A

Infinite systems

Γ countable graph, e.g. $\Gamma = \mathbb{Z}^\nu$,
with nearest neighbor edges,

d metric on Γ, e.g.
d = graph distance, on $\mathbb{Z}^\nu$ is ℓ^1-distance

$\mathcal{H}_\Lambda$, $\mathcal{A}_\Lambda = B(\mathcal{H}_\Lambda)$, $\forall x \in \Gamma$,
$\mathcal{H}_{\{x\}} = \mathbb{C}^{d_x}$

If $\Lambda_1 \subseteq \Lambda_2$ then $A \in \mathcal{A}_{\Lambda_1}$,
$A \otimes \mathbb{1}_{\Lambda_2 \setminus \Lambda_1} \in \mathcal{A}_{\Lambda_2}$

$$\mathcal{A}_{loc} = \bigcup_{\underbrace{\Lambda \subseteq \Gamma}_{\text{finite}}} \mathcal{A}_\Lambda$$

$$\mathcal{A}_\Gamma = \overline{\mathcal{A}_{loc}}^{\|\cdot\|}$$

$\mathcal{A}_\Gamma$ = algebra of quasi-local observables

(B)

Interactions:

$$\Phi : \mathcal{P}_0(\Gamma) \to \mathcal{A}_{loc}$$

finite subsets of Γ, $\forall X \in \mathcal{P}_0(\Gamma)$, $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$

local Hamiltonians

$$H_\Lambda = \sum_{X \subseteq \Lambda} \Phi(X)$$

F-function

$F : [0,\infty) \to (0,\infty)$, non-increasing

such that

(i) $\sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x,y)) = \|F\| < \infty$

(ii) $\exists\, C$ s.t. $\forall x,y \in \Gamma$,

$$\sum_{z\in\Gamma} F(d(x,z))F(d(z,y)) \le C \cdot F(d(x,y))$$

If $\Gamma = \mathbb{Z}^\nu$, one possibility is $F(r) = (1+r)^{-\nu-\varepsilon}$

with $C = 2^{\nu+\varepsilon+1}\|F\|$

If F is an F-function then $\forall a \geq 0$, $F_a(r) = e^{-ar} F(r)$, is too.

$$\|\Phi\|_F = \sup_{x,y\in\Gamma} \frac{1}{F(d(x,y))} \sum_{\substack{X \subseteq \Gamma \\ x,y\in X}} \|\Phi(X)\| < \infty$$

(Γ, d), Φ, F, $\|\Phi\|_F < \infty$

$$\partial_\Phi(X) = \{x \in X \mid \exists Z \subseteq \Gamma \text{ s.t. } x \in Z,\ X^c \cap Z \neq \emptyset,\ \Phi(Z) \neq 0\}$$

<u>Thm</u> (Lieb-Robinson bound) for finite $\Lambda \subseteq \Gamma$

Let $X, Y \subseteq \Lambda$, $X \cap Y = \emptyset$,

$A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$.

$$\|[\tau_t^{(\Lambda)}(A), B]\| \leq C^{-1} \cdot 2 \|A\| \cdot \|B\| \left(e^{2C\|\Phi\|_F |t|} - 1\right) D(X, Y)$$

$H_\Lambda = \sum_{X \subseteq \Lambda} \Phi(X)$

$\tau_t^{(\Lambda)}(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$

where

$D(X, Y) = \min\{\sum_{x \in \partial_\Phi(X)} \sum_{y \in Y} F(d(x,y)) : x \leftrightarrow y\}$

D

E.g.

$F_a(r) = e^{-ar} F(r)$

$\min\{|\partial_\Phi(X)|, |\partial_\Phi(Y)|\}$

$\min\{|X|,|Y|\}$

thm $D(X,Y) \le |X| \cancel{|Y|}\, e^{-a\, d(X,Y)} \|F\|$

$d(X,Y) = \min\{d(x,y) : x \in X, y \in Y\}$

$\rightarrow$ Lieb-Robinson bound

$$\|[\tau_t^{(\Lambda)}(A), B]\| \le C_{xy} \|A\| \cdot \|B\| e^{-a(d(X,Y) - v|t|)}$$

$v = a^{-1} 2C \|\Phi\|_{F_a}$: Lieb-Robinson velocity

Thm $\forall A \in \mathcal{A}_{loc}$,

$$\lim_{\Lambda \uparrow \Gamma} \tau_t^{(\Lambda)}(A) = \tau_t(A) \text{ in } \|\cdot\|$$

τ_t is a strongly continuous one parameter group of automorphism of $\mathcal{A}_\Delta$

$\Rightarrow$ there exists generator $\delta = \lim_{\Lambda \uparrow \Gamma} [H_\Lambda, \cdot\,]$

closed operator s.t. $\tau_t = e^{it\delta}$. $\mathcal{A}_{loc}$ is a core for δ.

(E)

Ground states

For finite ~~volume~~ $\Lambda \subseteq \Gamma$, g.s. correspond to eigenvectors belonging to minspec(H_Λ)

$\omega_\Lambda(A) = \langle \psi_\Lambda, A\psi_\Lambda \rangle$, $A \in \mathcal{A}_\Lambda$.

$\exists$ limit points of $\{\omega_\Lambda\}_{\Lambda \subseteq \Gamma}$

Proposition: Suppose $\forall A \in \mathcal{A}_{loc}$,

- $\langle \psi_\Lambda, A\psi_\Lambda \rangle \to \omega(A)$ a state on $\mathcal{A}_\Gamma$, and ψ_Λ belongs to minspec$(\tilde{H}_\Lambda)$ for each Λ
- $\delta_\Lambda(A) = [\tilde{H}_\Lambda, A]$ converges to the same $\delta(A)$ as before, for each $A \in \mathcal{A}_{loc}$

(E.g., possibly $\tilde{H}_\Lambda = H_\Lambda + V_{\partial\Lambda}$.)

Then $\omega(A^* \delta(A)) \geq 0$ for all $A \in \mathcal{A}_{loc}$.

(F)

<u>Proof:</u>

(i) $\langle \psi_\Lambda, A^* [H_\Lambda, A] \psi_\Lambda \rangle$

$$= \langle \psi_\Lambda, A^* H_\Lambda A \psi_\Lambda \rangle - \langle \psi_\Lambda, A^* A H_\Lambda \psi_\Lambda \rangle$$

$$= \langle \Psi_\Lambda, A^* (H_\Lambda - E_\Lambda) A \Psi_\Lambda \rangle$$

$$= \langle A\Psi_\Lambda, (H_\Lambda - E_\Lambda) A\Psi_\Lambda \rangle \geq 0$$

b/c $H_\Lambda - E_\Lambda \geq 0$

$\cancel{\|A\Psi_\Lambda}$

(ii) $\forall A \;\; \exists \; \Lambda_0$ s.t.

$\|A^* \delta(A) \cancel{\|} \cancel{-} \; A^* \delta_\Lambda(A)\| \leq \varepsilon$

$\forall \Lambda \geq \Lambda_0$

$\omega(A^* \delta_\Lambda(A))$

LIMIT ARGUMENT

Infinite Systems

∞-system dynamics $\{\tau_t\}_{t\in\mathbb{R}}$

∞-system g.s.'s ω

The GNS representations for states ω on the C^*-algebra $\mathcal{A}$.

Let $\pi : \mathcal{A} \to B(\mathcal{H})$ ~~be~~ a representation

$\Omega \in \mathcal{H}$ is called a <u>cyclic vector</u> for π

if $\{\pi(A)\Omega : A \in \mathcal{A}\}$ is dense in $\mathcal{H}$.

Thm (GNS)

Let ω be a state on $\mathcal{A}$.

Then there exists a Hilbert space $\mathcal{H}_\omega$, a representation π of $\mathcal{A}$ on $B(\mathcal{H}_\omega)$ and a unit vector $\Omega_\omega \in \mathcal{H}_\omega$, cyclic for π s.t. $\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$ for all $A \in \mathcal{A}$.

(Thm GNS continued...) $\textcircled{A}$

Moreover the triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is uniquely determined by ω up to unitary equivalence.

If $(\mathcal{H}'_\omega, \pi'_\omega, \Omega'_\omega)$ is another such triple then $\exists$ unitary op. $U: \mathcal{H}_\omega \to \mathcal{H}'_\omega$

s.t. $\Omega'_\omega = U\Omega_\omega$

and $\pi'_{\omega}(A) = U\pi_\omega(A)$.

Proof by a sequence of exercises.

Outline of the proof:

(i) Construct $\mathcal{H}_\omega$

- $\mathcal{A}$ with $\langle A, B\rangle = \omega(A^* B)$.
- make it nondegenerate by taking a quotient, dividing out $I = \{A \text{ s.t. } \omega(A^* A) = 0\}$.
- complete in norm

If $B^*B \le I$ then $A^*B^*BA \le A^*A$

(ii) I is a left ideal for $\mathcal{A}$. | (iii) $\Omega = \psi_1$

Let ψ_A = equiv. class for A

Def ω is a ground state for (I)

$(\mathcal{A}, \{\tau_t\})$ if $\forall A \in \mathcal{A}_{loc}$

$$\omega(A^* \delta(A)) \geq 0$$

$$= \lim_{\Lambda \uparrow \Gamma} \omega(A^* [H_\Lambda, A])$$

If ω is time-invariant (stationary)

then $\omega \tau_t = \omega \quad \forall t \in \mathbb{R}$

$$\frac{d}{dt} \omega(\tau_t(A)) \Big|_{t=0} = i\omega(\delta(A)) = 0$$

- ω is time invariat iff

$\omega(A^* \delta(A)) \in \mathbb{R}$ for all $A \in$ ~~$\mathcal{A}_{loc}$~~ $\mathcal{A}_\Gamma$

Let ω be a g.s. of $(\mathcal{A}, \tau_t)$.

Then $\omega \circ \tau_t = \omega$ for $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ GNS triplet of ω

So $(\mathcal{H}_\omega, \pi_\omega \circ \tau_t, \Omega_\omega)$ is also a GNS triple.

$\Rightarrow \exists\, U_t : \mathcal{H}_\omega \to \mathcal{H}_\omega$ unitary s.t. $U_t \Omega = \Omega$, and $\mathcal{U}_t^* \pi_\omega(A) \mathcal{U}_t = \pi_\omega(\tau_t(A))$.

$t \mapsto \tau_t(A)$ is norm-continuous

implies

$$\|\mathcal{U}_t^* \pi(A)\Omega - \pi(A)\Omega\|$$

$$= \|\pi(\tau_t(A) - A)\Omega\|$$

$$\to 0 \quad \text{as } t \to 0, \text{ for any } A.$$

So $\mathcal{U}_t$ is strongly continuous.

So, by Stone's theorem,

$\exists$ s.a. operator H_ω s.t.

$$\mathcal{U}_t = e^{-itH_\omega}.$$

(K)

$$0 = \frac{d}{dt} \mathcal{U}_t \Omega_\omega = -i H_\omega \Omega_\omega$$

Since ω is a g.s.,

$$0 \leq \omega(A^* \delta(A)) = \frac{1}{i} \frac{d}{dt} \omega(A^* \tau_t(A))\Big|_{t=0}$$

$$= \frac{1}{i} \frac{d}{dt} \langle \Omega, \pi(A)^* \mathcal{U}_t \pi(A) \mathcal{U}_t^* \Omega \rangle \Big|_{t=0}$$

$$= \langle \Omega, \pi(A)^* [H_\omega, \pi(A)] \Omega \rangle$$

$$= \langle \Omega, \pi(A)^* H_\omega \pi(A) \Omega \rangle + 0$$

$$= \langle \pi(A) \Omega, H_\omega \pi(A) \Omega \rangle$$

so $H_\omega \geq 0$.

Thermal equilibrium states

$\forall \beta \in [0, \infty)$, $T = k_B/\beta = \beta^{-1}$ in units where $k_B = 1$

~~Define~~ $\forall \Lambda \subseteq \Gamma$, finite,

define density matrix $\rho_{\Lambda,\beta} = \dfrac{e^{-\beta H_\Lambda}}{\mathrm{Tr}(e^{-\beta H_\Lambda})}$

$\omega_{\Lambda,\beta}(A) = \mathrm{Tr}[\rho_{\Lambda,\beta} A]$ for all $A \in \mathcal{A}_\Lambda$

<u>Ex</u> (i) $\rho_{\Lambda,\beta}$ is also the unique minimizer of the free energy functional F_β

$$F_\beta(\rho) = \mathrm{Tr}\, \rho H_\Lambda + \beta^{-1} \mathrm{Tr}\, \rho \log \rho$$

(ii) $\omega_{\Lambda,\beta}$ is the unique state

s.t.
$$\omega(A^*[H_\Lambda, A]) \geq \beta^{-1} \omega(A^*A) \log \frac{\omega(A^*A)}{\omega(AA^*)}$$

for all $A \in \mathcal{A}_\Lambda$.