Lecture 1

Quantum Spin Systems

A spin ~~is~~ a quantum system w/ a fin. dim Hilbert space $\mathbb{C}^d$.

physical spin, or an approximation, or abstract qdit
if $d=2$ called qbit

Physical spin $d = 2S+1$
where $S = (0), \frac{1}{2}, 1, \frac{3}{2}, \ldots$

Observables: physically ~~literature~~: $A = A^* \in \mathcal{B}(\mathcal{H})$
$$\mathcal{B}(\mathcal{H}) = M_d(\mathbb{C})$$

For us: $\mathcal{A} = \mathcal{B}(\mathcal{H})$

e.g. $d=2$, $\mathcal{A} = M_2(\mathbb{C})$

(real) basis: $\mathbb{1}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Hilbert-Schmidt inner product $(A, B) = \mathrm{Tr}(A^* B)$

o.n. with respect to $\frac{1}{2}$ H-S inner product.

(1st exercise of the day)

States: normalized, positive linear functionals on $\mathcal{A}$

$$\omega : \mathcal{A} \to \mathbb{C}, \ \omega(A^* A) \geq 0 \ \forall A \in \mathcal{A}, \ \omega(\mathbb{1}) = 1$$

E.g: vector states $\forall \psi \in \mathcal{H}$, $\|\psi\| = 1$

$\omega(A) = \langle \psi, A\psi \rangle$ defines a state

All states of M_d : convex hull of the vector states

$$\langle \psi, A\psi \rangle = \mathrm{Tr}\,(A \underbrace{|\psi\rangle\langle\psi|}_{\text{orth. proj. onto } \psi \in \mathcal{H}})$$

$t_i \geq 0$, $\sum_i t_i = 1$

$$\sum_i t_i \langle \psi_i, A\psi_i \rangle = \mathrm{Tr}\Big(A \underbrace{\sum_i t_i |\psi_i\rangle\langle\psi_i|}_{\substack{\rho \\ \text{density matrix} \\ \rho \geq 0 \text{ and } \mathrm{Tr}(\rho) = 1}}\Big)$$

Exercise Prove that the states on M_d are in 1-1 correspondence w/ the density matrices by $\omega(A) = \mathrm{Tr}(\rho A)$.

Exercise (i) $\rho \in M_2$ is a density matrix iff $\rho = \begin{pmatrix} r & \mu \\ \bar{\mu} & 1-r \end{pmatrix}$, $r \in [0,1]$ and $\mu \in \mathbb{C}$, $|\mu|^2 \leq r(1-r)$

(ii) Same ~~hypotheses~~ cond as before iff

$$\rho = \tfrac{1}{2}(\mathbb{1} + \bar{x} \cdot \bar{\sigma}), \quad \bar{x} \in \mathbb{R}^3, \ \|x\| \leq 1$$

$$\bar{x} \cdot \bar{\sigma} = x_1 \sigma^1 + x_2 \sigma^2 + x_3 \sigma^3$$

$$x_i = \mathrm{Tr}\, \rho \sigma^i$$

Def A state ω of $\mathcal{A}$ is called pure iff ω is an extreme point of the set of all states

$\Leftrightarrow$ $\forall t \in (0,1)$, states ω_1 and ω_2

then $\omega = t\omega_1 + (1-t)\omega_2 \Rightarrow \omega_1 = \omega_2$

Exercise A state ω of M_d is pure iff the corresponding density matrix is a rank-one projection. $\exists \psi \in \mathcal{H}$, $\rho = |\psi\rangle\langle\psi|$

$d=2$ The set of all states corresponds to

$$\left(\rho = \frac{1}{2}(\mathbb{1} + \bar{x}\cdot\bar{\sigma}),\ x \in \mathbb{R}^3,\ \|x\| \leq 1\right)$$

$$\xleftrightarrow[\text{correspondence}]{1:1} \{x \in \mathbb{R}^3 : \|x\| \leq 1\} = \mathbb{B}^3 \quad \text{(Bloch sphere)}$$

o the set of pure states $\xleftrightarrow[\text{correspondence}]{1-1} \{x \in \mathbb{R}^3 : \|x\| = 1\}$

$R \in SO(3)$, $\mathrm{Tr}(\rho_R \sigma^i) = (Rx)_i$

If ρ is pure, $\rho = |\psi\rangle\langle\psi|$, $\psi \in \mathbb{C}^2$

then $\rho_R = |\psi_R\rangle\langle\psi_R|$

Question: how to go $\psi \to \psi_R$? 2 dim rep of $SO(3)$: NO!

Projective representations are okay.
I.e. rep. of SU(2) the universal cover of SO(3).

$$\mathcal{H}_1 = \mathbb{C}^{d_1}, \ \mathcal{H}_2 = \mathbb{C}^{d_2}, \ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \cong \mathbb{C}^{d_1 d_2}$$

$$\mathcal{A} = \mathbb{B}(\mathcal{H}) = \mathcal{A}_1 \otimes \mathcal{A}_2$$

* PDF for tensor product

*Not all vectors $\psi \in \mathcal{H}$ are of the form $\psi = \psi_1 \otimes \psi_2$!

States of that kind are called product states

Convex hull of $|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|$ is the separable states.
(That is the definition.)

All other states are called entangled.
(This is a definition. The defin. of entangled states:
Entangled = not separable.)

Big difference between classical & quantum systems: (5)

Marginals of a pure state are not necessarily pure.

They are exactly not pure iff the state was entangled to begin with.

Example $d_1 = d_2 = 2$ o.n. basis $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$

$\psi = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)$ is maximally entangled

Take $A \in \mathcal{A}_1$. Then $A \otimes I \in \mathcal{A}$.

Then $\mathrm{Tr}(|\psi\rangle\langle\psi| (A \otimes I))$

$= \langle\psi| (A \otimes I) |\psi\rangle$

$= \langle\psi, A \otimes I\, \psi\rangle$

$=$ no cross terms 1's ~~match~~ match & 0's must match:

$= \frac{1}{2}(\langle 0|A|0\rangle + \langle 1|A|1\rangle)$

$= \frac{1}{2}\mathrm{Tr}(A)$ ($x = 0$ right at center of B^3)