Lecture 2 (Thermodynanic limit)

States on a C^*-algebra

M_d and $B(\mathcal{H})$ for an complex Hilbert space

are examples of C^*-algebras.

(First chapter of Bratelli & Robinson is very nice.)

C^*-algebras: Banach $*$-subalgebra of $B(\mathcal{H})$ with an anti-linear involution $*$.

$$(AB)^* = B^* A^*, \quad \|AB\| \leq \|A\| \cdot \|B\|,$$
$$\|A^*\| = \|A\|,$$
$$\|A^* A\| = \|A\|^2$$

$\mathbb{1} \in \mathcal{A}$ (there exist an identity element)

$\cancel{X} = A^* A$ positive elements

for $\mathcal{A}, \mathcal{B}$ two C^*-algebras,

- a morphism π, $\pi : \mathcal{A} \to \mathcal{B}$

$$\pi(A^*) = \pi(A)^*$$
$$\pi(AB) = \pi(A)\pi(B)$$
$$\pi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$$

- a representation is a morphism $\mathcal{A} \to B(\tilde{\mathcal{H}})$ for some Hilbert space $\tilde{\mathcal{H}}$

($\tilde{\mathcal{H}}$ does not need to equal $\mathcal{H}$)

- an automorphism of $\mathcal{A}$ is an invertible morphism $\alpha : \mathcal{A} \to \mathcal{A}$

- ω is a state on $\mathcal{A}$ means ω is: linear, positive, normalized functional

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Thm: $\|\omega\| = \|\pi\| = \|\alpha\| = 1$

Rk. $|\omega(A)| \leq \|A\|$, $|\omega(A^*B)|^2 \leq \omega(A^*A)\,\omega(B^*B)$ Cauchy-Schwarz

$$|\omega(A^*BA)| \leq \omega(A^*A)\,\|B\|$$

System of N spins. all fin.-dim.

$\mathcal{H}^{(N)} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$

$\mathcal{A}^{(N)} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_N$ where $\mathcal{A}_i = \mathcal{B}(\mathcal{H}_i)$ for $i = 1, \ldots, N$.

Hamiltonian $H = H^* \in \mathcal{A}^{(N)}$

$$\begin{cases} i \frac{d}{dt} \psi(t) = H\psi(t) & \text{Schrödinger equation} \\ \psi(0) = \psi_0 \in \mathcal{H}^{(N)} \end{cases}$$

$U_t = e^{-itH}$

Solution: $\psi(t) = e^{-itH}\psi_0$ (unitary evolution on $\mathcal{H}$)

Schrödinger (Liouville equation)
for the density matrix

$$\rho_0 = |\psi_0\rangle\langle\psi_0|$$

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = \mathcal{U}_t\, \rho_0\, \mathcal{U}_t^*$$

So $\quad i \frac{d}{dt}\rho(t) = [H, \rho(t)]$

$\uparrow$ $\rho(t)$ solves this equation
(Schrödinger - (Liouville) equation)

Heisenberg picture

$$\mathrm{Tr}[\rho(t) A] = \mathrm{Tr}[\mathcal{U}_t\, \rho_0\, \mathcal{U}_t^* A]$$

$$= \mathrm{Tr}[\rho_0 \underbrace{\mathcal{U}_t^* A\, \mathcal{U}_t}]$$ by cyclicity of trace

$$= \mathrm{Tr}[\rho_0 A(t)]$$

Heisenberg equation: $\begin{cases} \frac{d}{dt} A(t) = i[H, A(t)] \\ A(0) = A \end{cases}$

Heisenberg (Quantum) Spin Chain.

$\forall x \in \mathbb{Z}$, $\mathcal{H}_x = \mathbb{C}^2$, $\Lambda = [a,b] \subseteq \mathbb{Z}$

$$\sigma^i_x = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma^i \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

$a, \quad \cdots x-1, \quad x, \quad x+1 \quad \cdots, \quad b$

$$\underset{[a,b]}{\overset{\shortparallel}{\mathcal{H}_\Lambda}} = -J \sum_{x=a}^{b-1} \vec{\sigma}_x \cdot \vec{\sigma}_{x+1}$$ Heisenberg model

$J > 0$ ferromagnetic Heis. model

$J < 0$ antiferromag Heis. model

$$\sum_{i=1}^{3} \sigma^i \otimes \sigma^i = 2t - \mathbb{1}$$ where

t is the transposition operator

$$t(\psi \otimes \varphi) = \varphi \otimes \psi \quad \forall \psi, \varphi \in \mathbb{C}^2$$

ground states $\leftrightarrow$ eigenvector of H_Λ belonging to the smallest eigenvalue

If $J > 0$, $H_{[1,2]}$ has ground state space
= the symmetric vectors
in $\mathbb{C}^2 \otimes \mathbb{C}^2$

$\{|0\rangle \otimes |0\rangle, \ |1\rangle \otimes |1\rangle, \ |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\}$

For any $a < b$,
ground states of H_Λ is the symmetric
subspace of $\mathcal{H}^{(N)}$

E.g. $|0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$

$$[h, \mathcal{U} \otimes \mathcal{U}] = 0 \quad \forall \mathcal{U} \in SU(2)$$

So can rotate $|0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$
to $|\varphi\rangle \otimes |\varphi\rangle \otimes \cdots \otimes |\varphi\rangle$
for any $\varphi \in \mathbb{C}^2$

$$\mathcal{U}_t^{(\Lambda)} = e^{-itH_\Lambda}$$

$$A(t) = e^{itH_\Lambda} A e^{-itH_\Lambda} = e^{it[H, \cdot]} A$$

$$= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} [H, [H, \ldots [H, A] \ldots]]$$

$h_{x,x+1} = \vec\sigma_x \cdot \vec\sigma_{x+1}$

E.g. $A \in \mathcal{A}_{\{0\}}$, $[H,A] = -J\{[\vec\sigma_0\cdot\vec\sigma_1, A] + [\vec\sigma_{-1}\cdot\vec\sigma_0, A]\}$

$= -J\{\underbrace{[h_{-1,0},A] + [h_{0,1},A]}_{\text{elem. of } \mathcal{A}_{\{-1,0,1\}}}\}$

$$A(t) = \underset{\mathcal{A}_{\{0\}}}{\underset{\cap}{A}} + it \underset{\mathcal{A}_{\{-1,0,1\}}}{\underset{\cap}{[H,A]}} + \frac{(it)^2}{2!} \underset{\mathcal{A}_{\{-2,-1,0,1,2\}}}{\underset{\cap}{[H,[H,A]]}} + \dots$$

etc

b/c

$[H,[H,A]]$

$= [h_{-2,-1} + h_{-1,0} + h_{0,1} + h_{1,2}, [H,A]]$

$\in \mathcal{A}_{[-2,2]}$

$B \in \mathcal{A}_{\{x\}} \in \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes \overset{x}{B} \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}$

$[A(t), B] = O(|t|^{|x|})$

Lemma Let $\mathcal{H}_1$ & $\mathcal{H}_2$ be two Hilbert spaces, $\varepsilon \geq 0$, $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ s.t.

$\| [A, \mathbb{1} \otimes B] \| \leq \varepsilon \|B\|$

Then $\exists\, A' \in \mathcal{B}(\mathcal{H}_1)$ s.t.

$\| A' \otimes \mathbb{1} - A \| \leq \varepsilon$

If $\dim(\mathcal{H}_2) < \infty$ then take $A' = \frac{1}{\dim(\mathcal{H}_1)} \mathrm{Tr}_{\mathcal{H}_2} A$

$\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, $A \in \mathcal{B}(\mathcal{H})$

$\mathrm{Tr}_{\mathcal{H}_2} A \in \mathcal{B}(\mathcal{H}_1)$ such that

$$\mathrm{Tr}\left[A(B \otimes \mathbb{1})\right] = \mathrm{Tr}_{\mathcal{H}_1}\left[(\mathrm{Tr}_{\mathcal{H}_2} A) B\right]$$