_\delta^-)$. By Proposition 3.9, each point of N lies in an element $F_x \in \mathcal{F}$ with diameter at most 2ε. Since F_x is small and meets Y, $F_x \cap (P \cup M) \neq \emptyset$ for each $x \in N$. It follows from Proposition 3.10 that there exists $x \in N$ such that $F_x = F$ meets both P and M and, hence, F separates b from ∞ in $\mathbb{C} \setminus Y_\delta^-$. We may assume that $F \in \mathcal{G}$ since, if $F = V(z)$ is a simplex for some z, we can replace F by one of the components of $\overline{V(z)} \setminus (Y \cup V(z))$.

Now, $\operatorname{var}(f, F, X) \leq 0$. For if $\operatorname{var}(f, F, X) > 0$, there exists a positive geometric outchannel $\mathcal{E}_s$ for which a defining chain starts with F. But, if the end points of F are x', y', then R_s would cross some chord $G \subset \langle x', y'\rangle \subset K$ essentially. This is a contradiction since K contains no crosscuts of positive variation. So $\operatorname{var}(f, F, X) \leq 0$.

It follows that $F \subset Y_\delta^-$ and $T(Y \cup F) \subset Y_\delta^-$. This contradicts that $b \in \mathbb{C} \setminus Y_\delta^-$. ☐

4. Uniqueness of the Outchannel

Theorem 3.23 asserts the existence of at least one negative geometric outchannel which is dense in X. We show below that there is exactly one geometric outchannel, and that its variation is -1. Of course, X could have other dense channels, but they are "neutral" as far as variation is concerned.

Theorem 4.1 (Unique Outchannel)**.** *Assume the standing hypothesis 1.1. Then there exists a unique geometric outchannel $\mathcal{E}_t$ for X, which is dense in $X = \partial Y$. Moreover, for any sufficiently small chord Q in any chain defining $\mathcal{E}_t$,* $\operatorname{var}(f, Q, X) = -1$*, and for any sufficiently small chord Q' not crossing R_t essentially,* $\operatorname{var}(f, Q', X) = 0$.

Proof. Suppose by way of contradiction that X has a positive outchannel. Let $\delta > 0$ such that $T(B(Y, 2\delta))$ contains no fixed points of f and such that,

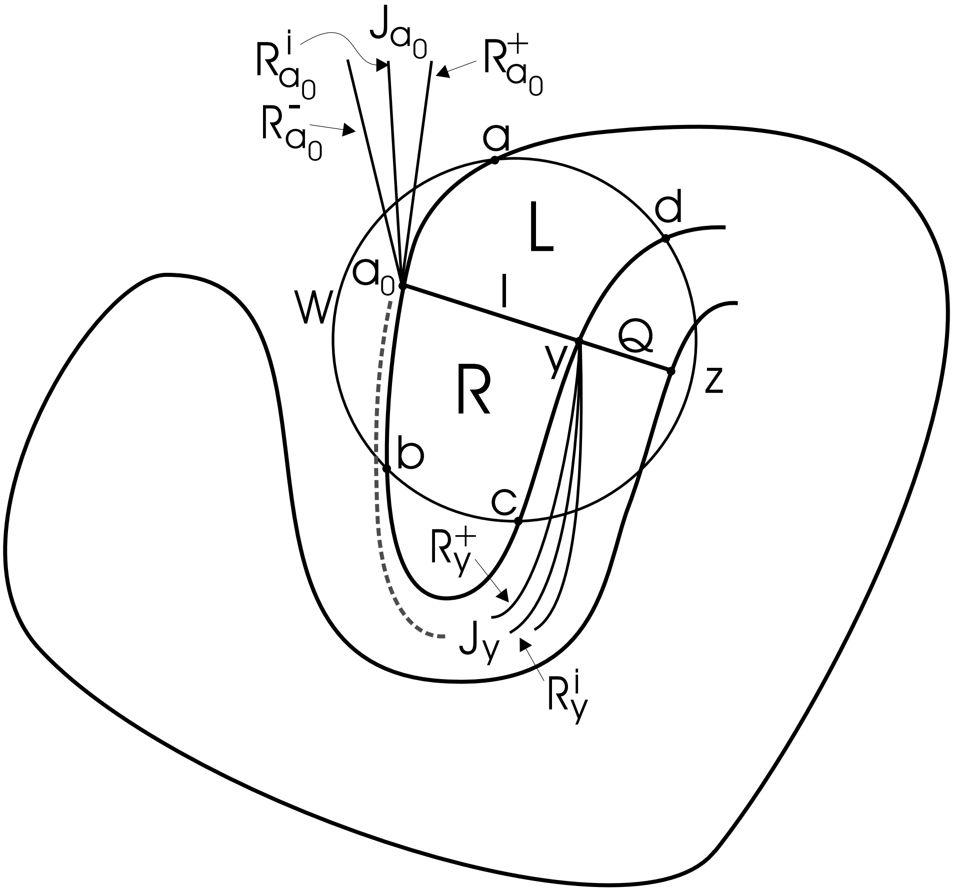

FIGURE 4. Uniqueness of the negative outchannel.

if $M \subset B(Y, 2\delta)$ with $\operatorname{diam}(M) < 2\delta$, then $f(M) \cap M = \emptyset$. Since X has a positive outchannel, ∂Y^-_δ is not a simple closed curve. By Theorem 3.24 ∂Y^-_δ contains an arc component K which is the one-to-one continuous image of $\mathbb{R}$. Note that each point of K is accessible.

Let $\varphi : \Delta_\infty \to U_\infty = \mathbb{C} \setminus Y^-_\delta$ a conformal map. By Theorem 3.24, and its proof, φ extends continuously and injectively to a map $\tilde{\varphi} : \tilde{\Delta}_\infty \to \tilde{U}_\infty$, where $\tilde{\Delta}_\infty \setminus \Delta_\infty$ is a dense and open subset of S^1 which contains K in its image. Then $\tilde{\varphi}^{-1}(K) = (t', t) \subset S^1$ is an open arc with $t' < t$ in the counterclockwise order on S^1. Let $<$ denote the order in K induced by $\tilde{\varphi}$ and for $x < y$ in K, denote the arc in K with endpoints x and y by $\langle x, y\rangle$. Let $\langle x, \infty\rangle = \cup_{y>x}\langle x, y\rangle$

Let $\mathcal{E}_t$ be the prime-end corresponding to t. Then $\Pr(\mathcal{E}_t)$ is a positive geometric outchannel and, hence, by Lemma 3.22, $\Pr(\mathcal{E}_t) = X$. Let $R_t = \varphi(re^{it})$, $r > 1$, be the external conformal ray corresponding to the prime-end

$\mathcal{E}_t$. Since $\overline{R_t} \setminus R_t = X$ and the small chords which define $\Pr(\mathcal{E}_t)$ have one end point in K (c.f., Proposition 3.9), $\overline{\langle x, \infty\rangle} \cap Y_\delta^- = X$.

Let $\varepsilon > 0$ such that $T(B(Y,\varepsilon)) \subset B(Y,\delta)$ (by Proposition 3.8). It follows from Propositions 3.8, 3.10 and 3.13, there exists $x \in K$ such that in each arc $M \subset \langle x, \infty\rangle$ with $\operatorname{diam}(M) > \varepsilon/4$, there exists $y \in M$ and a chord $G \in \mathcal{G}_\delta$ with end point y which crosses R_t essentially.

Let $a_0 \in K \cap X$ so that $a_0 > x$ and J_{a_0} a junction of Y_δ^-. Let W be a topological disk about a_0 with simple closed curve boundary of diameter less than ε so that the component of $K \cap W$ containing a_0 has closure $\langle a, b\rangle$, $a < a_0 < b$ in K and $f(\overline{W}) \cap (\overline{W} \cup J_{a_0}) = \emptyset$. We may suppose that $(K \cap W) \setminus \langle a, b\rangle$ is contained in one component of $W \setminus \langle a, b\rangle$ since one side of K is accessible from ∞ in $\mathbb{C} \setminus Y_\delta^-$. Since $X \subset \overline{\langle a_0, \infty\rangle}$, there are components of $W \cap \langle b, \infty\rangle$ which pass arbitrarily close to a_0. Choose $\langle c, d\rangle$ to be the closure of a component of $W \cap \langle b, \infty\rangle$ such that:

(1) a and d lie in the same component of $\partial W \setminus \{b, c\}$,
(2) there exists $y \in \langle c, d\rangle \cap X \cap W$ and an arc $I \subset (W \setminus \langle a, d\rangle) \cup \{a_0, y\}$ joining a_0 to y, and
(3) there is a chord $Q \subset W$ with y and z as endpoints which crosses R_t essentially.

To see the above, note that there are small chords or simplexes which cross R_t essentially through each point of R_t. By Proposition 3.10 given an arc A that crosses R_t essentially and is sufficiently close to X, there is a small diameter chord that essentially crosses R_t and meets A. By Lemma 3.25, it follows that if two small chords both cross R_t essentially and both meet a small diameter arc in R_t, then they both meet a small diameter arc in K. Thus we can satisfy (3) on any arc in $W \cap K$ which gets close to a_0. Note that (2) holds for any point of $\langle c, d\rangle$ for which (3) holds.

Let B be the arc in $\partial W \setminus \{b\}$ with end points a and d. Let A be a bumping arc in $(\mathbb{C} \setminus [J_{a_0} \cup T(\langle a, d\rangle \cup B)]) \cup \{a, d\}$ with end-points a and d such that $Y \setminus T(\langle a, d\rangle \cup B) \subset T(A \cup B)$. Hence, $S = \langle a, d\rangle \cup A$ is a simple closed curve and $Y \subset T(S)$. We may suppose that $A \subset B(Y, \varepsilon)$ so that f is fixed point free on $T(S)$ and each component of $A \setminus X$ has diameter less than δ so that variation is defined on each such component.

Since $\overline{Q} \cap Y = \{y, z\}$ we may suppose that $A \cap \overline{Q} = \{z\}$. Note that I is an arc in $T(S)$ which meets S only at its end points a_0 and y. Since $I \subset W$, $f(I) \cap J_{a_0} = \emptyset$. Let $R = T(\langle a_0, y\rangle \cup I)$ and let $L = T(\langle y, d\rangle \cup I \cup A \cup \langle a, a_0\rangle)$. Let J_y be a junction for S such that $J_y \cap \overline{Q} = \{y\}$, $J_{a_0} \setminus W \subset J_y$ so that $R^*_{a_0} \setminus W \subset R^*_y$ for each $* \in \{+, i, -\}$ and J_y runs very close to $\langle a_0, y\rangle \cup J_{a_0}$.

Note that the order $<$ on K coincides with the counterclockwise order on S. It follows that $W \cup R^i_y$ separates $L \cup R^-_y \setminus J_{a_0}$ from $R \cup R^+_y \setminus J_{a_0}$. Since Q crosses R_t essentially, we know that $\operatorname{var}(f, Q, Y) > 0$. We will use this information to show that $f(y) \in R$. To compute $\operatorname{var}(f, Q, Y) = \operatorname{var}(f, Q, S)$ we will use the fact that the variation is invariant under a homotopy which keeps y and z in $h(U_y)$ (see Proposition 2.10 and the remark following that proposition). Hence, if we homotope $f|_{\overline{Q}}$ to a map $f' : \overline{Q} \to \mathbb{C} \setminus W$ such that $f|_{f^{-1}(T(S))\cap\overline{Q}} = f'|_{f^{-1}(T(S))\cap\overline{Q}}$, then $\operatorname{var}(f', Q, S) = \operatorname{var}(f, Q, S)$. Moreover, we can choose f' such that the number of components of $f'^{-1}(\mathbb{C} \setminus T(S))$ is minimal (the set of components of $\overline{Q} \cap f^{-1}(\mathbb{C} \setminus T(S))$ whose closures meet both $f^{-1}(T(L))$ and $f^{-1}(T(R))$ is finite since $f(\overline{Q}) \cap W = \emptyset$). Then to compute $\operatorname{var}(f', Q, S)$ we use the following recipe: As we go along Q from y to z, each time the image of f' goes from R to L count $+1$. Each time the image goes from L to R count -1. Make no other counts. Then it follows that if $f'(y) = f(y) \in R$ and $f(z) \in L$, then $\operatorname{var}(f', Q, S) = +1$, if $f'(y) \in L$ and $f'(z) \in R$, then $\operatorname{var}(f', Q, S) = -1$. Otherwise $\operatorname{var}(f', Q, S) = 0$. Since $\operatorname{var}(f', Q, S) > 0$, $f(y) \in R$.

The Lollipop Lemma, Theorem 2.14, applies to S and the arc I. Also, since $Y \subset T(S)$, $\operatorname{var}(f, C, S) = \operatorname{var}(f, C, Y)$ for each chord C contained in S. Hence, there exists a chord $Q_1 \subset \langle a_0, y\rangle$ such that $\operatorname{var}(f, Q_1, Y) < 0$. Since there are no chords of positive variation on $\langle a_0, y\rangle$ and

$$0 = \operatorname{ind}(f, I \cup \langle a_0, y\rangle) = \sum_{C\in\mathcal{G},\ C\subset\langle a_0,y\rangle} \operatorname{var}(f, C, Y) + 1,$$

we know that $\operatorname{var}(f, Q_1, Y) = -1$.

We repeat the above argument starting with $y \in K$ in place of a_0 and J_y in place of J_{a_0} and an open disk $V \subset W$ about y to find a second chord $Q_2 \subset \langle y, \infty\rangle$ with $\operatorname{var}(f, Q_2, Y) = -1$.

We will now show that the existence of chords Q_1 and Q_2 in K with variation -1 on each leads to a contradiction. Choose $c' < d' \in K$ such that $\langle c', d'\rangle$ is the closure of a component of $K \cap W$ satisfying the following conditions.

1. $Q_1, Q_2 \subset \langle a_0, c'\rangle \subset K$,
2. $\{a, d'\}$ is contained in one component of $\partial W \setminus \{b, c'\}$,
3. there exist $y' \in \langle c', d'\rangle \cap Y \cap W$ and an arc I' from a_0 to y' in $\{a_0, y'\} \cup (W \setminus \langle a, d'\rangle)$, and
4. there exists a chord $Q' \subset W$ with endpoints y' and z' such that Q' crosses R_t essentially.

Let B' be the arc in $\partial W \setminus \{b\}$ with endpoints $\{a\}$ and $\{d'\}$. Let A' be an arc in $\{a, d'\} \cup \mathbb{C} \setminus T(\langle a, d'\rangle \cup B')$ such that $Y \setminus T(\langle a, d'\rangle \cup B') \subset T(A' \cup B')$ and such that the components of $A' \setminus X$ have diameter less than δ. We may suppose that $\overline{Q}' \cap A' = \{z'\}$. We can prove, as above, that $f(y') \in R' = T(\langle a_0, y'\rangle \cup I')$ and, hence all conditions of the Lollipop Lemma 2.14 are again satisfied for S' and I' and

$$\operatorname{ind}(f, \langle a_0, y'\rangle \cup I') = \sum_{C\in\mathcal{G},\ C\subset\langle a_0,y'\rangle} \operatorname{var}(f, C, S') + 1.$$

Since $\langle a_0, y'\rangle$ contains Q_1 and Q_2, $\operatorname{var}(f, Q_i, S') = \operatorname{var}(f, Q_i, Y) < 0$ and contains no chords of positive variation, $\sum \operatorname{var}(f, C, S') + 1 \le -1$.

Since f is fixed point free on R', $\text{ind}(f, \langle a_0, y' \rangle \cup I') = 0$ by Theorem 2.5. This contradiction shows that X has no positive outchannels.

By Theorems 3.23 and 2.13, X has exactly one negative outchannel and its variation is -1. □

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