WANDERING GAPS FOR WEAKLY HYPERBOLIC CUBIC POLYNOMIALS

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ABSTRACT. Thurston introduced σ_d-invariant laminations (where $\sigma_d = z^d : S^1 \to S^1$, $d \geq 2$) and defined *wandering k-gons* as sets $T \subset S^1$ such that $\sigma_d^n(T)$ consists of $k \geq 3$ distinct points for all $n \geq 0$ and the convex hulls of all the sets $\sigma_d^n(T)$ in the plane are pairwise disjoint. He proved that σ_2 has no wandering k-gons.

We show that there exist uncountably many cubic laminations with a wandering 3-gon (triangle) and pairwise non-conjugate induced maps on the corresponding quotient spaces J. Then we show that these dynamical systems are realizable as weakly hyperbolic, cubic polynomials on their Julia sets (which are locally connected).

1. INTRODUCTION

Laminations were introduced by Thurston [24] as a tool for studying both individual complex polynomials and the space of all of them, especially in degree 2. The former is achieved as follows. Let $P : \mathbb{C}^* \to \mathbb{C}^*$ be a degree d polynomial with a connected Julia set J_P acting on the complex sphere $\mathbb{C}^*$. Denote by K_P the corresponding filled-in Julia set. Let $\theta = z^d : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ ($\mathbb{D} \subset \mathbb{C}$ is the open unit disk). There exists a conformal isomorphism $\Psi : \mathbb{D} \to \mathbb{C}^* \setminus K_P$ with $\Psi \circ \theta = P \circ \Psi$ [11]. If J_P is locally connected, then Ψ extends to a continuous function $\overline{\Psi} : \overline{\mathbb{D}} \to \overline{\mathbb{C}^* \setminus K_P}$ and $\overline{\Psi} \circ \theta = P \circ \overline{\Psi}$. Let $S^1 = \partial\mathbb{D}$, $\sigma_d = \theta|_{S^1}$, $\psi = \overline{\Psi}|_{S^1}$. Define an equivalence relation $\sim_P$ on S^1 by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$. The equivalence $\sim_P$ is called the *(d-invariant) lamination (generated by P)*. The quotient space $S^1/\sim_P = J_{\sim_P}$ is homeomorphic to J_P and the map $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ induced by σ_d is topologically conjugate to P.

Kiwi [13] extended this construction to *all* polynomials P with connected Julia set and no irrational neutral cycles. For such polynomials he obtained

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a d-invariant lamination $\sim_P$ on S^1. Then $J_{\sim_P} = S^1/\sim_P$ is a locally connected continuum and the induced map $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ is semi-conjugate to $P|_{J_P}$ under a monotone map $m : J_P \to J_{\sim_P}$ (by *monotone* we mean a map whose point preimages are connected). The lamination $\sim_P$ generated by P provides a combinatorial description of the dynamics of $P|_{J_P}$. One can introduce laminations abstractly as equivalence relations on S^1 having certain properties similar to those of laminations generated by polynomials as above (we give detailed definitions below); in the case of such an abstract lamination $\sim$ we call $S^1/\sim = J_\sim$ a *topological Julia set* and denote the map *induced* by σ_d on $J_\sim$ by $f_\sim$.

On the other hand, studying the space of all polynomials of degree d reduces to studying the space $\mathcal{P}_d \cong \mathbb{C}^{d-1}$ of degree $d \ge 2$ monic centered polynomials $z \mapsto z^d + a_{d-2}z^{d-2} + \cdots + a_0$ [6]. The *connectedness locus* is the set $\mathcal{C}_d$ of parameters in $\mathcal{P}_d$ for which the Julia set is connected (by [6, 15] $\mathcal{C}_d$ is compact and connected). If $d = 2$, the set $\mathcal{C}_d$ is called the *Mandelbrot set* and is denoted by $\mathcal{M}$. Thurston [24] defined a "meta-lamination" QML and showed that the space of all 2-invariant (*quadratic*) laminations can be thought of as the quotient space $S^1/QML = \mathcal{M}_c$, a combinatorial counterpart of $\mathcal{M}$. A crucial role in this was played by the No Wandering Triangle Theorem [24] which states that quadratic laminations have no wandering k-gons; Thurston [24] posed the problem of extending it to the higher degree case and emphasized its importance. Let us also mention here, that in a recent preprint [7] Bruin and Schleicher study various combinatorial invariants of quadratic polynomials and discuss relations between them; in particular, they study the Mandelbrot set and its combinatorics.

In the language of induced maps, the No Wandering Triangle Theorem states the non-existence of wandering non-precritical branch points of induced maps of quadratic topological Julia set and extends a simple property of continuous maps of finite graphs according to which all branch points of graphs are either preperiodic or precritical. In other words, induced maps of quadratic topological Julia sets still possess certain properties of maps of finite graphs (which is quite surprising since topological Julia sets are much more complicated and in general have infinitely many branch points).

The aim of this paper is to prove that wandering classes of three points (triples) *exist* for an uncountable family of essentially distinct 3-invariant (*cubic*) laminations (see [5] for a sketch of the construction) corresponding to weakly hyperbolic, cubic polynomials with locally connected Julia sets. We start with some definitions. By the *positive* direction on S^1 we mean the *counterclockwise* direction and by the arc $(p, q) \subset S^1$ we mean the positively oriented arc from p to q.

Consider an equivalence relation $\sim$ on the unit circle S^1 such that:
(E1) $\sim$ is *closed*: the graph of $\sim$ is a closed set in $S^1 \times S^1$;

(E2) $\sim$ defines a *lamination*, i.e., it is *unlinked*: if g_1 and g_2 are distinct equivalence classes, then the convex hulls of these equivalence classes in the unit disk $\mathbb{D}$ are disjoint,
(E3) each equivalence class of $\sim$ is *totally disconnected*.

We always assume that $\sim$ has a class of at least two points. Equivalence classes of $\sim$ are called *($\sim$-)classes*. A class consisting of two points is called a *leaf*; a class consisting of at least three points is called a *gap* (this is more restrictive than Thurston's definition in [24]). Fix an integer $d > 1$. The equivalence relation $\sim$ is called *(d-)invariant* if:
(D1) $\sim$ is *forward invariant*: for a class g, the set $\sigma_d(g)$ is a class too which implies that
(D2) $\sim$ is *backward invariant*: for a class g, its preimage $\sigma_d^{-1}(g) = \{x \in S^1 : \sigma_d(x) \in g\}$ is a union of classes;
(D3) for any gap g, the map $\sigma_d|_g : g \to \sigma_d(g)$ is a *covering map with positive orientation*, i.e., for every connected component (s,t) of $S^1 \setminus g$ the arc $(\sigma_d(s), \sigma_d(t))$ is a connected component of $S^1 \setminus \sigma_d(g)$.

Call a class g *critical* if $\sigma_d|_g : g \to \sigma(g)$ is not one-to-one, and *precritical* if $\sigma_d^j(g)$ is critical for some $j \geq 0$. Call g *preperiodic* if $\sigma_d^i(g) = \sigma_d^j(g)$ for some $0 \leq i < j$. A gap g is *wandering* if g is neither preperiodic nor precritical. Let $p : S^1 \to J_\sim = S^1/\sim$ be the standard projection of S^1 onto its quotient space $J_\sim$ and let $f_\sim : J_\sim \to J_\sim$ be the map induced by σ_d.

J. Kiwi [12] extended Thurston's theorem by showing that a wandering gap in a d-invariant lamination is at most a d-gon. In [16] G. Levin showed that laminations with one critical class have no wandering gaps. Let $k_\sim$ be the maximal number of critical $\sim$-classes g with pairwise disjoint infinite σ_d-orbits and $|\sigma_d(g)| = 1$.

Theorem 1 ([3])**.** *Let $\sim$ be a d-invariant lamination and let Γ be a non-empty collection of wandering d_j-gons ($j = 1, 2, \dots$) with distinct grand orbits. Then $\sum_j (d_j - 2) \leq k_\sim - 1 \leq d - 2$.*

Let us call laminations with wandering k-gons *WT-laminations*. Until recently it had not been known if WT-laminations existed, even in case the degree $d = 3$. In the language of topological Julia sets their existence would mean the existence of cubic topological Julia sets with wandering branch points; this could serve as a step towards the completion of the description of the combinatorial portrait of topological Julia sets. Our main theorem shows the existence of WT-laminations realized by polynomials with *Topological Collet-Eckmann property* (*TCE-polynomials*).

Theorem 2. *There is an uncountable family $\{P_\alpha\}_{\alpha\in\mathcal{A}}$ of cubic TCE polynomials P_α such that for every α the lamination $\sim_{P_\alpha}$ generated by P_α is WT-lamination, the Julia set J_{P_α} is a dendrite containing a wandering branch point x of J_{P_α} of order 3 and the maps $P_\alpha|_{J_{P_\alpha}}$ are pairwise non-conjugate.*

Thus, weak hyperbolicity of cubic polynomials does not prevent their Julia sets from exhibiting such "pathology" as having wandering branch points.

Let us describe how we organize the paper. In Section 2 we study (discontinuous) self-mappings of certain sets $A \subset S^1$ and give sufficient conditions under which such maps can be embedded into $\sigma_d : S^1 \to S^1$. In Section 3 a preliminary version of the main theorem is proven; in this version we establish the existence of an uncountable family of cubic WT-laminations. The proof was inspired by [3] and [18]; the result was announced in [5]. We construct a set $A \subset S^1$ and a function $g : A \to A$ of degree 3 so that (1) A is the g-orbit of a wandering triple T and (2) the set A with the map g can be embedded into S^1 by means of an embedding $\varphi : A \to S^1$ so that the induced map on $\varphi(A)$ is σ_3. Standard arguments show that $\varphi(T)$ is a wandering gap in a cubic invariant lamination. Flexibility in the construction allows us to fine tune it in Section 4 to prove our main theorem.

2. CIRCULAR MAPS WHICH ARE σ-EXTENDABLE

In this section we introduce the notion of a *topologically exact* dynamical system $f : A \to A, A \subset S^1$ of *degree* n. A dynamical system which can be embedded into $\sigma_n : S^1 \to S^1$ is said to be *σ-extendable (of degree n)*. We show that a topologically exact countable dynamical system of degree 3 without fixed points is σ-extendable of degree 3.

A subset of S^1 is said to be a *circular* set. An ordered circular triple $\{x, y, z\}$ is *positive* if $y \in (x, z)$. Given $X \subset S^1$, a function $f : X \to S^1$ is *order preserving* if for any positive triple $\{x, y, z\} \subset X$ the triple $\{f(x), f(y), f(z)\}$ is positive too. Given a set $A \subset S^1$, a (possibly discontinuous) function $f : A \to B$ is *of degree* d if d is the minimal number for which there exists a partition $x_0 < x_1 < \cdots < x_d = x_0$ of S^1 such that for each i, $f|_{[x_i,x_{i+1})\cap A}$ is order preserving. If A is finite, one can extend f to a map on S^1 which maps each arc complementary to A forward as an increasing map and is one-to-one inside the arc - the degree of the extension is equal to that of f. Thus, if A is finite then $d < \infty$, but d may be finite even if A is infinite. If $d < \infty$, we denote it by $\deg(f)$.

If $A = B$ and $\deg(f) < \infty$, we call f a *circular map*. An order preserving bijection $h : X \to Y$ (with $X, Y \subset S^1$) is called an *isomorphism*. Two circular maps $f : A \to A$ and $g : A' \to A'$ are *conjugate* if they are conjugate in the set-theoretic sense by an isomorphism $h : A \to A'$. The degree of a circular map is invariant under conjugacy. A circular map $f : A \to A$ is *σ-extendable* if for some $\sigma_{\deg(f)}$-invariant set $A' \subset S^1$ the map $f|_A$ is conjugate to the function $\sigma_{\deg(f)}|_{A'} : A' \to A'$. We prove that a version of *topological exactness* (i.e., the property that all arcs eventually expand and "cover" the entire circle) implies that a circular map is σ-extendable.

We need a few other definitions. An *arc in a circular set X (or X-arc)* is the intersection of an arc in S^1 and X. Every arc I in X (or in S^1) has the *positive* order $<_I$ determined by the positive orientation on S^1 (if it is clear from the context what I is, we omit the subscript I). Given sets A and B contained in an arc $J \subset S^1$ we write $A < B$ if $a < b$ for each $a \in A$ and each $b \in B$. Arcs in the circle may be open, closed or include only one of the two endpoints and will be denoted (a, b), $[a, b]$ etc. Corresponding arcs in a circular set X will be denoted by $(a, b)_X, [a, b]_X$ etc. If $X, Y \subset S^1$ are two disjoint closed arcs then by (X, Y) we mean the open arc enclosed between X and Y so that the movement from X to Y within this arc is in the positive direction. We always assume that a circular set A contains at least two points.

Definition 3. *Let $f : A \to A$ be a circular map. Then f is said to be* topologically exact *if for each $x \neq y$ in A there exists an $n \geq 1$ such that either $f^n(x) = f^n(y)$ or $f([f^n(x), f^n(y)]_A) \not\subset [f^{n+1}(x), f^{n+1}(y)]_A$.*

A circular map $f : A \to A$ may not admit a continuous extension over $\overline{A}$. However we define a class of set-valued functions which help in dealing with f anyway. Namely, a set-valued function $F : S^1 \to S^1$ is called an *arc-valued map* if for each $x \in S^1$, $F(x) = [a_x, b_x]$ (with $a_x \leq b_x \in S^1$ in the positive order) and for each sequence $z_i \to z$ in S^1, $\limsup F(z_i) \subset F(z)$; clearly, this is equivalent to the fact that the graph of F is closed as a subset of the 2-torus $\mathbb{T}^2 = S^1 \times S^1$.

Definition 4. *We say that an arc-valued map $F : S^1 \to S^1$ is* locally increasing *if for* ***any*** *$z \in S^1$ there exists an arc $I = [x_z, y_z], x_z <_I z <_I y_z$ with 1) $F(x_z) \cap F(y_z) = \emptyset$, 2) for each $u <_I w \in (x_z, y_z)$ the arcs $F(u), F(w)$ are contained in the open arc $(F(x_z), F(y_z))$ and $F(u) < F(w)$. The* degree *of a locally increasing arc-valued map F, denoted by $\deg(F)$, is the number of components of $F^{-1}(z)$ (by $F^{-1}(z)$ we mean the set of all $y \in S^1$ such that $z \in F(y)$). It is easy to see (by choosing a finite cover of S^1 by intervals (x_z, y_z)) that $\deg(F)$ is well-defined and finite.*

Let F be a locally increasing arc-valued map and $f : A \to A$, $A \subset S^1$, be a circular map; we say that F is an *arc-valued extension* of f if $f(a) \in F(a)$ for each $a \in A$. Now we prove the main result of this subsection; the statement is far from the most general one, but sufficient for our purpose.

Theorem 5. *Suppose that $f : A \to A$ is a topologically exact circular map of degree 3 such that A is countable and does not contain a fixed point. Then f is σ-extendable.*

Proof. We may assume that points of A are isolated (otherwise replace each point of A with a small enough interval and put the point of A in the middle of it) and, hence, that points of $\overline{A} \setminus A$ do not belong to A. Define an arc-valued extension F of f as follows. First, for each $z \in \overline{A}$ define

$$L(z) = \begin{cases} f(z) & \text{if } z \in A; \\ \lim f(a_i), & \text{if there exists } a_i \in A \text{ such that } a_i \nearrow z; \\ \lim f(b_i), & \text{for a sequence } b_i \in A \text{ such that } b_i \searrow z \text{ otherwise.} \end{cases}$$

The map $L(z)$ is well-defined, and is easy to see that $L(z)$ maps $\overline{A}$ into $\overline{A}$ and that $L(z)$ is still of degree 3. Given a map $g : S^1 \to S^1$ defined at points a, b, let the *linear extension* of g on (a, b) be the map which maps the interval (a, b) linearly onto the interval $(g(a), g(b))$. Extend $L(z)$ on each component of $S^1 \setminus \overline{A}$ linearly. For each point $x \in S^1$ define $F(x)$ as the interval $[\lim_{t\nearrow x} L(t), \lim_{t\searrow x} L(t)]$. Then F is a locally increasing arc-valued map with $f(z) \in F(z)$ for each $z \in A$ and $\deg(f) = \deg(F) = 3$. Note that for each $a \in A$, $F(a) = \{f(a)\}$ and the set of points with non-degenerate image is countable.

Let $p : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ be a standard projection of the real line onto the circle. We may assume that $F(0)$ is a point. Choose a lifting G of F such that $G(0)$ is a point between 0 and 1. Then the graph of $G|_{[0,1]}$ stretches from the point $(0, G(0))$ to $(1, G(1))$ and $G(1) = G(0) + 3$. Hence the graph of $G|_{[0,1]}$ intersects the graph of $y = x + 1$, and we can change the projection p so that $0 \in G(0)$. Then $0 \notin A'$ (A' is the lifting of A) because otherwise $a = p(0)$ would be a fixed point in A.

Since the graph of G intersects each horizontal line at exactly one point, there are two points $0 < b' < c' < 1$ with $1 \in G(b'), 2 \in G(c')$. Then $b', c' \notin A'$ because otherwise $b = p(b') \in A$ or $c = p(c') \in A$ and so $a \in A$, a contradiction. Hence $a \in F(a) \cap F(b) \cap F(c)$. Consider the arcs $[a, b] = I_0, [b, c] = I_1$ and $[c, a] = I_2$ and associate to every point $x \in A$ its itinerary itin(x) in the sense of this partition. Then $F^k(x), k \geq 0$ is a point for a point $x \in A$, and $F^k(x) \neq a, b, c$ for any k (because $f|_A$ has no fixed points). Hence itin(x) is well-defined.

Let us show that any two points of A have distinct itineraries. Define pullbacks of the arcs I_0, I_1, I_2 by taking preimages of points a, b, c inside I_0, I_1 and I_2 appropriately and considering arcs between these preimages. This can be done arbitrarily many times, hence every point $x \in A$ belongs to the intersection $I(\text{itin}(x))$ of the appropriate pullbacks of I_0, I_1 and I_2.

If points $x, y \in A$ had the same itinerary $\bar{r}$ then they would both belong to the same interval $I(\bar{r})$. Let J be the arc between x and y contained in $I(\bar{r})$. Then: a) J and all its F-images have well-defined endpoints (i.e. the endpoints of every F-image of J are such that their F-images are points, not intervals), and b) every F-image of J is contained in I_0, or I_1, or I_2. This contradicts the topological exactness of $f|_A$ and shows that $\text{itin}(x) \neq \text{itin}(y)$. Hence no point $z \in A$ can have itinerary $\text{itin}(z) = (iii\dots)$ for some $i = 0, 1, 2$ (otherwise z and $f(z) \neq z$ have the same itinerary).

The same construction applies to σ_3. Set $K_0 = [0, 1/3], K_1 = [1/3, 2/3]$ and $K_2 = [2/3, 1]$ (here 0 and 1 are identified and the full angle is assumed to be 1) and use the notation $K(\bar{r})$ for the point x with σ_3-itinerary $\bar{r}$. Given $x \in A$ define $h(x)$ as $K(\text{itin}(x))$. Then h is a one-to-one map from A onto a σ-invariant set $B \subset S^1$. Since on each finite step the circular order among the F-pullbacks of I_0, I_1 and I_2 is the same as the circular order among the σ-pullbacks of K_0, K_1 and K_2 then the map h is an isomorphism between the circular sets A and B, hence h conjugates $f|_A$ and $\sigma|_B$. □

3. CUBIC LAMINATIONS WITH WANDERING TRIANGLES

In Section 3 we prove a preliminary version of Theorem 2. The construction must satisfy necessary conditions for a cubic lamination $\sim$ to be a WT-lamination which follow from [12] or from [3]. Indeed, by Theorem 1 if $\sim$ is a cubic WT-lamination then $k_\sim = 2$, the two critical classes of $\sim$ are leaves, and $J_\sim$ is a *dendrite*, i.e. a locally connected continuum without subsets homeomorphic to the circle. The two critical leaves in $\sim$ correspond to two critical points in $J_\sim$. By [2] (see also [9] for laminations of any degree) both critical points in $J_\sim$ must be recurrent with the same limit set under the induced map $f_\sim$.

Set $\sigma_3 = \sigma$. The circle S^1 is identified with the quotient space $\mathbb{R}/\mathbb{Z}$; points of S^1 are denoted by real numbers $x \in [0, 1)$. Let $B = \{0 < c' < s_0 < u_0 < \frac{1}{2} < d' < v_0 < t_0 < 1\}$ with $v_0 - u_0 = 1/3$ and $t_0 - s_0 = 2/3$, $\bar{c}_0$ be the chord with the endpoints u_0, v_0, and $\bar{d}_0$ be the chord with the endpoints s_0, t_0. Let u_{-k} be the point with $u_{-k} \in (u_0, v_0)$, $\sigma(u_{-k}) \in (u_0, v_0), \dots, \sigma^k(u_{-k}) = u_0$. Similarly we define points v_{-k}, s_{-k}, t_{-k}. Then $\lim u_{-n} = \frac{1}{2}$ and $\sigma(u_{-i}) = u_{-i+1}$; analogous facts hold for v_{-n}, s_{-n}, and t_{-n}. All these points together with the set B form the set B'. The chord connecting u_{-k}, v_{-k} is denoted by $\bar{c}_{-k}$, and the chord connecting s_{-k}, t_{-k} is denoted by $\bar{d}_{-k}$. Also, let $d' \in (v_{-1}, t_{-1})$.

Below we define a triple $T_1 = \{\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1\}$ and the set $X_1 = B' \cup T_1$. On each step a new triple $T_n = \{\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n\}$ is added and the set $X_n = X_{n-1} \cup \{\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n\}$ is defined. Denote new points added on each step by boldface letters. This explains the following notation: the function g on points x_{n-1},

y_{n-1}, z_{n-1} is defined as $g(x_{n-1}) = \mathbf{x}_n$, $g(y_{n-1}) = \mathbf{y}_n$, $g(z_{n-1}) = \mathbf{z}_n$. Below a "triple" means one of the sets T_i and a "triangle" means the convex hull of a triple. By "the triangle (of the triple) T_i" we mean "the convex hull of the triple T_i". Define A as $\cup T_i$ and A' as $A \cup B' \setminus \{c', d'\}$.

Suppose X_{i-1} is defined. The location of the i-th triple T_i is determined by points $p, q, r \in X_{i-1}$ with $p < \mathbf{x}_i < q < \mathbf{y}_i < r < \mathbf{z}_i$ and $[(p, \mathbf{x}_i) \cup (q, \mathbf{y}_i) \cup (r, \mathbf{z}_i)] \cap X_{i-1} = \emptyset$; then we write $T_i = T(p, \mathbf{x}_i, q, \mathbf{y}_i, r, \mathbf{z}_i)$. If 2 or 3 points of a triple lie between two adjacent points of X_{i-1}, we need less than 6 points to denote T_i - e.g., $T(p, \mathbf{x}_i, \mathbf{y}_i, q, \mathbf{z}_i)(p, q \in X_{i-1})$, means that $p < \mathbf{x}_i < \mathbf{y}_i < q < \mathbf{z}_i$, and $[(p, \mathbf{y}_i) \cup (q, \mathbf{z}_i)] \cap X_{i-1} = \emptyset$. Define the function g on all points of $[(B' \cap [s_{-1}, t_{-1}]) \setminus \{d'\}] \cup \{0\}$ as σ. Set $g(u_0) = g(v_0) = c'$, $g(s_0) = g(t_0) = d'$. This gives a function $g : B' \setminus \{c', d'\} \to B'$. The function g is constructed step by step to satisfy Rule A below.

Rule A. *All triples T_i are pairwise unlinked and disjoint from the set B'. The map g is order preserving on $[s_0, u_0]_{A'}, [u_0, v_0)_{A'}, [v_0, t_0]_{A'}, [t_0, s_0)_{A'}$ (which implies that the degree of $g|_{A'}$ is 3).*

Now we introduce locations of the initial triples:
$T_1 = T(0, \mathbf{x}_1, c', \mathbf{y}_1, t_0, \mathbf{z}_1)$, $T_2 = T(s_{-1}, \mathbf{x}_2, v_{-1}, \mathbf{y}_2, d', \mathbf{z}_2)$,
$T_3 = T(s_0, \mathbf{x}_3, v_0, \mathbf{y}_3, \mathbf{z}_3)$, $T_4 = T(x_1, \mathbf{x}_4, c', \mathbf{y}_4, \mathbf{z}_4)$,
$T_5 = T(u_{-1}, \mathbf{x}_5, \mathbf{y}_5, t_{-2}, \mathbf{z}_5)$, $T_6 = T(u_0, \mathbf{x}_6, \mathbf{y}_6, t_{-1}, \mathbf{z}_6)$,
$T_7 = T(y_1, \mathbf{x}_7, \mathbf{y}_7, t_0, \mathbf{z}_7)$.

Rule A forces the location of some triples. For two disjoint chords p, q denote by $S(p, q)$ be the strip enclosed by p, q and arcs of the circle. Then the boundary A'-arcs of the strip $S(\bar{d}_{-1}, \bar{c}_0)$ must map one-to-one into the arcs $(t_0, c')_{A'}$ and $(c', s_0)_{A'}$. Also, the boundary A'-arcs of the strip $S(\bar{c}_{-i}, \bar{d}_{-i})$ map into the boundary arcs of the strip $S(\bar{c}_{-i+1}, \bar{d}_{-i+1})$ one-to-one, and the boundary A'-arcs of the strip $S(\bar{d}_{-i-1}, \bar{c}_{-i})$ map into the boundary arcs of the strip $S(\bar{d}_{-i}, \bar{c}_{-i+1})$ one-to-one. Observe, that $T_2 \subset S(\bar{c}_{-1}, \bar{d}_{-1})$ and so by Rule A $T_3 \subset S(\bar{c}_0, \bar{d}_0)$ (the point x_3 must belong to (s_0, u_0) while the points y_3, z_3 must belong to (v_0, t_0)). The segment of triples $T_1, \dots, T_7$ is the basis of induction (see Figure 1).

Clearly, T_7 separates the chord $\bar{d}_0$ from T_1. Our Rules then force the location of forthcoming triples $T_8, T_9, \dots$ with respect to $X_7, X_8, \dots$ for some time. More precisely, $T_8 = T(y_2, \mathbf{x}_8, \mathbf{y}_8, d', \mathbf{z}_8)$, $T_9 = T(y_3, \mathbf{x}_9, \mathbf{y}_9, \mathbf{z}_9)$, $T_{10} = T(y_4, \mathbf{x}_{10}, \mathbf{y}_{10}, \mathbf{z}_{10})$. The first time when the location of a triple with respect to the previously constructed triples and points of $B' \setminus \{c', d'\}$ is not forced is when T_{10} is mapped onto T_{11}. At this moment Rule A guarantee that T_{11} must be located in the arc (y_5, z_5), but otherwise the location of T_{11} is not forced. In particular, the location of the triangle T_{11} with respect to $\frac{1}{2}$ is not forced. The freedom of choice of the location of T_{11} at this moment, and the similar variety of options available later on at similar moments, is

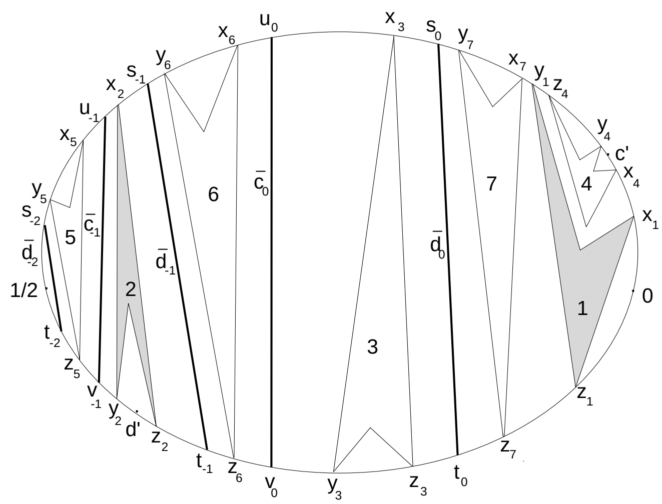

FIGURE 1. The first seven triangles

the reason why the construction yields not just one, but uncountably many types of behavior of a wandering triangle.

Let us pass on to the step of induction. It depends on a sequence of natural numbers $n_1 < m_1 < n_2 < m_2 < \dots$ (each pair of numbers n_i, m_i corresponds to the i-th step of induction). The inductive assumptions are of dynamical nature and deal with the locations of triples on the circle.

Next we introduce a general rule which will be enforced throughout the construction and will help us determine the location of the triples.

Rule B. *Points of any triple T_i are ordered in the arc $(0, 0)$ as follows: $\mathbf{x}_i < \mathbf{y}_i < \mathbf{z}_i$. All triangles are disjoint from the chords $\bar{c}_0, \bar{d}_0$.*

Since Rule B deals with the order of points on the arc $(0, 0)$, it establishes more than mere fact that the cyclic order among points $\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i$ is kept. Denote by UP the upper semicircle $(0, \frac{1}{2})$ and by LO the lower semicircle $(\frac{1}{2}, 0)$. By Rule B there are three types of triples:

(1) *up triples*, or *triples of $\triangle$-type*: triples with $\mathbf{x}_i \in UP, \mathbf{y}_i < \mathbf{z}_i \in LO$, denoted by $\triangle(\cdot)$ (the standard notation is $T(\cdot)$);
(2) *down triples*, or *triples of $\bigtriangledown$-type*: triples with $\mathbf{x}_i < \mathbf{y}_i \in UP, \mathbf{z}_i \in LO$, denoted by $\bigtriangledown(\cdot)$;
(3) *horizontal triples*, or *triples of $\triangleleft$-type*: triples with $\mathbf{x}_i < \mathbf{y}_i < \mathbf{z}_i$ contained entirely either in UP or in LO, denoted by $\triangleleft(\cdot)$.

Up triples and down triples are called *vertical triples*. Convex hulls of up, down, vertical and horizontal triples are said to be *up, down, vertical* and *horizontal* triangles. Chords with endpoints in UP and LO are *vertical* (e.g., $\bar{c}_0$ and $\bar{d}_0$ are vertical), otherwise they are *horizontal* (all sides of a horizontal triangle are horizontal). Let us discuss properties of vertical triples. A *proper* arc is an arc which contains none of the points $0, \frac{1}{2}, s_0, t_0, u_0, v_0$. Given a triple $T_i = \{x_i, y_i, z_i\}$, call the arcs $(x_i, y_i), (y_i, z_i)$ and (z_i, x_i) *xy-arc, yz-arc*, and *zx-arc*; all such arcs are said to be *generated* by the corresponding triples (or simply arcs *of* that triple). An up triple generates only one proper arc contained in LO and a down triple generates only one proper arc contained in UP. A vertical triangle has two vertical sides. Also, if vertical triples T', T'' are unlinked then none of them contains the other in its proper arc (this is not true for horizontal triples).

In the construction there will be *crucial moments* at which the Rules leave open the choice for the location of a new triple T_{n+1} with respect to X_n; the dynamics of a triangle at a crucial moment is called a *crucial event*. Crucial events are of the 4 types: an $h_\bigtriangledown$-event (the next closest approach of the triple to $\frac{1}{2}$ while the triple is of $\bigtriangledown$-type), a d-event (the next closest approach to the *entire* chord $\bar{d}_0$ from the right), an $h_\triangle$-event (the next closest approach to $\frac{1}{2}$ while the triple is of $\triangle$-type), and a c-event (the next closest approach to $\bar{c}_0$-chord from the right). The crucial moments of these types are denoted $h_\bigtriangledown(i), d(i), h_\triangle(i)$ and $c(i)$; the number i indicates that the crucial event takes place at the corresponding crucial moment during the i-th inductive step of the construction. We are now ready to state our Rule C.

Rule C. *Vertical triples have the following properties:*

(1) *up triples can only be contained in the strips* $S(\bar{c}_0, \bar{d}_0)$, $S(\bar{c}_{-1}, \bar{d}_{-1})$, $\dots, S(\bar{c}_{-i}, \bar{d}_{-i}), \dots$.
(2) *down triples can only be located to the right of the chord* $\bar{d}_0$ *as well as in the strips* $S(\bar{d}_{-1}, \bar{c}_0), S(\bar{d}_{-2}, \bar{c}_{-1}), \dots, S(\bar{d}_{-i-1}, \bar{c}_{-i}), \dots$.

The Rules allow us to explain how we choose the location of a triple; giving the order of the points without mentioning the Rules would significantly lengthen the verification. Crucial moments always happen in the order $d(i) < h_\triangle(i) < c(i) < h_\bigtriangledown(i) < d(i+1) < \dots$. Let us describe the i-th segment of the triples in the set A' from the moment $d(i)$ through the moment $d(i+1)-1$. A triple T_k is *minimal* if it contains no triples $T_i, i < k$ in its proper arcs. Set $d(0) = 1, h_\triangle(0) = 2, c(0) = 3, h_\bigtriangledown(0) = 5, d(1) = 7$.

Inductive assumptions for step i

(a) The i-th segment begins at the crucial moment $d(i)$ when the triple $T_{d(i)}$ is a down triple closest from the right to the chord $\bar{d}_0$:

$$T_{d(i)} = \bigtriangledown(y_{d(i-1)}, \mathbf{x}_{d(i)}, \mathbf{y}_{d(i)}, t_0, \mathbf{z}_{d(i)})$$

(b) Between the moments $d(i)+1$ and $h_\triangle(i)-1$ all triples are horizontal and minimal. Their location is determined by our Rules and existing triples.

(c) At the crucial moment $h_\triangle(i)$ the triple $T_{h_\triangle(i)}$ is an up triple closest to $\frac{1}{2}$ and contained in the strip $S(\bar{c}_{-n_i}, \bar{d}_{-n_i})$:

$$T_{h_\triangle(i)} = \triangle(s_{-n_i}, \mathbf{x}_{h_\triangle(i)}, v_{-n_i}, \mathbf{y}_{h_\triangle(i)}, \mathbf{z}_{h_\triangle(i)})$$

(d) We set
$$c(i) = h_\triangle(i) + n_i \tag{1}$$

For each $1 \le j \le n_i - 1$ we have that

$$T_{h_\triangle(i)+j} = \triangle(s_{-n_i+j}, \mathbf{x}_{h_\triangle(i)+j}, v_{-n_i+j}, \mathbf{y}_{h_\triangle(i)+j}, \mathbf{z}_{h_\triangle(i)+j})$$

if the triple $T_{h_\triangle(i)+j}$ is the first triple entering the strip $S(\bar{c}_{-n_i+j}, \bar{d}_{-n_i+j})$. If this triple enters a strip of type $S(\bar{c}_{-r}, \bar{d}_{-r})$ already containing other triples, then we locate it to be an up triple closest to $\bar{c}_{-r}$.

(e) At the crucial moment $c(i)$ the triple $T_{c(i)}$ is an up triple closest from the right to the chord $\bar{c}_0$:

$$T_{c(i)} = \triangle(x_{c(i-1)}, \mathbf{x}_{c(i)}, v_0, \mathbf{y}_{c(i)}, \mathbf{z}_{c(i)})$$

(f) Between the moments $c(i)+1$ and $h_\bigtriangledown(i)-1$ all triples are horizontal and minimal. Their location is determined by our Rules and existing triples.

(g) At the crucial moment $h_\bigtriangledown(i)$ the triple $T_{h_\bigtriangledown(i)}$ is a down triple closest to $\frac{1}{2}$ and contained in the strip $S(\bar{d}_{-m_i}, \bar{c}_{-m_i+1})$:

$$T_{h_\bigtriangledown(i)} = \bigtriangledown(u_{-m_i+1}, \mathbf{x}_{h_\bigtriangledown(i)}, \mathbf{y}_{h_\bigtriangledown(i)}, t_{-m_i}, \mathbf{z}_{h_\bigtriangledown(i)})$$

(h) We set
$$d(i+1) = h_\bigtriangledown(i) + m_i \tag{2}$$

For each $1 \le j \le m_i - 1$ we have that

$$T_{h_\bigtriangledown(i)+j} = \bigtriangledown(u_{-m_i+j+1}, \mathbf{x}_{h_\bigtriangledown(i)+j}, \mathbf{y}_{h_\bigtriangledown(i)+j}, t_{-m_i+j}, \mathbf{z}_{h_\bigtriangledown(i)+j})$$

if the triple $T_{h_\bigtriangledown(i)+j}$ is the first triple entering the strip $S(\bar{d}_{-m_i+j}, \bar{c}_{-m_i+j+1})$. If this triple enters a strip of type $S(\bar{d}_{-r}, \bar{c}_{-r+1})$ already containing other triples, then we locate it to be a down triple closest to $\bar{d}_{-r}$.

The properties (a) - (h) are exhibited at the basic step from $d(0)$ to $d(1)$. The step of induction can be made to satisfy the same properties.

Step of induction

(a) The $i+1$-st segment begins at the crucial moment $d(i+1)$ when the triple $T_{d(i+1)}$ is a down triple closest from the right to the chord $\bar{d}_0$:

$$T_{d(i+1)} = \bigtriangledown(y_{d(i)}, \mathbf{x}_{d(i+1)}, \mathbf{y}_{d(i+1)}, t_0, \mathbf{z}_{d(i+1)})$$

Then $T_{d(i+1)}$ lies between $T_{d(i)}$ and $\bar{d}_0$. The Rules and inductive assumptions determine the next few locations of the triple. We call $T_{d(i)}$ the *forcing* triple and $T_{d(i+1)}$ the *current* triple (this terminology applies to their images too).

(b) By the Rules on the next step the current triple $T_{d(i+1)+1}$ is contained in the arc $(y_{d(i)+1}, z_{d(i)+1})$, and for some time the triples $T_{d(i+1)+j}$ are contained inside yz-arcs of the images of the forcing triple. The containment holds at least until, at the crucial moment $h_\triangle(i)$, the crucial event of $h_\triangle$-type takes place for the forcing triple. However since the yz-arc of the forcing triple then is not exposed to $\frac{1}{2}$, we see that yet for a while the current triple stays inside the yz-arcs of the forcing triple and remains minimal. In fact, it remains minimal until, at the crucial moment $h_\bigtriangledown(i)$, the i-th crucial event of type $h_\bigtriangledown$ takes place for the forcing triple. Then the location of the current triple with respect to B' and existing triples is not fully determined because the yz-arc of the forcing triple is "exposed" to $\frac{1}{2}$ for the first time. Choose this to be the crucial moment $h_\triangle(i+1)$ for our current triple. Then

$$h_\triangle(i+1) = d(i+1) + h_\bigtriangledown(i) - d(i) \tag{3}$$

(c) At the crucial moment $h_\triangle(i+1)$ the triple $T_{h_\triangle(i+1)}$ is an up triple closest to $\frac{1}{2}$ and contained in the strip $S(\bar{c}_{-n_{i+1}}, \bar{d}_{-n_{i+1}})$:

$$T_{h_\triangle(i+1)} = \triangle(s_{-n_{i+1}}, \mathbf{x}_{h_\triangle(i+1)}, v_{-n_{i+1}}, \mathbf{y}_{h_\triangle(i+1)}, \mathbf{z}_{h_\triangle(i+1)})$$

(d) We set $c(i+1) = h_\triangle(i+1) + n_{i+1}$ (see (1)). Between the crucial moments $h_\triangle(i+1)$ and $c(i+1)$ the locations of the triples are almost completely determined by the Rules. For each $1 \le j \le n_{i+1} - 1$ we have

$$T_{h_\triangle(i+1)+j} = \triangle(s_{-n_{i+1}+j}, \mathbf{x}_{h_\triangle(i+1)+j}, v_{-n_{i+1}+j}, \mathbf{y}_{h_\triangle(i+1)+j}, \mathbf{z}_{h_\triangle(i+1)+j})$$

if the triple $T_{h_\triangle(i+1)+j}$ is the first triple entering the corresponding strip. If this triple enters a strip of type $S(\bar{c}_{-r}, \bar{d}_{-r})$ already containing other triples, then we locate it to be an up triple closest to $\bar{c}_{-r}$.

(e) At the crucial moment $c(i+1)$ the triple $T_{c(i+1)}$ is an up triple closest from the right to the chord $\bar{c}_0$:

$$T_{c(i+1)} = \triangle(x_{c(i)}, \mathbf{x}_{c(i+1)}, v_0, \mathbf{y}_{c(i+1)}, \mathbf{z}_{c(i+1)}).$$

Then $T_{c(i+1)}$ lies between $T_{c(i)}$ and $\bar{c}_0$. The Rules and inductive assumptions determine the next few locations of the triple. We call $T_{c(i)}$ the *forcing* triple and $T_{c(i+1)}$ the *current* triple (this applies to their images too).

(f) By the Rules on the next step the current triple $T_{c(i+1)+1}$ is contained in the arc $(x_{c(i)+1}, y_{c(i)+1})$, and for some time the triples $T_{c(i+1)+j}$ are contained inside the xy-arcs of the images of the forcing triple. The containment holds at least until, at the crucial moment $h_\bigtriangledown(i)$, the crucial event of $h_\bigtriangledown$-type takes place for the forcing triple. However since the xy-arc of the forcing triple then is not exposed to $\frac{1}{2}$, we see that yet for a while the current triple stays inside the the xy-arcs of the forcing triple and remains minimal. In fact, it remains minimal until, at the crucial moment $h_\triangle(i)$, the i-th crucial event of type $h_\triangle$ takes place for the forcing triple. Then the location of the current triple with respect to B' and existing triples is not fully determined because the xy-arc of the forcing triple is "exposed" to $\frac{1}{2}$ for the first time. Choose this to be the crucial moment $h_\bigtriangledown(i+1)$ for our current triple. Then

$$h_\bigtriangledown(i+1) = c(i+1) + h_\triangle(i+1) - c(i) \quad (4)$$

(g) At the crucial moment $h_\bigtriangledown(i+1)$ the triple $T_{h_\bigtriangledown(i+1)}$ is a down triple closest to $\frac{1}{2}$ and contained in the strip $S(\bar{d}_{-m_{i+1}}, \bar{c}_{-m_{i+1}+1})$:

$$T_{h_\bigtriangledown(i+1)} = \bigtriangledown(u_{-m_{i+1}+1}, \mathbf{x}_{h_\bigtriangledown(i+1)}, \mathbf{y}_{h_\bigtriangledown(i+1)}, t_{-m_{i+1}}, \mathbf{z}_{h_\bigtriangledown(i+1)})$$

.

(h) We set $d(i+2) = h_\bigtriangledown(i+1) + m_{i+1}$ (see (2)). Between the crucial moments $h_\bigtriangledown(i+1)$ and $d(i+2)$ the locations of the triples are almost completely determined by the Rules. For each $1 \le j \le m_{i+1} - 1$ we have

$$T_{h_\bigtriangledown(i+1)+j} = \bigtriangledown(u_{-m_{i+1}+j+1}, \mathbf{x}_{h_\bigtriangledown(i+1)+j}, \mathbf{y}_{h_\bigtriangledown(i+1)+j}, t_{-m_{i+1}+j}, \mathbf{z}_{h_\bigtriangledown(i+1)+j})$$

if the triple $T_{h_\bigtriangledown(i+1)+j}$ is the first triple entering the corresponding strip. If this triple enters a strip of type $S(\bar{d}_{-r}, \bar{c}_{-r+1})$ already containing other triples, then we locate it to be a down triple closest to $\bar{d}_{-r}$.

This concludes the induction. It is easy to check that the time between two consecutive crucial events grows to infinity. Let us check if these examples generate an uncountable family of cubic WT-laminations with pairwise non-conjugate induced maps.

Lemma 6. *The function $g|_A$ is σ-extendable of degree 3 (here $A = \cup_{i=1}^\infty T_i$).*

Proof. It is easy to see that the degree of g is 3. By Theorem 5 we need to check that for $a \ne b \in A$ there exists $n \ge 0$ such that $g([g^n(a), g^n(b)]_A) \not\subset [g^{n+1}(a), g^{n+1}(b)]$. This is obvious if $(a,b) \supset [s_0, u_0]$, or $(a,b) \supset [u_0, v_0]$, or $(a,b) \supset [v_0, t_0]$, or $(a,b) \supset [t_0, s_0]$. Suppose first that a and b are in the same triangle T_i. If $a = x_i$, $b = y_i$, and the next crucial moment of c-type is $c(j)$, then the arc $(f^{c(j)-i}(x_i), f^{c(j)-i}(y_i))$ contains (u_0, v_0) as desired. If $a = y_i$, $b = z_i$, and the next crucial moment of d-type is $d(l)$, then the arc $(g^{d(l)-i}(y_i), g^{d(l)-i}(z_i))$ contains (u_0, v_0) as desired. Now, let $a = z_i$ and $b = x_i$. Then if T_i is located to the left of $\bar{d}_0$, then $(a,b) \supset (t_0, s_0)$ and we are done. Otherwise it follows from the construction that T_{i+1} is located to the left of $\bar{d}_0$, and we are done too. Now assume that $a \in T_p$ and $b \in T_q$ with $p < q$. Since $q - p$ is finite and $m_i \to \infty$, we may assume that there exist k and i with $h_\bigtriangledown(i) \le p+k < q+k < d(i+1)$ and both T_{p+k} and T_{q+k} are down triples located in the arc (u_0, v_0). Then $[s_0, u_0] \cup [v_0, t_0]$ separates the points $g^{d(i+1)-q}(b)$ and $g^{d(i+1)-q}(a)$, and the result follows. □

By Lemma 6 from now on we assume that $T_1, T_2, \dots$ is the σ-orbit of a triple T_1 with the order among points of $A = \cup_{i=1}^\infty T_i$ exactly as before.

Lemma 7. *Let* $\hat{s}_0 = \lim_{i\to\infty} y_{d(i)}, \hat{t}_0 = \lim_{i\to\infty} z_{d(i)}, \hat{u}_0 = \lim_{i\to\infty} x_{c(i)}$ *and* $\hat{t}_0 = \lim_{i\to\infty} y_{c(i)}$. *Then the points* $\hat{s}_0, \hat{u}_0, \hat{v}_0, \hat{t}_0$ *are all distinct,* $\sigma(\hat{s}_0) = \sigma(\hat{t}_0)$ *and* $\sigma(\hat{u}_0) = \sigma(\hat{v}_0)$.

Proof. The limits in the lemma are well defined and for every i there are points of A in the arcs $(y_{d(i)}, x_{c(i)}),\ (x_{c(i)}, y_{c(i)}),\ (y_{c(i)}, z_{d(i)}),\ (z_{d(i)}, y_{d(i)})$. Hence the points $\hat{s}_0, \hat{u}_0, \hat{v}_0, \hat{t}_0$ are all distinct. To see that $\sigma(\hat{s}_0) = \sigma(\hat{t}_0)$ we show that $\alpha = \lim_{i\to\infty} y_{d(i)+1}$ and $\beta = \lim_{i\to\infty} z_{d(i)+1}$ are the same. Indeed, otherwise the arc $[\alpha, \beta]$ is non-degenerate and there exists the least $l \geq 0$ such that $\sigma^l[\alpha, \beta] = [\sigma^l(\alpha), \sigma^l(\beta)] = I$ is an arc of length at least $1/3$. The chord connecting the endpoints of I is the limit of chords connecting $y_{d(i)+1+l}, z_{d(i)+1+l} \in T_{d(i)+1+l}$, and the endpoints of $T_{d(i)+1+l}$ are outside I. Let us show that $A \cap I = \emptyset$. Suppose otherwise. Then there is a triple $T_k \subset I$ because if a point of T_k is in I then $T_k \subset I$ (if $T_k \not\subset I$ then a chord connecting points of T_k intersects chords connecting $y_{d(i)+1+l}$ and $z_{d(i)+1+l}$ with large i, a contradiction). Now we choose a big i so that $d(i)+l+1 > k$ is between the crucial moments $d(i)$ and $h_\triangle(i)$. Then the triple $T_{d(i)+l+1}$ must be minimal among the already existing triples, a contradiction with $T_k \subset I$. So, I contains no points of A which contradicts the fact that $g|_A$ is of degree 3 and implies that $\sigma(\hat{s}_0) = \sigma(\hat{t}_0)$. Similarly, $\sigma(\hat{u}_0) = \sigma(\hat{v}_0)$. □

Let $\bar{c}_0$ be the chord connecting $\hat{s}_0$ with $\hat{t}_0$, $\bar{d}_0$ be the chord connecting $\hat{v}_0$ with $\hat{u}_0$. To associate a lamination to $\Xi = \{\bar{c}_0, \bar{d}_0\}$ we rely on Kiwi [14]. A collection $\Theta = \{X_1, \dots, X_{d-1}\}$ of pairwise disjoint σ_d-critical chords (whose endpoints form a set $R = R_\Theta$) is called a *critical portrait* (e.g., Ξ is a critical portrait). The chords $X_1, \dots, X_{d-1}$ divide $\overline{\mathbb{D}}$ into components $B_1, \dots, B_d$ whose intersections with S^1 are finite unions of open arcs with endpoints in R. Given $t \in S^1$, its *itinerary i(t)* is the sequence $I_0, I_1, \dots$ of sets $B_1, \dots, B_d$, R with $\sigma_d^n(t) \in I_n (n \geq 0)$. A critical portrait Θ such that $i(t), t \in R_\Theta$ is not preperiodic is said to have a *non-periodic kneading*. Denote the family of all critical portraits with non-periodic kneadings by $\mathcal{Y}_d$. A lamination $\sim$ is Θ*-compatible* if the endpoints of every chord from Θ are $\sim$-equivalent. Theorem 8 is a particular case of Proposition 4.7 [13].

Theorem 8. *To each* $\Theta \in \mathcal{Y}_d$ *one can associate a* Θ*-compatible lamination* $\sim$ *such that all* $\sim$*-classes are finite,* $J_\sim$ *is a dendrite, and the following holds:* (1) *any two points with the same itinerary, which does not contain* R*, are* $\sim$*-equivalent;* (2) *any two points whose itineraries are different at infinitely many places are not* $\sim$*-equivalent.*

Denote the family of laminations from Theorem 8 by $\mathcal{K}_d$.

Lemma 9. *We have* $\Xi \in \mathcal{Y}_3$. *There is a lamination* $\sim$ *from* $\mathcal{K}_3$ *compatible with* Ξ *such that* T_1 *forms a* $\sim$*-class, and* $\bar{c}_0, \bar{d}_0$ *are the* $\sim$*-critical leaves.*

Proof. We prove that $\hat{s}_0, \hat{u}_0, \hat{v}_0, \hat{t}_0$ have non-preperiodic itineraries and never map into one another. We have $\sigma^l(\hat{s}_0) \in (\hat{u}_0, \hat{v}_0), h_{\triangle(i)} \le l \le c_i - 1$. Since $c_i - h_{\triangle(i)} \to \infty$, the only way $i(\hat{s}_0)$ can be preperiodic is if $\hat{s}_0$ eventually stays in $(\hat{u}_0, \hat{v}_0)$ forever, a contradiction to the construction. Assume that $\hat{s}_0$ maps into $\hat{v}_0$ by σ^r. Choose j with $h_\triangle(j) - d(j) > r$. Then the triangle $T_{d(j)+r}$ intersects $\bar{c}_0$, a contradiction to the construction. The claim for $\hat{s}_0$ is proven; the claims for other points of R_Ξ can be proven similarly.

By Theorem 8 there exists a lamination $\sim$ in $\mathcal{K}_3$ compatible with Ξ; since points in any T_i have the same itinerary avoiding R_Ξ, they are $\sim$-equivalent. Let us show that $\{\hat{u}_0, \hat{v}_0\}$ is a $\sim$-class. Indeed, otherwise the $\sim$-class of $\sigma(\hat{u}_0)$ is non-degenerate, hence it includes all triples $T_{c(j)+1}$ with big enough j and is infinite, a contradiction with Theorem 8. Similarly, $\{\hat{s}_0, \hat{t}_0\}$ is a $\sim$-class, and these are the only two critical $\sim$-classes. It follows that T_1 is a $\sim$-class. Indeed, otherwise T_1 is a proper subset of a $\sim$-class Q. Then Q contains more than 3 points. Hence by Theorem 1 - or by [12] - Q is preperiodic or precritical. If for some $i \ge 0$ the class $f^i(Q)$ is periodic, then, since the triple T_1 is wandering, $f^i(Q)$ must be infinite, a contradiction to Theorem 8. If for some minimal $i \ge 0$ the class $f^i(Q)$ is critical then it has to consist of $|Q| > 3$ elements, a contradiction to the above. □

By Lemma 9 for a sequence $\mathcal{T} = n_1 < m_1 < \dots$ we construct a WT-lamination $\sim$ in $\mathcal{K}_3$; the family $\mathcal{W}$ of all such laminations is uncountable.

Theorem 10. *Laminations $\sim$ in $\mathcal{W}$ have pairwise non-conjugate induced maps $f_\sim|_{J_\sim}$.*

Proof. Let sequences $\mathcal{T} = n_1 < m_1 < \dots, \mathcal{T}' = n'_1 < m'_1 < \dots$ be distinct, $\sim$ and $\sim'$ be the corresponding laminations from Lemma 9, p and p' be the corresponding quotient maps, and the topological Julia sets with induced maps be $f : J \to J$ and $f' : J' \to J'$, resp. All the points and leaves from our construction are denoted as before for f (e.g., $\hat{u}_0, \hat{v}_0, \dots, \bar{c}_0, \bar{d}_0, \dots$) while in the case of f' we add $'$ to the notation $(\hat{u}'_0, \hat{v}'_0, \dots)$.

Assume that a homeomorphism $\varphi : J \to J'$ conjugates f and f'. The critical points $p(\hat{u}_0) = C, p(\hat{s}_0) = D \in J$ of f are cutpoints each of which cuts J into 2 pieces. Moreover, the set $J \setminus (C \cup D)$ consists of 3 components: $L = p((\hat{u}_0, \hat{v}_0)), M = p((\hat{s}_0, \hat{u}_0) \cup (\hat{v}_0, \hat{t}_0))$ and $R = p((\hat{t}_0, \hat{s}_0))$. Similarly, the critical points $p'(\hat{u}'_0) = C', p'(\hat{s}'_0) = D' \in J'$ of f' are cutpoints each of which cuts J' into 2 pieces. Moreover, the set $J' \setminus (C' \cup D')$ consists of 3 components: $L' = p'((\hat{u}'_0, \hat{v}'_0)), M' = p'((\hat{s}'_0, \hat{u}'_0) \cup (\hat{v}'_0, \hat{t}'_0))$ and $R' = p'((\hat{t}'_0, \hat{s}'_0))$. Clearly, φ maps points C, D onto points C', D'.

For a $\sim$-class g the point $p(g) \in J$ divides J into $|g|$ components (the same holds for $\sim'$). Let us show that the $\sim$-class of 0 is $\{0\}$. Indeed, no point outside $(\hat{t}_0, \hat{s}_0)$ can be $\sim$-equivalent to 0 because the leaf $\bar{d}_0$ cuts them off 0. Since all points of $(\hat{t}_0, \hat{s}_0)$ but 0 eventually leave $(\hat{t}_0, \hat{s}_0)$ then no point not equal to 0 can be $\sim$-equivalent to 0. Similarly, $\{0\}$ is a $\sim'$-class, and $\{\frac{1}{2}\}$ is a $\sim$-class and a $\sim'$-class. Hence $a = p(\frac{1}{2}), b = p(0)$ are non-dividing f-fixed points, and $a' = p'(\frac{1}{2}), b' = p'(0)$ are non-dividing f'-fixed points. These are all the non-dividing fixed points, so φ maps points a, b onto points a', b'. By the construction a is the only non-dividing f-fixed belonging to the limit sets of f-critical points, and a' is the unique non-dividing f'-fixed point belonging to the limit sets of the f'-critical points. Hence $\varphi(a) = a'$ which implies that $\varphi(b) = b'$, and therefore $\varphi(C) = C', \varphi(D) = D'$. Thus, $\varphi(L) = L, \varphi(M) = M', \varphi(R) = R'$.

Assume that the first time the sequences $\mathcal{T}, \mathcal{T}'$ are different is $n_i > n_i'$. Then $h_\triangle(i) = h'_\triangle(i) = h$, and up until that moment all corresponding crucial moments for the two laminations are equal: $d(r) = d'(r), h_\triangle(r) = h'_\triangle(r), c(r) = c'(r), h_\bigtriangledown(r) = h'_\bigtriangledown(r)(0 \le r \le i-1)$, and $d(i) = d'(i)$). Before the crucial moment h the behavior of the triples relative to the chords $\bar{c}_0, \bar{d}_0$ (resp. $\bar{c}_0', \bar{d}_0'$) is the same. Consider the triple $T_{d(i)}$ (the closest approach to $\bar{d}_0$ preceding h), and the corresponding triple $T'_{d'(i)}$. Then the dynamics of $T_{d(i)}$ ($T'_{d'(i)}$) forces the same dynamics on $\bar{d}_0$ ($\bar{d}_0'$) until $T_{d(i)}$ ($T'_{d'(i)}$) maps onto $T_{c(i)}$ ($T'_{c'(i)}$). Hence $\sigma^{h-d(i)+n_i'}(\hat{s}_0')$ already belongs to the arc $(\hat{v}_0', \hat{t}_0')$ while $\sigma^{h-d(i)+n_i}(\hat{s}_0)$ still belongs to the arc $(\frac{1}{2}, \hat{v}_0)$. Therefore $f^{h-d(i)+n_i'}(D) \in L$ whereas $(f)'^{h-d(i)+n_i'}(D') \in M$. Since $\varphi(D) = D'$ and $\varphi(M) = M'$ we get a contradiction which shows that φ does not exist and the maps $f|_J$ and $f'|_{J'}$ are not conjugate. □

4. TCE-POLYNOMIALS WITH WANDERING BRANCH POINTS

In Section 4 we show that there exists an uncountable family of TCE-polynomials P whose induced laminations $\sim_P$ are WT-laminations (since by [19] the Julia set of a TCE-polynomial is locally connected then the polynomial on its Julia set and the induced map on the corresponding topological Julia set are conjugate). The *Topological Collet-Eckmann (TCE)* condition is studied in a number of papers (e.g., [10, 19, 20, 21, 22]; a list of references can be found in a nice recent paper [20]). It is considered a form of non-uniform (weak) hyperbolicity. By [20] several standard conditions of non-uniform hyperbolicity of rational maps, including the TCE-condition, are equivalent. By Proposition 5.2 [19] (see also [10, 21]) the Julia set of a TCE polynomial is Hölder (i.e., the Riemann map extends over the boundary as Hölder), hence locally connected. Non-uniformly hyperbolic dynamics was introduced by Ya. G. Sinai in the context of billiards; we deal with this notion in the context of 1-dimensional complex dynamics.

The plan is to construct WT-laminations $\sim$ from $\mathcal{W}$ corresponding to specific sequences $\mathcal{T}$ whose induced maps $f_\sim|_{J_\sim}$ satisfy the TCE condition (the definitions are below). Since $\mathcal{W} \subset \mathcal{K}_3$, by the results of Kiwi [13, 14] to each such lamination $\sim$ a polynomial $P_\sim$ is associated, and $P_\sim|_{J_{P_\sim}}$ is monotonically semiconjugate to the induced map $f_\sim|_{J_\sim}$. This implies that $P_\sim$ satisfies the TCE condition, by [19] its Julia set is locally connected (actually Hölder) and $P_\sim|_{J_{P_\sim}}$ is in fact conjugate to $f_\sim : J_\sim \to J_\sim$.

A continuum $K \subset S^2$ is *unshielded* if it is the boundary of one of its complementary domains (see, e.g., [4]). Below K is either S^2 or a locally connected unshielded continuum in S^2 (then we choose a metric in K such that all balls are connected; the existence of such a metric is proven in [1], see also [17]). Given a set $A \subset K$ and a point $z \in A$ we denote by $\mathrm{Comp}_z A$ the component of A containing z. Consider a branched covering map $f : K \to K$ (in the case when $K = S^2$ we assume that f is a rational function). Then the set of critical points, Cr_f, is finite.

Take a point $x \in K$ and the ball $B(f^n(x), r)$. For each i, $0 \le i \le n$, consider $\mathrm{Comp}_{f^i(x)} f^{-(n-i)}(B(f^n(x), r))$ and call it a *pull-back of* $B(f^n(x), r)$ *(along the orbit of* x*)*. Denote by $\Delta_f(x, r, n)$ all the moments i such that $\mathrm{Comp}_{f^i(x)} f^{-(n-i)}(B(f^n(x), r)) \cap \mathrm{Cr}_f \neq \emptyset$. A map $f : K \to K$ is said to satisfy the *TCE condition* (or to be a *TCE map*, or just *TCE*) iff there are $M > 0, P > 1$ and $r > 0$ such that for every $x \in K$ (if $K = S^2$ then we consider every $x \in J_f$) there is an increasing sequence $n_j \le Pj$ of numbers with $\Delta_f(x, r, n_j) \le M$. Therefore if a map is not TCE then for *any* $M > 0, P > 1$ and $r > 0$ there exist $x \in K$ (if $K = S^2$ then we consider every $x \in J_f$) and $N > 0$ with

$$\frac{|\{n \in [0, N] \mid \Delta_f(x, r, n) > M\}|}{N + 1} > 1 - \frac{1}{P}$$

A continuous map $f : X \to X$ of a metric space is *backward stable* at $x \in X$ if for any δ there is ε such that for any connected set $K \subset B(x, \varepsilon)$, any $n \ge 0$ and any component M of $f^{-n}(K)$, $\mathrm{diam}(M) \le \delta$; f is *backward stable* if it is backward stable at all points. If X is compact then f is *backward stable* iff for any δ there is ε such that for any continuum T with $\mathrm{diam}(T) \le \varepsilon$, any $n \ge 0$ and any component M of $f^{-n}(T)$, $\mathrm{diam}(M) \le \delta$. Essentially, the notion is due to Fatou. Classic results (see, e.g., Fatou, [8]) imply that R is backward stable outside the critical limit sets and is not backward stable at parabolic or attracting periodic points. In an important paper [16] Levin showed that polynomials with one critical point and locally connected Julia set are backward stable on their Julia sets. Later [4] this result was extended to all induced maps on their topological Julia set.

Orbit segments $\{z, f(z), \dots, f^n(z)\}$ and $\{y, f(y), \dots, f^n(y)\}$ *δ-shadow (each other)* if $d(f^i(z), f^i(y)) \le \delta$ for $0 \le i \le n$. Denote the orbit of z by orb(z); an ($[i,j]$-)*segment* of orb(z) is the set $\{f^i(z), \dots, f^j(z)\}$. Given a point z, an integer n and an $\varepsilon > 0$, say that $f^n(z)$ is *critically ε-shadowed of order k* if there are precisely k distinct pairs (each pair consists of a critical point u and an iteration s) such that $f^s(z), \dots, f^n(z)$ is ε-shadowed by $u, \dots, f^{n-s}(u)$. If so, we call n a *critical ε-shadowing time of order k (for z)*. Lemma 11 is inspired by Lemma 2.2 of the paper [22] by Smirnov.

Lemma 11. *Suppose that $f : K \to K$ is a branched covering, backward stable map, and there exist $\varepsilon' > 0$, s' and $\tau' > 0$ such that for any critical point u and any integer $N > 0$ there are more than $\tau'(N+1)$ critically ε'-shadowed times of order less than s' in $[0, N]$ for u. Then f satisfies the TCE condition.*

Proof. We prove that if f is not TCE condition then for any given $\varepsilon > 0$, s and $\tau > 0$, there is $N > 0$ and a critical point u such that there are less than $\tau(N+1)$ critically ε-shadowed times of order less than s in $[0, N]$ for u. Since f is not TCE, for any $P > 1, r > 0, M > 0$ there exist $x \in K$ and $N > 0$ such that for a set H of more than $\frac{(P-1)(N+1)}{P}$ integers $l \in [0, N]$ we have $\Delta_f(x, r, l) > M$. Let the distance between any two critical points be more than $R > 0$ and choose $M > \frac{sP}{(P-1)\tau}$. Since f is backward stable, we can find $\delta < \min\{\varepsilon/2, R/2\}$ and $r > 0$ so that any pull back of an r-ball is of diameter less than δ. For $x \in K$ let $c(x)$ be a closest to x critical point.

Define a collection $\mathcal{I}$ of *intervals of integers*. For an integer $j, 0 \le j \le N$ define (if possible) the largest number $k = k_j, j \le k \le N$ with

$$\text{Comp}_{f^j(x)} f^{-(k-j)}(B(f^k(x), r)) \cap \text{Cr}_f \neq \emptyset.$$

Let A be the set of all j for which k_j exists, and $\mathcal{I}$ be the family of all intervals of integers $\{[j, k_j] : j \in A\}$. The $[j, k_j]$-segment of orb(x) is δ-shadowed by the $[0, k_j - j]$-segment of the critical point $c(f^j(x))$. If a critical point belongs to the pullback $U = \text{Comp}_{f^i(x)} f^{-(l-i)}(B(f^l(x), r))$ of $B(f^l(x), r)$ along the orbit of x then $i \in A$ and $l \in [i, k_i]$. Hence, if $l \in H$ then more than M intervals from $\mathcal{I}$ contain l. Since $|H| \ge \frac{(P-1)(N+1)}{P}$ then

$$\sum_{I \in \mathcal{I}} |I| \ge \frac{(P-1)(N+1)M}{P} > \frac{s(N+1)}{\tau}.$$

Let $i, j \in A, u = c(f^i(x)), v = c(f^j(x))$. If $j \ge i$ and $[i, k_i] \cap [j, k_j] = [j, l]$ ($l = k_i$ or $l = k_j$) then the $[j-i, l-i]$-segment of orb(u) and $[0, l-j]$-segment of orb(v) 2δ-shadow each other. Since $2\delta < \varepsilon$ then if $t \in [i, k_i]$ is covered by at least s intervals of the form $[j, k_j] \in \mathcal{I}$ with $i \le j$ then $f^{t-i}(u)$ is critically ε-shadowed of order at least s. Let us show that in some interval

$I = [i, k_i] \in \mathcal{I}$ there are $h > (1-\tau)|I|$ integers $t_1, \dots, t_h$ covered by at least s intervals of the form $[j, k_j] \in \mathcal{I}$ with $i \le j$.

Let us show that such an interval $[i, k_i] \in \mathcal{I}$ exists. Indeed, otherwise in each interval $I = [i, k_i] \in \mathcal{I}$ at most $(1-\tau)|I|$ points are covered by s intervals of the form $[j, k_j] \in \mathcal{I}$ with $i \le j$. Let us call a pair (I, l) *admissible* if $I \in \mathcal{I}, l \in I$ and there are at least s intervals $[j, k_j] \in \mathcal{I}$ with $i \le j \le l \le k_j$. Denote the number of all admissible pairs by L and count it in two ways: over intervals I from $\mathcal{I}$, and over points l. If we count L over intervals from $\mathcal{I}$ then, since by the assumption each interval $I \in \mathcal{I}$ contains at most $(1-\tau)|I|$ numbers l such that (I, l) is admissible, we see that $L \le (1-\tau)\sum_{I\in\mathcal{I}} |I|$. For each $l \in [0, N]$ let $m(l)$ be the number of intervals from $\mathcal{I}$ containing l. Then $\sum_{I\in\mathcal{I}} |I| = \sum m(l)$. Define two sets $A \subset [0, N], B \subset [0, N]$ as follows: A is the set of all integers $l \in [0, N]$ with $m(l) \le s-1$, and B is the set of all integers $l \in [0, N]$ with $m(l) \ge s$. Then it is easy to see that $L = \sum_{l\in B}(m(l) - s + 1)$. Hence

$$\sum_{I\in\mathcal{I}} |I| = \sum_{l=0}^{N} m(l) = (s-1)|B| + \sum_{l\in B}[m(l) - s + 1] + \sum_{l\in A} m(l) = \\ = (s-1)|B| + L + \sum_{l\in A} m(l)$$

Since $L \le (1-\tau)\sum_{I\in\mathcal{I}} |I|$ and $m(l) \le s-1$ for $l \in A$ then

$$\sum_{I\in\mathcal{I}} |I| \le (s-1)(|B|+|A|)+(1-\tau)\sum_{I\in\mathcal{I}} |I| = (s-1)(N+1)+(1-\tau)\sum_{I\in\mathcal{I}} |I|$$

which implies that

$$\sum_{I\in\mathcal{I}} |I| \le \frac{(s-1)(N+1)}{\tau},$$

a contradiction. Hence there exists an interval $I = [i, k_i] \in \mathcal{I}$ with $h > (1-\tau)|I|$ integers $t_1, \dots, t_h$ covered by at least s intervals of the form $[j, k_j] \in \mathcal{I}$ with $i \le j$. Set $N = k_i - i$; then h integers $t_1 - i \in [0, N], \dots, t_h - i \in [0, N]$ are critically ε-shadowing times of order at least s for u. Hence there are less than $N + 1 - h < \tau(N + 1)$ integers in $[0, N]$ which are critically ε-shadowing times of order less than s for u. Doing this for $\varepsilon = \varepsilon', s = s', \tau = \tau'$ from the lemma, we get a contradiction to the assumptions of the lemma and complete its proof. □

Let $\sim$ be a lamination constructed as in Section 3 for a sequence $\mathcal{T} = n_1 < m_1 < \dots$, and let $f|_J$ be its induced map. Let us state some facts about the construction in terms of the map f. Let $p : S^1 \to J$ be the corresponding quotient map and $I \subset J$ be the arc connecting $p(1/2) = b$ and $p(0) = a$. A $\sim$-class g contains points of the upper semicircle UP and the lower one LO iff $p(g) \in I$. Put $p(T_i) = t_i, p(\hat{u}_0) = C, p(\hat{d}_0) = D, f^i(C) = C_i$ and $f^i(D) = D_i$. Assume that $J \subset \mathbb{C}$ and that the orientation of J fits into that on the unit circle. Visualize I as a subsegment of the x-axis such that b is the "leftmost" point of the entire J (its x-coordinate is the least), a is the "rightmost" point of J (its x-coordinate is the greatest), the points of J corresponding to angles from UP belong the upper half-plane, and the points of J corresponding to angles from LO belong the lower half-plane.

By the construction $d(0) = 1, h_\triangle(0) = 2, c(0) = 3, h_\triangledown(0) = 5$, $d(1) = 7$. The crucial moments $d(i), h_\triangle(i), c(i), h_\triangledown(i)$ are the *moments of the closest approach of images of* t_1 (or just the *closest approaches of* t_1) to $D, b, C, b, \dots$ in this order. To explain the term "closer" we need the following notation: if $m, n \in J$ then $S(m, n)$ is the component of $J \setminus \{m, n\}$ which contains the unique arc in J connecting m and n. Say that a point $x \in J$ is *closer* to a point $w \in J$ than a point $y \in J$ if $y \notin S(x, w)$ (this notion is specific to the closest approaches of t_1 to C, D, b taking place on I). We distinguish between two types of closest approach to b depending on which critical point is approached next (equivalently, depending on the type of the triangle which approaches $1/2$). Thus, $h_\triangle(i)$ is a closest approach to b after which t_1 will have the next closest approach to C ($h_\triangle(i)$ is the i-th *such* closest approach to b). Similarly, $h_\triangledown(i)$ is a closest approach to b after which t_1 will have the next closest approach to D ($h_\triangledown(i)$ is the i-th *such* closest approach to b).

We apply Lemma 11 to f choosing $\mathcal{T}$ appropriately. The behavior of C, D is forced by that of t_1. The three germs of J at t_1 corresponding to the arcs $(x_1, y_1), (y_1, z_1)$ and (z_1, x_1) in S^1 are denoted X, Y, Z; call their images X-germs, Y-germs, or Z-germs resp. (at t_k). The dynamics of the arcs is reflected by the behavior of the germs, and helps one see where in J images of C, D are located. We use terms "the X-germ (at t_k) points up", "the Y-germ (at t_k) points to the left" etc which are self-explanatory if $t_k \in I$. To the X-, Y-, and Z-germs at t_k correspond the components $C_X(t_k), C_Y(t_k), C_Z(t_k)$ of $J \setminus t_k$ containing the corresponding germs at t_k; the components are called the X-, Y-, Z-*components* (of J at t_k) resp.

For $t_k \in I$ the Z-germ at t_k always points to the right, so we only talk about X- and Y-germs at points $t_k \in I$. At the moment $d(i)$ the point $t_{d(i)} \in I$ is to the right of D in $S(D, t_{d(i-1)})$, its X-germ points up, and its Y-germ points to the left. Then it leaves I, and between the moments $d(i) + 1$ and $h_\triangle(i) - 1$ all its images avoid $I \cup S(D, t_{d(i)}) \cup S(C, t_{c(i-1)}) \cup S(b, t_{h_\triangledown(i-1)})$ (its images are farther away from D, C, b than the three previous closest approaches to these points). The next crucial moment is $h_\triangle(i)$ when t_1 maps into $I \cap S(b, t_{h_\triangledown(i-1)})$ (so it is the next closest approach to b), its X-germ points to the left, and its Y-germ points down. The map locally "rotates" J: the X-germ, which was pointing up, now points to the left, and the Y-germ, which was pointing to the left, now points down. Thus, D (which belongs to the Y-component at $t_{d(i)}$) maps by $f^{h_\triangle(i)-d(i)}$ inside the Y-component at $t_{h_\triangle(i)}$ ($D_{h_\triangle(i)-d(i)}$ and $t_{h_\triangle(i)}$ are very close).

For the next n_i steps t_1 stays in I while being repelled from b to the right with no "rotation" (the X-germ points to the left, the Y-germ points down). For these n_i steps the images of t_1 and D stay close while being repelled "together" from b. At the next crucial moment $c(i) = h_\triangle(i)+n_i$ the images of t_1 and D map inside $S(C, t_{c(i-1)})$ (this is the next closest approach of t_1 and to C), and the process is repeated with obvious changes.

At the moment $c(i)$ the point $t_{c(i)} \in I$ is to the right of C in $S(c, t_{c(i-1)})$, its X-germ points to the left, and its Y-germ points down. Then it leaves I, and between the moments $c(i) + 1$ and $h_\triangledown(i) - 1$ all its images avoid $I \cup S(D, t_{d(i)}) \cup S(C, t_{c(i)}) \cup S(b, t_{h_\triangle(i)})$ (its images are farther away from D, C, b than the three previous closest approaches to these points). The next crucial moment is $h_\triangledown(i)$ when t_1 maps into $I \cap S(b, t_{h_\triangle(i)})$ (so it is the next closest approach to b), its X-germ points up, and its Y-germ points to the left. The map locally "rotates" J: the X-germ, which pointed to the left, now points up, and the Y-germ, which pointed down, now points to the left.

Thus, $D_{c(i)-d(i)}$ (which belongs to the Y-component at $t_{c(i)}$) maps by $f^{h_\triangledown(i)-c(i)}$ inside the Y-component at $t_{h_\triangledown(i)}$, and all the points from the appropriate segments of the orbits of t_1 and D are very close. Now the behaviors of t_1 and D differ. In terms of t_1, for the next m_i steps $t_{h_\triangledown(i)}$ stays in I while being repelled from b to the right with no rotation (the X-germ points up, the Y-germ points to the left). At the next crucial moment $d(i+1) = h_\triangledown(i) + m_i$ the point t_1 maps inside $S(D, t_{d(i)})$ (this is the next closest approach to D), and the process for t_1 is repeated inductively (the segments of the constructed orbit repeat the same structure as the one described above). However the dynamics of D is more important.

The point $D_{h_\triangledown(i)-d(i)}$ corresponds to the point $t_{h_\triangledown(i)}$. Since by (3) $h_\triangledown(i)-d(i) = h_\triangle(i+1) - d(i+1) = q$ then D_q belongs to the Y-component at $t_{h_\triangle(i+1)}$. Now the next segment of the orbit of D begins which includes n_{i+1} steps when D is repelled away from b on I, and then $h_\triangledown(i+1) - c(i+1)$ steps when D is shadowed by the orbit of C. So, the orbit of D can be divided into countably many pairs of segments described below.

(d1) Segment D'_i from $h_\triangle(i)-d(i) = h_\triangledown(i-1)-d(i-1)$-th to $c(i)-d(i)-1$-th iteration of D of length n_i when D is repelled from b with the images $t_{h_\triangle(i)}, \dots, t_{c(i)-1}$ of t_1 so that the images of D belong to the Y-components of the appropriate images of t_1 which belong to I and stay to the left of C while the images of D are below the images of t_1.

(d2) Segment D''_i from $c(i) - d(i)$-th to $h_\bigtriangledown(i) - d(i) - 1 = h_\triangle(i+1) - d(i+1) - 1$-th iteration of D of length $h_\bigtriangledown(i) - c(i) = h_\triangle(i) - c(i-1)$ when D is closely shadowed by the orbit of C and has no closest approaches to b, C, D; $h_\bigtriangledown(i) - c(i) = h_\triangle(i) - c(i-1)$ by (4).

Since the construction is symmetric with respect to D and C, the orbit of C can be divided into countably many pairs of segments described below.

(c1) Segment C'_i from $h_\bigtriangledown(i) - c(i) = h_\triangle(i) - c(i-1)$-th to $d(i+1) - c(i) - 1$-th iteration of C of length m_i when C is repelled from b with the images $t_{h_\bigtriangledown(i)}, \dots, t_{d(i+1)-1}$ of t_1 so that the images of C belong to the X-components of the appropriate images of t_1 which belong to I and stay to the left of C while the images of C are above the images of t_1.

(c2) Segment C''_i from $d(i+1) - c(i)$-th to $h_\bigtriangledown(i+1) - c(i+1) - 1 = h_\triangle(i+1) - c(i) - 1$-th iteration of C of length $h_\triangle(i+1) - d(i+1) = h_\bigtriangledown(i) - d(i)$ when C is closely shadowed by the orbit of D and no closest approaches to b, C, D.

By **(c1)** the segment C'_i begins at $h_\bigtriangledown(i) - c(i) = h_\triangle(i) - c(i-1)$; since $c(i-1) < d(i)$ then $h_\triangle(i) - d(i) < h_\triangle(i) - c(i-1)$ and the segment C'_i begins after the segment D'_i does. By **(d1)** the segment D'_{i+1} begins at $h_\triangle(i+1) - d(i+1) = h_\bigtriangledown(i) - d(i)$; since $d(i) < c(i)$ then $h_\bigtriangledown(i) - c(i) < h_\bigtriangledown(i) - d(i)$ and the segment D'_{i+1} begins after the segment C'_i does.

The length of the segment D''_i does not depend on n_i, m_i. Indeed, the length of D''_i is $h_\bigtriangledown(i) - c(i) = h_\triangle(i) - c(i-1)$ by (4). However both $h_\triangle(i)$ and $c(i-1)$ are defined before n_i, m_i need to be defined. Likewise, the length of C''_i equals $h_\triangle(i+1) - d(i+1) = h_\bigtriangledown(i) - d(i)$, see (3). Since both $h_\bigtriangledown(i), d(i)$ are defined before m_i, n_{i+1} need to be defined, the length of the segment C''_i does not depend on m_i and n_{i+1}.

Lemma 12. *Suppose that $\mathcal{T} = n_1 < m_1 < \dots$ is such that $n_i > 9h_\triangle(i)$ and $m_i > 9h_\bigtriangledown(i)$. Then the corresponding map f is TCE.*

Proof. By Lemma 11 we need to show that there exist $\varepsilon > 0, s$ and $\tau < 1$ such that for any N and any critical point u there are more than $\tau(N+1)$ critically ε-shadowed times of order less than s in $[0, N]$ for u. Set $\tau = .4$ and $s = 2$; ε will be chosen later.

The segment D'_{i+1} begins at $h_\bigtriangledown(i) - d(i)$ while the segment C'_i ends at $m_i + (h_\bigtriangledown(i) - c(i)) - 1$; since $m_i > 9h_\bigtriangledown(i)$ then C'_i ends after D'_{i+1} begins. The segment C'_{i+1} begins at $h_\triangle(i+1) - c(i)$ while the segment D'_{i+1} ends at $n_{i+1} + (h_\bigtriangledown(i) - d(i)) - 1$; since $n_{i+1} > 9h_\triangle(i+1)$ then D'_{i+1} ends after C'_{i+1} begins. Thus, C'_i ends inside D'_{i+1}. Likewise, D'_i ends inside C'_i. All these segments form a "linked" sequence in which (1) each D'-segment begins and ends inside the appropriate consecutive C' segments,

(2) each C'-segment begins and ends inside the appropriate consecutive D'-segments, (3) $D''_i \subset C'_i$, and (4) $C''_i \subset D'_{i+1}$

The segment D'_i is at least 9 times longer than any segment $D''_q, q \le i$-segment (the length of D''_i is $h_\triangle(i) - c(i-1)$ and the length of D'_i is n_i); D'_i is also at least 9 times longer than any segment $C''_q, q < i$ since all these segments are shorter than $h_\triangle(i)$ by the construction. Similarly, the segment C'_i is at least 9 times longer than any C''-segment before it *and* the segment C''_i (the length of C''_i is $h_\bigtriangledown(i) - d(i)$ and the length of C'_i is m_i); C'_i is also at least 9 times longer than any segment $D''_q, q \le i$ since all these segments are shorter than $h_\bigtriangledown(i)$ by the construction.

It is easy to check that the construction and the choice of the constants imply the following. Let $u = C$ or $u = D$. Each D''-segment begins when the image of D is to the right of C close to C, and ends also when the image of D is to the right of C close to a preimage of b not equal to b. Each C''-segment begins when the image of C is to the right of C close to D, and ends also when the image of C is to the right of C close to a preimage of b not equal to b. Within segments D'_i and C'_i critical points are repelled from b while staying to the left of C. In the beginning of a segment the appropriate image of a critical point is close to b while on the first step after the end of a segment it maps very close to either C or D. Hence there exists $\varepsilon > 0$ such that within any segment D'_i, C'_i the images of critical points are more than 3ε-distant from the closure of the component of $J \setminus \{C\}$ located to the right of C, in particular from both critical points. Assume also that 3ε is less than the distance between any two points from the set $\{C, D, f(C), f(D)\}$. This completes the choice of constants.

Consider the critical point D and show that all times in the subsegment $E_i = [h_\triangle(i) - d(i) + n_{i-1}, c(i) - d(i) - 1]$ of $D'_i = [h_\triangle(i) - d(i), c(i) - d(i) - 1]$ are critically ε-shadowed of order at most 2. One such shadowing is trivial - the point D shadows itself. Let show that there is no more than 1 *non-trivial* shadowing for the described above times. Choose $t \in E_i$. Suppose that for some q and a critical point u the $[q, t]$-segment of orb(D) is shadowed by the $[0, t-q]$-segment of u. Then $f^q(D)$ is ε-close to u. Hence $1 \le q < h_\triangle(i) - d(i)$ by the choice of ε. Thus, u stays to the left of C for $t - [h_\triangle(i) - d(i)] + 1 > n_{i-1}$ consecutive iterations of f as it shadows $f^{h_\triangle(i)-d(i)}(D), \dots, f^t(D)$ within the $[h_\triangle(i) - d(i) - q, t-q]$-segment Q of its orbit. The segment Q begins before the segment D'_i, consists of images of u located to the left of C, and is at least $n_{i-1} + 1$ long. Hence it must be contained in a segment of one of the four listed above types of length at least $n_{i-1} + 1$. There is only one such segment, namely the C'_{i-1}-segment of the orbit of C, and so $u = C$ and $Q \subset C'_{i-1}$.

Let us show that $q = c(i-1) - d(i-1)$ coincides with the beginning of D''_{i-1}. If $q < c(i-1) - d(i-1)$ then, as the orbit of C ε-shadows the orbit of $f^q(D)$, an iteration of C from the C'_{i-1}-segment of the orbit of C will correspond to the last iteration of D in the segment D''_{i-1} which is impossible since this image of D is to the right of C and is therefore more than ε-distant from any image of C from C'_{i-1}. On the other hand, if $q > c(i-1) - d(i-1)$ then, as the orbit of C ε-shadows the orbit of $f^q(D)$, the last iteration of C in the segment C''_{i-2} of the orbit of C will correspond to an iteration of D from D'_i, a contradiction because this iteration of C is to the right of C and is therefore more than ε-distant from any image of D from D'_i. Thus, the only non-trivial critical ε-shadowing which may take place for a time $t \in E_i$ is by the orbit of C which ε-shadows the $[f^{c(i-1)-d(i-1)}, t]$-segment of the orbit of D, and so any $t \in E_i$ is critically ε-shadowed of order at most 2.

Let us estimate which part of any segment $[0, N]$ is occupied by the times which are critically ε-shadowed of order at most 2 for D. Assume that N belongs to $F_i = [h_\triangle(i) - d(i) + n_{i-1}, h_\triangle(i+1) - d(i+1) + n_i - 1]$ for some i. The segment E_i lies in the beginning of F_i and forms a significant portion of F_i. Indeed, $n_{i-1} < h_\triangle(i) < 9h_\triangle(i) < n_i$. Hence $|E_i| > \frac{8}{9}n_i$. After E_i the segment $D''_i \subset F_i$ follows, and by **(d2)** we have $|D''_i| < h_\triangle(i) < \frac{n_i}{9}$. Finally, the last part of F_i is occupied by $n_i - 1$ initial times from D'_{i+1}. Hence, $\frac{|E_i|}{|F_i|} > \frac{4}{9}$ which implies that the times which are critically ε-shadowed of order at most 2 for D form at least $\frac{4}{9}$ of the entire number of times in $[0, N]$. Similar arguments show that the times which are critically ε-shadowed of order at most 2 for C form at least $\frac{4}{9}$ of the entire number of times in $[0, N]$. By Lemma 11 this implies that f is TCE as desired. □

So far we have dealt with the dynamics of induced maps $f = f_\sim$ of laminations $\sim$. However our goal is to establish corresponding facts concerning polynomials. To "translate" our results from the language of induced maps of laminations into that of polynomials we need an important result of Kiwi [13, 14]. In Section 3 we define the family $\mathcal{Y}_d$ of collections of σ_d-critical chords whose endpoints have non-preperiodic itineraries, and the corresponding family $\mathcal{K}_d$ of laminations whose properties are described in [13, 14] (see Theorem 8 in Section 3). The following theorem is a version of results of Kiwi [13, 14] which is sufficient for our purpose.

Theorem 13. *Let $\sim$ be a lamination from $\mathcal{K}_d$; then there exists a polynomial P of degree d such that its Julia set J_P is a non-separating continuum on the plane and $P|_{J_P}$ is monotonically semiconjugate to $f_\sim|_{J_\sim}$ by a map ψ_P. Moreover, $J_\sim$ is a dendrite, ψ_P-images of critical points of P are critical* *points of $f_\sim$, ψ_P-preimages of preperiodic points of $f_\sim$ are points, and J_P is locally connected at all its preperiodic points.*

We combine Lemma 12 and Theorem 13 to prove Theorem 2.

Proof of Theorem 2. Let a sequence $\mathcal{T}$ satisfy conditions of Lemma 12. By Lemma 12 the induced map $f_\sim = f$ of the corresponding lamination $\sim$ is TCE. The lamination $\sim$ belongs to $\mathcal{W} \subset \mathcal{K}_3$, hence by Theorem 13 there is a polynomial P such that the Julia set J_P is a non-separating continuum on the plane and $P|_{J_P}$ is monotonically semiconjugate to $f|_{J_\sim}$ by a map ψ_P. Let $M \geq 0, L \geq 1, r' > 0$ be constants for which f exhibits TCE-property, i.e. such that for every $x \in J_\sim$ and every positive integer N we have

$$\frac{|\{n \in [0, N] \mid \Delta_f(x, r', n) \leq M\}|}{N+1} \geq \frac{1}{L}$$

Clearly, for some $r > 0$ and any point $z \in J_P$ we have $\psi_P(B(z, r)) \subset B(\psi_P(z), r')$. Let $z \in J_P$. To estimate the number of integers $n \in [0, N]$ with $\Delta_P(z, r, n) \leq M$, take $x = \psi_P(z)$. The number of integers $n \in [0, N]$ with $\Delta_f(x, r', n) \leq M$ is at least $(N+1)/L$. Let n be one of such numbers and estimate $\Delta_P(z, r, n)$. Observe that if $\text{Comp}_{f^i(x)} f^{-(n-i)}(B(f^n(x), r')) \cap \text{Cr}_f = \emptyset$, then $\text{Comp}_{P^i(z)} P^{-(n-i)}(B(f^n(z), r)) \cap \text{Cr}_P = \emptyset$ because ψ_P maps critical points of P to critical points of f. Hence $\Delta_P(z, r, n) \leq \Delta_f(x, r', n) \leq M$, and there are at least $(N+1)/L$ numbers $n \in [0, N]$ with $\Delta_P(z, r, n) \leq M$. So, P is TCE, and by Proposition 5.2 [19] (cf [10, 21]) it follows that the Julia set of P is Hölder and hence locally connected.

By the Carathéodory theory it means that for any sequence $\mathcal{T}$ satisfying the conditions of Lemma 12 and the corresponding lamination $\sim$ there exists a TCE-polynomial P such that J_P and $J_\sim$ are homeomorphic and $P|_{J_p}$ and $f_\sim|_{J_\sim}$ are topologically conjugate. It is easy to see that there are uncountably many sequences $\mathcal{T}$ inductively constructed so that $n_i > 9h_\triangle(i), m_i > 9h_\bigtriangledown(i)$, i.e. satisfying conditions of Lemma 12. This completes the proof of Theorem 2. □

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REFERENCES

[1] Bing, R.H.: A convex metric for a locally connected continuum, Bull. Amer. Math. Soc. **55**, 812–819 (1949)

[2] Blokh, A.: Necessary conditions for the existence of wandering triangles for cubic laminations, Disc. and Cont. Dyn. Syst., **13**, 13–34 (2005)

[3] Blokh, A., Levin, G.: An inequality for laminations, Julia sets and "growing trees", Erg. Th. and Dyn. Sys., **22**, 63–97 (2002)

[4] Blokh, A., Oversteegen, L.: Backward stability for polynomial maps with locally connected Julia sets, Trans. Amer. Math. Soc., **356** 119–133 (2004)
[5] Blokh, A., Oversteegen, L.: Wandering triangles exist, C. R. Math. (Acad. Sci., Paris), **339**, 365–370 (2004)
[6] Branner, B., Hubbard, J. H.: The iteration of cubic polynomials. Part I: The global topology of parameter space, Acta Math. **160**, 143–206 (1988)
[7] Bruin, H., Schleicher, D.: Symbolic Dynamics of Quadratic Polynomials, Institute Mittag-Leffler, Preprint #7 (2001/02)
[8] Carleson, L., Gamelin T. W.: Complex dynamics, Universitext: Tracts in Mathematics. Springer-Verlag, New York, NY (1993)
[9] Childers, D.: Wandering vertices and recurrent critical points, Erg. Th. and Dyn. Syst., to appear
[10] Graczyk, J., Smirnov, S.: Collet, Eckmann and Hölder, Invent.Math. **131** 69-96, (1998)
[11] Douady, A., Hubbard, J. H.: Étude dynamique des polynômes complexes I, II, Publications Mathématiques d'Orsay **84-02** (1984), **85-04** (1985)
[12] Kiwi, J.: Wandering orbit portraits, Trans. Amer. Math. Soc. **354**, 1473–1485 (2002)
[13] Kiwi, J.: $\mathbb{R}$eal laminations and the topological dynamics of complex polynomials, Advances in Math. **184**, 207–267 (2004)
[14] Kiwi, J.: Combinatorial continuity, Proc. London Math. Soc. **91**, 215-248 (2005)
[15] Lavaurs, P.: Systemes dynamiques holomorphes: explosion de points periodiques, Phd Thesis, Université Paris-Sud (1989)
[16] Levin, G.: On backward stability of holomorphic dynamical systems, Fund. Math. **158**, 97–107 (1998)
[17] Mayer, J. C., Mohler, L. K., Oversteegen, L. G., Tymchatyn, E. D.: Characterization of separable metric R-trees, Proc. Amer. Math. Soc. **115**, 257–264 (1992)
[18] Oversteegen, L., Rogers, J. Jr.: An inverse limit description of an atriodic tree-like continuum and an induced map without a fixed point, Houston J. of Math **6** 549–564 (1980)
[19] Przytycki, F.: Hölder implies Collet-Eckmann, Asterisque **261** 385-403 (2000)
[20] Przytycki, F., Rivera-Letelier, J., Smirnov, S.: Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps, Invent. Math. **151** 29–63 (2003)
[21] Przytycki, F., Rohde, S.: Porosity of Collet-Eckmann Julia sets, Fund.Math. **155** 189-199 (1998)
[22] Smirnov, S.: Symbolic dynamics and Collet-Eckmann conditions, Internat. Math. Res. Notices **2000**, 333–351 (2000)
[23] van Strien, S., de Melo, W.: One-dimensional dynamics. Springer-Verlag, Berlin (1993)
[24] Thurston W.: The combinatorics of iterated rational maps, Preprint (1985)

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