The Sets of ground states, $\mathcal{S}_0$ and thermal equilibrium states at inverse temperature $\beta \in [0, \infty]$, $\mathcal{S}^{(\beta)}$. ①

$(\mathcal{A}, \{\tau_t\}_{t \in \mathbb{R}})$ a QSS with a SCOG of automorphisms

on (Γ, d) $\quad$ $\tau_t = e^{it\delta}$, $\delta = \lim_{\Lambda \uparrow \Gamma} [H_\Lambda, \cdot\,]$

<u>Def^n</u> (i) a state ω on $\mathcal{A}$ is a ground state iff $\omega(A^* \delta(A)) \geq 0$, for all $A \in \mathcal{A}_{loc}$

(ii) ω is an eq. state at β iff

$$\omega(A^* \delta(A)) \geq \beta^{-1} \omega(A^* A) \log\left(\frac{\omega(A^* A)}{\omega(A A^*)}\right)$$

for all $A \in \mathcal{A}_{loc}$.

- $\mathcal{S}_0$ and $\mathcal{S}_\beta$ are convex sets
- extreme points of $\mathcal{S}_0$ are <u>pure</u> states ($(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ in the GNS triple of ω)

$\longrightarrow$ and have $\overline{\pi_\omega(\mathcal{A})}^{weak} = \mathcal{B}(\mathcal{H}_\omega)$

- the extreme points of $\mathcal{S}_\beta$ are factor states: meaning center of $\overline{\pi_\omega(\mathcal{A})}^{w} = \mathbb{C}\mathbb{1}$.

- $\mathcal{S}^{(\beta)}$ is a simplex

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local symmetries / gauge symmetries

Suppose $\forall x \in \Gamma$, you have $\mathcal{U}_x \in \mathcal{B}(\mathcal{H}_x)$ unitary

$\forall \Lambda \subseteq \Gamma$, $\alpha^{(\Lambda)}(A) = \left(\bigotimes_{x\in\Lambda} \mathcal{U}_x^*\right) A \left(\bigotimes_{x\in\Lambda} \mathcal{U}_x\right)$

for each $A \in \mathcal{A}_\Lambda$

$\alpha = \lim_{\Lambda \uparrow \Gamma} \alpha^{(\Lambda)}$ is an automorphism of $\mathcal{A}$.

If G is a group and $g \mapsto \mathcal{U}_x(g)$ is a unitary rep. of G, then this construction yields a rep $g \mapsto \alpha_g$ of G on $\mathcal{A}$.

α is a symmetry of $\{\tau_t\}_{t\in\mathbb{R}}$ if $\tau_t \circ \alpha = \alpha \circ \tau_t$, $\forall t \in \mathbb{R}$

sufficient condition is that

$$\left[H_\Lambda, \bigotimes_{x\in\Lambda} \alpha_x\right] = 0$$

for each Λ.

Space symmetries / Point symmetries

$\Gamma = \mathbb{Z}^\nu$, acts on itself as translations

$\forall x$ the mapping $y \mapsto x + y$

Assuming $\mathcal{H}_x \cong \mathbb{C}^d$ the same d, for all $x \in \mathbb{Z}^\nu$

then $\forall x \in \mathbb{Z}^\nu$ define T_x on $\mathcal{A}$ by

extension of

$$A \in \mathcal{A}_{\{y\}} \mapsto T_x(A) \in \mathcal{A}_{\{x+y\}}$$

$$\ldots \mathbb{1} \otimes \underset{\substack{\uparrow \\ y}}{\sigma^i} \otimes \mathbb{1} \longrightarrow \mathbb{1} \otimes \ldots \otimes \underset{\substack{\uparrow \\ x+y}}{\sigma^i} \otimes \ldots$$

An interaction Φ is translation invariant if

$$\Phi(X + z) = T_x(\Phi(X))$$

$$\Rightarrow \tau_t \circ T_x = T_x \circ \tau_t$$

- ~~If~~ $\forall \omega \in \mathcal{S}_0$ or $\mathcal{S}^{(\beta)}$ are α-invariant i.e. $\omega \circ \alpha = \omega$ then the symmetry is unbroken
- ~~Use it in~~ Else it is (spontaneously) broken.

(Proof on board for g.s.'s)

$$\omega \circ \alpha(A^* \delta(A)) = \omega(\alpha(A^* \delta(A)))$$
$$= \omega(\alpha(A^*)\,\alpha(\delta(A)))$$
$$= \omega(\alpha(A^*)\,\delta(\alpha(A)))$$
$$= \omega(\alpha(A)^* \delta(\alpha(A))) \geq 0.$$

General Theorem

① $\Gamma = \mathbb{Z}$ and short range Φ

~~&~~ $\mathcal{S}^{(\beta)} = \{\omega\}$, $\exists!$ equilibrium state

$\forall \beta \in [0, \infty)$. Araki (1968)

② $\underbrace{\text{Mermin-Wagner}}_{1966} - \underbrace{\text{Hohenberg}}_{1967}$

Frohlich & Pfister 1981

Continuous symmetries are unbroken for $\beta < \infty$, $d \leq 2$

and not-too-long-range interaction.

③ The Goldstone-Theorem (Landau-Perez-Wreszinski JSP 1981)

For translation invariant models on $\mathbb{Z}^\nu$,

ground states: continuous symmetry breaking implies gapless excitation spectrum

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excitation spectrum $= \mathrm{spec}(H_\omega) \setminus \{0\}$

"spectral gap" above the g.s.

$$\gamma = \sup\{\delta > 0 \mid (0,\delta) \cap \mathrm{spec}(H_\omega) = \emptyset\}$$

$(\gamma = 0$ if $\{\ \} = \emptyset)$

Equilibrium states continuous symmetry breaking $\Rightarrow$ slow decay of correlations

④ Exponential Clustering Theorem (2006 Hastings & Koma)

If $\mathcal{S}_0 = \{\omega\}$, Φ has exponential decay and $\gamma > 0$, then all correlations in ω decay exponentially in space.

Frustration free spin chains

$\Gamma = [1,\infty) \subseteq \mathbb{Z}$, $\mathcal{H}_x \cong \mathbb{C}^d$ for all $x \in \Gamma$

Finite range interactions $\Phi([x,x+R]) \in \mathcal{A}_{[x,x+R]}$

$R=1$: nearest neighbor interactions

call this h_x

$$H_{[1,N]} = \sum_{x=1}^{N-R} h_x$$

WLOG assume : $\inf \operatorname{spec}(h_x) = 0$

Frustration free = FF

FF $\inf \operatorname{spec}(H_{[1,N]}) = 0$ for all $N \geq R$

$\iff \ker(H_{[1,N]}) \neq \{0\} \quad \forall N \geq R+1$

$\mathcal{G}_x = \ker(h_x)$

$$\iff \bigcap_x (\mathbb{C}^d)^{\otimes x-1} \otimes \underset{\underset{\mathcal{H}_{[x,x+R]}}{\wedge}}{\mathcal{G}_x} \otimes (\mathbb{C}^d)^{\otimes N-R-x-1} \neq \{0\}$$

It has dimension > 0.

Let ω_N be a ground state of $H_{[1,N]}$

$\Rightarrow$ any weak limit point of $\{\omega_N\}$,

ω will be a g.s. of ~~the~~ $\mathcal{A}_{[1,\infty)}$,

and $\omega(h_x) = 0$, for each $x \in \Gamma$.

Cuntz algebra $\mathcal{O}(d)$ Is the unique

C^*-algebra with $\mathbb{1}, S_1, \ldots, S_d,$
$S_1^*, \ldots, S_d^*,$

$$\sum_{\alpha=1}^{d} S_\alpha^* S_\alpha = \mathbb{1}, \quad S_\alpha S_\beta^* = \delta_{\alpha\beta} \mathbb{1} \text{ for } \alpha, \beta \in \{1, \ldots d\}$$

$\sigma : \mathcal{A}_{[1,\infty)} \to \mathcal{O}(d)$

$\{ |1\rangle, \ldots, |d\rangle \}$ o.n. basis of $\mathbb{C}^d$

$$\sigma(|\alpha_1 \cdots \alpha_n\rangle\langle \beta_1, \ldots \beta_n|) = S_{\alpha_1}^* \cdots S_{\alpha_n}^* S_{\beta_n} \cdots S_{\beta_1}$$

$\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$

$S_{\underline{\alpha}} = S_{\alpha_n} \cdots S_{\alpha_1}$ $\qquad \sigma(|\underline{\alpha}\rangle\langle\underline{\beta}|) = S_{\underline{\alpha}}^* S_{\underline{\beta}}$

$$|\alpha\rangle\langle\beta| \cdot |\gamma\rangle\langle\delta| = \delta_{\beta\gamma}\, |\gamma\rangle\langle\delta|$$

$$S_\alpha^* S_\beta S_\gamma^* S_\delta = \delta_{\beta\gamma}\, S_\alpha^* S_\gamma$$

$$\sum_\alpha |\alpha\rangle\langle\alpha| = \mathbb{1}$$

Let ω be a pure zero-energy ground stat i.e. $\omega(h_x) = 0$, $\forall x \in [1,\infty)$

Define a state $\tilde{\omega}$ on $\mathcal{O}(d)$

$$\tilde{\omega}(S_{\underline{\alpha}}^* S_{\underline{\beta}}) = \omega(|\underline{\alpha}\rangle\langle\underline{\beta}| \otimes \mathbb{1} \cdots)$$

$$\tilde{\omega}(\prod_i S_i^\#) = 0 \quad \text{if } \# S^* \neq \# S\text{'s}$$

$\{|1\rangle, \dots, |d\rangle\}$ is o.n. basis of $\mathbb{C}^d$

<u>Thm 1</u> For any state ω on $\mathcal{A}_{[1,\infty)}$, $\exists$ a Hilbert space $\mathcal{K}$, a unit $\Omega \in \mathcal{K}$ and $V_1, \dots, V_d \in \mathcal{B}(\mathcal{K})$ s.t.

(i) $\sum_{\alpha=1}^{d} V_\alpha^* V_\alpha = \mathbb{1}$, (ii) $\mathcal{K} = \overline{\mathrm{span}\{V_{\alpha_n} \cdots V_{\alpha_1}\Omega \mid \alpha_1, \dots, \alpha_n\}}^{\,n \geq 0}$

(iii) $\omega(|\underline{\alpha}\rangle\langle\underline{\beta}|) = \langle V_{\underline{\alpha}}\Omega | V_{\underline{\beta}}\Omega\rangle$

$\mathcal{K} \subseteq \mathcal{H}_{\bar{\omega}}$, $V_\alpha = \pi_\omega(S_\alpha)|_{\mathcal{K}}$

$$\mathcal{K} = \overline{\{\pi(S_{\underline{\alpha}})\Omega : n \geq 0, \alpha_1, \ldots, \alpha_n\}} \subseteq \mathcal{H}_{\bar{\omega}}$$

Thm 2. Let ω be a pure zero-energy g.s. on $\mathcal{A}_{[1,\infty)}$

with the $\mathcal{H}$ and $\{V_\alpha\}_{\alpha=1,\ldots,d}$ of Thm 1,

iff (i) $\forall \psi \in \mathcal{G}_x^\perp$, $\psi = \sum_{\alpha_0 \cdots \alpha_k} \psi_{\alpha_0 \cdots \alpha_k} |\alpha_0 \cdots \alpha_k\rangle$ ($\psi_{\alpha_0 \cdots \alpha_k}$ coeff.'s)

we have the condition

$$\sum_{\alpha_0 \cdots \alpha_k} \overline{\psi}_{\alpha_0 \cdots \alpha_k} V_{\alpha_0} \cdots V_{\alpha_k} = 0$$

(ii) for any $X \in \mathcal{B}(\mathcal{K})$, $\sum_\alpha V_\alpha^* X V_\alpha = X$ implies $X = \lambda \mathbb{1}$, $\lambda \in \mathbb{C}$.

Proof $\forall \psi \in \mathcal{G}_x^{\perp}$,

$$0 = \omega\left(\mathbb{1}_{[1,\dots,x-1]} \otimes |\psi\rangle\langle\psi| \otimes \mathbb{1}_{[x+R+1,\infty)}\right)$$

$$0 = \sum_{\underline{\gamma},\underline{\alpha},\underline{\beta}} \psi_{\underline{\alpha}} \overline{\psi_{\underline{\beta}}}\, \omega\left(|\underline{\gamma},\underline{\alpha}\rangle\langle\underline{\gamma},\underline{\beta}|\right)$$

$x-1$ indices

R indices each

$$0 = \sum_{\underline{\gamma}} \left\langle V_{\underline{\gamma}} \Omega, \left(\sum_{\underline{\alpha}} \psi_{\underline{\alpha}} V_{\underline{\alpha}}^{*}\right)\left(\sum_{\underline{\beta}} \overline{\psi_{\underline{\beta}}} V_{\underline{\beta}}\right) V_{\underline{\gamma}} \Omega \right\rangle$$

$$= \sum_{\underline{\gamma}} \left\langle V_{\underline{\gamma}} \Omega, \left(\sum_{\underline{\alpha}} \overline{\psi_{\underline{\alpha}}} V_{\underline{\alpha}}\right)^{*}\left(\sum_{\underline{\beta}} \overline{\psi_{\underline{\beta}}} V_{\underline{\beta}}\right) V_{\underline{\gamma}} \Omega \right\rangle$$

$$\Rightarrow \left(\sum_{\underline{\alpha}} \overline{\psi_{\underline{\alpha}}} V_{\underline{\alpha}}\right)^{*}\left(\sum_{\underline{\beta}} \overline{\psi_{\underline{\beta}}} V_{\underline{\beta}}\right) = 0$$

$$\Rightarrow \sum_{\underline{\beta}} \overline{\psi_{\underline{\beta}}} V_{\underline{\beta}} = 0$$ so (i) is established

(Note all arrow can be reversed for iff not just only if)

To prove (ii), Define $\hat{E}(X) = \sum_{\alpha=1}^{d} V_\alpha^* X V_\alpha$

Suppose $X \neq \lambda \mathbb{1}$ for any $\lambda \in \mathbb{C}$ and $\hat{E}(X) = X$

$$\Rightarrow \hat{E}(X^*) = X^* \text{ and } \hat{E}(\mathbb{1}) = \mathbb{1}$$

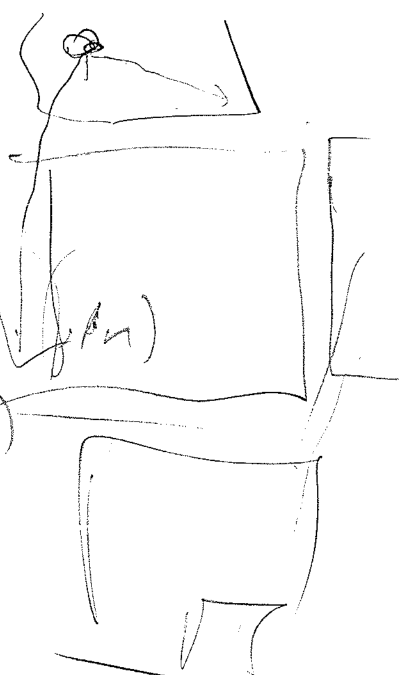
$$\exists\, Y = c_1 X + c_2 X^* + c_3 \mathbb{1} \text{ s.t. } 0 \le Y \le \mathbb{1}$$

and $Y \neq \lambda \mathbb{1}$

Make a new state ω_Y:

$$\omega_Y(|\underline{\alpha}\rangle\langle\underline{\beta}|) = \langle V_\alpha \Omega, Y V_\beta \Omega \rangle$$

Can also make $\omega_{\mathbb{1}-Y}$.



Compatibility means

$$\forall\, \underline{\alpha} = (\alpha_1, \dots, \alpha_n), \ \forall\, \underline{\beta} = (\beta_1, \dots, \beta_n)$$

$$\sum_\gamma \omega_Y(|\underline{\alpha}\gamma\rangle\langle\underline{\beta}\gamma|) = \omega(|\underline{\alpha}\rangle\langle\underline{\beta}|)$$