(A)

Frustration-free Chains

Matrix product states

Finitely Correlated states

$0 \leq h \in \mathcal{B}(\underbrace{\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d}_{R+1})$, $\Gamma = [1,\infty) \subseteq \mathbb{Z}$

$R \geq 1$, $h_x \in \mathcal{A}_{[x,x+R]}$, $H_{[1,N]} = \sum_{x=1}^{N-R} h_x$

$\mathcal{G} = \ker(h) \subseteq \underbrace{\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d}_{R+1}$

$\{|1\rangle, \ldots, |d\rangle\}$ o.n. basis of $\mathbb{C}^d$

Thm: Let ω on $\mathcal{A}_{[1,\infty)}$ be a pure zero-energy state for h ($\omega(h_x) = 0$ for all x)

Then $\exists$ Hilbert space $\mathcal{K}$ and $V_1, \ldots, V_d \in \mathcal{B}(\mathcal{K})$ and $\Omega \in \mathcal{K}$ s.t.

(i) $\omega(|\alpha_1 \cdots \alpha_n\rangle\langle\beta_1 \cdots \beta_n| \otimes \mathbb{1}) = \langle V_{\alpha_n} \cdots V_{\alpha_1} \Omega, V_{\beta_n} \cdots V_{\beta_1} \Omega\rangle$

(ii) $\forall \psi \in \mathcal{G}^\perp$, $\psi = \sum_{\alpha_0 \cdots \alpha_R} \psi_{\alpha_0 \cdots \alpha_R} |\alpha_0 \ldots \alpha_R\rangle$ ← coeff.'s

it happens $\sum_{\alpha_0 \cdots \alpha_R} \overline{\psi_{\alpha_0 \cdots \alpha_R}} V_{\alpha_R} \cdots V_{\alpha_1} = 0$

(iii) With $\hat{\mathbb{E}}(X) = \sum_{\alpha=1}^{d} V_\alpha^* X V_\alpha$, $\mathbb{1}$ is the only eigenvector w/ eigenvalue 1 of $\hat{\mathbb{E}}$

From now on, assume $\dim(\mathcal{K}) = k < \infty$ (B)

$\Rightarrow V_1, \dots, V_d \in M_k$

$$\omega(|\alpha_1 \cdots \alpha_n\rangle\langle\beta_1 \cdots \beta_n| \otimes \cdots) = \langle \Omega, V_{\alpha_1}^* \cdots V_{\alpha_n}^* V_{\beta_n} \cdots V_{\beta_1} \Omega\rangle$$

$$= \mathrm{Tr}\left[|\Omega\rangle\langle\Omega| \cdot (V_{\alpha_1}^* \cdots V_{\alpha_n}^* V_{\beta_n} \cdots V_{\beta_1})\right]$$

$$\omega(\mathbb{1} \otimes |\alpha_1 \cdots \alpha_n\rangle\langle\beta_1 \cdots \beta_n| \otimes \mathbb{1} \cdots) = \mathrm{Tr}\Big[|\Omega\rangle\langle\Omega| \underbrace{\sum_{\gamma=1}^{d} V_\gamma^* (V_{\alpha_1}^* \cdots V_{\alpha_n}^* V_{\beta_n} \cdots V_{\beta_1}) V_\gamma}_{\hat{\mathbb{E}}(V_{\alpha_1}^* \cdots V_{\alpha_n}^* V_{\beta_n} \cdots V_{\beta_1})}\Big]$$

$$\mathbb{1} = \sum_{\gamma=1}^{d} |\gamma\rangle\langle\gamma|$$

Define $\hat{\mathbb{E}}^t$ by $\mathrm{Tr}[\rho\, \hat{\mathbb{E}}(B)] = \mathrm{Tr}[\hat{\mathbb{E}}^t(\rho) B]$ for all $\rho, B \in M_k$

(transpose) $\hat{\mathbb{E}}^t(\rho) = \sum_\gamma V_\gamma \rho V_\gamma^*$

So $\omega(\mathbb{1} \otimes |\alpha_1 \cdots \alpha_n\rangle\langle\beta_1 \cdots \beta_n| \otimes \mathbb{1} \otimes \cdots)$

$$= \mathrm{Tr}\left(\hat{\mathbb{E}}^t(|\Omega\rangle\langle\Omega|) V_{\alpha_1}^* \cdots V_{\alpha_n}^* V_{\beta_n} \cdots V_{\beta_1}\right)$$

P $\Rightarrow \exists!\, \rho$, density matrix such that

$\hat{\mathbb{E}}^t(\rho) = \rho$ (by Perron-Frobenius extended to completely positive maps)

Now define a translation invariant state of ~~the algebra for the whole chain~~ $\mathcal{A}_{\mathbb{Z}}$ by

$$\omega_\rho(|\alpha_1 \cdots \alpha_n\rangle\langle\beta_1 \cdots \beta_n|) = \mathrm{Tr}\left[\rho\, (V_{\alpha_1}^* \cdots V_{\alpha_n}^* V_{\beta_n} \cdots V_{\beta_1})\right]$$

is also the zero energy state for h.

Define $\mathbb{E}_{|\alpha\rangle\langle\beta|}(B) = V_\alpha^* B V_\beta$

so $\quad \hat{\mathbb{E}} = \sum_\alpha \mathbb{E}_{|\alpha\rangle\langle\alpha|}$

$$\mathbb{E}_A = \sum_{\alpha,\beta} \langle\alpha|A|\beta\rangle \, \mathbb{E}_{|\alpha\rangle\langle\beta|}$$

$$\omega_\rho\left(|\alpha_1 \cdots \alpha_n\rangle\langle\beta_1 \cdots \beta_n|\right) = \mathrm{Tr}\left[\rho \cdot \mathbb{E}_{|\alpha_1\rangle\langle\beta_1|} \circ \cdots \mathbb{E}_{|\alpha_n\rangle\langle\beta_n|}(\mathbb{1})\right]$$

$$\omega_\rho(A_1 \otimes \cdots \otimes A_n) = \mathrm{Tr}\left[\rho \cdot \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n}(\mathbb{1})\right]$$

For $A \in M_d$, $B \in M_k$

$$\mathbb{E}_A(B) = \mathbb{E}\left[(A \otimes B) V^* (A \otimes B)\right]$$

$$V = \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} \quad dk \times k \text{ matrix} \quad V^* V = \mathbb{1}$$

Affleck - Kennedy - Lièb - Tasaki (1987-88)

<u>Example</u>: (AKLT model)

Spin 1 chain, $d=3$, $k=2$

Pick V the intertwiner of $D^{(1/2)}$ and $D^{(1)} \otimes D^{(1/2)}$
$\cong D^{(3/2)} \oplus D^{(1/2)}$

$$V : \mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^2$$

$\forall g \in SU(2)$, $\quad V D^{(1/2)}(g) = D^{(1)}(g) \otimes D^{(1/2)}(g) V$

(D)

$$V = \begin{bmatrix} 0 & 0 \\ \sqrt{2/3} & 0 \\ \hline -\sqrt{1/3} & 0 \\ 0 & \sqrt{1/3} \\ \hline 0 & -\sqrt{2/3} \\ 0 & 0 \end{bmatrix} \begin{matrix} = v_0 \\ \\ = v_1 \\ \\ = v_{-1} \end{matrix}$$

Call these V_1, V_0, V_{-1}

$R = 1$ (nearest-neighbor interaction)

$h = P^{(2)} \in \mathcal{B}(\mathbb{C}^3 \otimes \mathbb{C}^3)$

$D^{(1)} \otimes D^{(1)} \cong D^{(0)} \oplus D^{(1)} \oplus D^{(2)}$

$\mathcal{G} = \ker(h) = $ spin 0 + spin 1 vectors in $D^{(1)} \otimes D^{(1)}$

verify $\hat{E}(B) = V^*(\mathbb{1} \otimes B) V = \left(\frac{1}{2} \operatorname{Tr} B\right)\mathbb{1}$

$$- \frac{1}{3}\left(B - \left(\frac{1}{2}\operatorname{Tr} B\right)\mathbb{1}\right)$$

eigenvalues $1\ (1)$, $-\frac{1}{3}\ (3)$.

(and no other eigenvalues of abs. val $= 1$: "$\hat{E}$ has trivial peripheral spectrum" TPS

<u>Thm</u> ω is pure iff $\hat{E}$ has TPS.

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restriction of ω to $\mathcal{A}_{[1,n]} \to$ density matrix $\rho^{(n)}$

$$\langle \beta_1 \cdots \beta_n | \rho^{(n)} | \alpha_1 \cdots \alpha_n \rangle = \operatorname{Tr}\left[\rho^{(n)} |\alpha_1 \cdots \alpha_n\rangle\langle \beta_1 \cdots \beta_n|\right]$$

$$= \omega(|\alpha_1 \cdots \alpha_n\rangle\langle \beta_1 \cdots \beta_n|)$$

$$= \operatorname{Tr}\left[\rho V_{\alpha_1}^* \cdots V_{\alpha_n}^* V_{\beta_n} \cdots V_{\beta_1}\right]$$

$\forall \varphi \in (\mathbb{C}^d)^{\otimes n}$, $\varphi = \sum_{\alpha_1 \cdots \alpha_n} \varphi_{\alpha_1 \ldots \alpha_n} |\alpha_1 \cdots \alpha_n\rangle$

$$\rho^{(n)} \varphi = \sum_{\substack{\alpha_1 \cdots \alpha_n \\ \beta_1 \cdots \beta_n}} \rho^{(n)}_{\beta_1 \cdots \beta_n, \alpha_1 \cdots \alpha_n} \varphi_{\alpha_1 \ldots \alpha_n} |\beta_1 \cdots \beta_n\rangle$$

E

$$= \sum_{\beta_1 \cdots \beta_n} \left(\mathrm{Tr} \left(\underbrace{\left[\sum_{\alpha_1 \cdots \alpha_n} \rho V_{\alpha_1}^* \cdots V_{\alpha_n}^* \varphi_{\alpha_1 \cdots \alpha_n} \right.}_{A(\varphi) \in \mathcal{B}(\mathcal{K}) = M_k} \right] V_{\beta_n} \cdots V_{\beta_1} \right) |\beta_1 \cdots \beta_n\rangle$$

$$\Gamma_n : \mathcal{B}(\mathcal{K}) \to (\mathbb{C}^d)^{\otimes n}$$

$$\Gamma_n(A) = \sum_{\beta_1 \cdots \beta_n} \left(\mathrm{Tr}\left[A V_{\beta_n} \cdots V_{\beta_1} \right] |\beta_1 \cdots \beta_n\rangle \right)$$

Then $\mathrm{ran}(\rho^{(n)}) \subseteq \mathrm{ran}(\Gamma_n)$

One can show that ~~there~~ $\exists n_0$ s.t. $\forall n \geq n_0$,

$\mathrm{ran}(\rho^{(n)}) = \mathrm{ran}(\Gamma_n)$.

F

$$\underbrace{H_{[a,b]}}_{\Lambda} = \sum_{x=a}^{b-R} h_x$$

G_Λ = orth. proj. onto $\mathcal{G}_\Lambda$

where $\mathcal{G}_\Lambda = \ker(H_\Lambda)$

$\ell \geq R-1$, $G_\Lambda \in \mathcal{B}(\mathcal{H}_{[1,N]})$ (for N arbitrary, large #)

assuming $\gamma_{\ell+2} > 0$ strict

Assumption 1

$$H_{[n-\ell, n+1]} \geq \gamma_{\ell+2}\left(\mathbb{1} - G_{[n-\ell,n+1]}\right)$$

Assumption 2

$$\sum_{n=\ell+1}^{N-1} H_{[n-\ell,n+1]} \leq d_\ell H_{[1,N]}$$

(n.n. interactions $d_\ell = \ell+1$)

- if $\Lambda_1 \subseteq \Lambda_2$, then $G_{\Lambda_2} G_{\Lambda_1} = G_{\Lambda_1} G_{\Lambda_2} = G_{\Lambda_2}$
- if $\Lambda_1 \cap \Lambda_2 = \emptyset$ then $G_{\Lambda_1} G_{\Lambda_2} = G_{\Lambda_2} G_{\Lambda_1}$

for $\ell+1 \leq n \leq N$, $E_n = \begin{cases} \mathbb{1} - G_{[1,\ell+2]}, & n = \ell+1 \\ G_{[1,n]} - G_{[1,n+1]}, & \ell+2 \leq n \leq N-1 \\ G_{[1,N]}, & n = N. \end{cases}$

$$E_n^* = E_n, \quad E_n E_m = \delta_{n,m} E_m, \quad \sum_{n=\ell+1}^{N} E_n = \mathbb{1}$$

Assumption 3 $\exists \varepsilon_\ell > 0$, $\varepsilon_\ell < \frac{1}{\sqrt{\ell+2}}$

s.t. $\forall n \geq \ell+1$,

$\| G_{[n-\ell, n+1]} E_n \| \leq \varepsilon_\ell$

Thm Under the 3 assumptions we have

$$\text{gap}(H_{[1,N]}) \geq \frac{1}{2} \cdot \frac{\gamma_{\ell+2}}{d_\ell} \left(1 - \sqrt{\ell+2}\, \varepsilon_\ell\right)^2$$

Proof. $\psi \perp \mathcal{G}_{[1,N]}$ so $E_N \psi = 0$

need to lower bound $\langle \psi, H_{[1,N]} \psi \rangle$
want $\geq \gamma \|\psi\|^2$

So $\|\psi\|^2 = \sum_{n=\ell+1}^{N-1} \|E_n \psi\|^2$ (only up to $N-1$ b/c $E_N \psi = 0$)

$$\|E_n \psi\|^2 = \langle \psi | (\mathbb{1} - G_{[n-\ell, n+1]}) E_n \psi \rangle + \langle \psi | \underbrace{\left(\sum_{m=\ell+1}^{N-1} E_m \right)}_{\uparrow \text{insert identity}} G_{[n-\ell, n+1]} E_n \psi \rangle$$

don't need N b/c $\langle \psi | E_N = (E_N \psi)^* = 0^* = 0$

$[E_m, G_{[n-\ell, n+1]}] = 0$ if $m \leq n-\ell-2$ or $m \geq n+1$

and $E_m E_n = 0$ for $m \neq n$

(H)

$$= \langle \psi | (\mathbb{1} - G_{[n-\ell, n+1]}) E_n \psi \rangle$$

$$+ \langle \sum_{m=n-\ell-1}^{n} E_m \psi, G_{[n-\ell, n+1]} E_n \psi \rangle$$

use $|\langle \varphi_1, \varphi_2 \rangle| \le \frac{1}{2c} \|\varphi_1\|^2 + \frac{c}{2} \|\varphi_2\|^2$

$$\le \frac{1}{2c_1} \langle \psi | (\mathbb{1} - G_{[n-\ell, n+1]}) \psi \rangle + \frac{c_1}{2} \langle \psi, E_n \psi \rangle$$

$$+ \frac{1}{2c_2} \langle \psi | E_n G_{[n-\ell, n+1]} E_n \psi \rangle + \frac{c_2}{2} \langle \psi | \sum_{m=n-\ell}^{n} E_m \psi \rangle$$

Assumption 3

$$\boxed{\|G_{[n-\ell, n+1]} E_n\| \le \varepsilon_\ell \iff E_n G_{[n-\ell, n+1]} E_n \le \varepsilon_\ell^2 E_n}$$

So $\sum_{n=\ell+1}^{N-1} \|E_n \psi\|^2 = \|\psi\|^2$

and

$$\wedge\!\!|$$

$$\frac{1}{2c_1} \cdot \frac{d_\ell}{\gamma_{\ell+2}} \langle \psi, H_{[1,N]} \psi \rangle$$

$$+ \frac{1}{2c_2} \cdot \varepsilon_\ell^2 \|\psi\|^2 + \frac{c_1}{2} \|\psi\|^2 + \frac{c_2}{2} (\ell+1) \|\psi\|^2$$

So

$$2c_1 \|\psi\|^2 - 2c_1 \left(\frac{1}{2c_2} \varepsilon_\ell^2 \|\psi\|^2 + \frac{c_1}{2} \|\psi\|^2 + \frac{c_2}{2} (\ell+1) \|\psi\|^2 \right)$$

$$\le \frac{d_\ell}{\gamma_{\ell+2}} \langle \psi, H_{[1,N]} \psi \rangle \qquad c_2 = \frac{\varepsilon}{\sqrt{\ell+2}}, \; c_1 = 1 - \sqrt{\ell+2}\,\varepsilon_\ell$$

For any $A \in M_k \in \mathcal{B}(\mathcal{K})$

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$$\Gamma_n(A) = \sum_{\beta_1 \cdots \beta_n} \mathrm{Tr}\left[A\, V_{\beta_1} \cdots V_{\beta_n}\right] |\beta_1 \cdots \beta_n\rangle$$

Called Matrix Product vector (MPS)

Need to check

$$\ker(H_\Lambda) = \mathcal{G}_\Lambda = \mathrm{ran}(\rho_\Lambda)$$

$\hat{\mathbb{E}}$ has TPS

So $(\hat{\mathbb{E}})^n \to |\mathbb{1}\rangle\langle\rho|$

$\exists\, c > 0,\ \lambda \in [0,1]$ s.t.

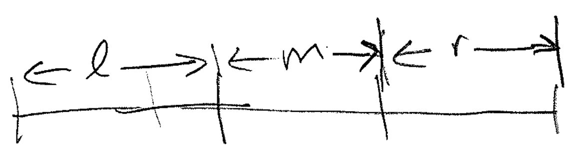
$$\underbrace{\|\hat{\mathbb{E}}^n - |\mathbb{1}\rangle\langle\rho|\,\|}_{\text{Call this } a(n)} \le c\lambda^n$$

$\mathcal{G}_n = \mathrm{ran}(\rho_n)$

G_n = orth. proj. onto $\mathcal{G}_n$

$$b(m) = \frac{k}{\rho_{min}}\, a(m)$$

ρ_{min} = smallest pos. eigenvalue of ρ



Thm:

$\forall \varphi, \psi \in \mathcal{G}^{\perp}_{l+m+r}$, $\varphi \in \mathcal{G}_{l+m} \otimes (\mathbb{C}^d)^{\otimes r}$

$\psi \in (\mathbb{C}^d)^{\otimes l} \otimes \mathcal{G}_{m+r}$

$$|\langle \varphi, \psi\rangle| \le \frac{b(m) + b(m)^2}{1 - b(m)} \|\varphi\| \cdot \|\psi\|$$

J

Exercise

Lemma $\forall\ B, C \in M_k$

$$\left| \langle \Gamma_n(B), \Gamma_n(C) \rangle - \langle B, C \rangle_\rho \right| \leq b(n) \|B\|_\rho \|C\|_\rho$$

$\langle B, C \rangle_\rho = \mathrm{Tr}[\rho B^* C]$, $\|B\|_\rho = \langle B, B \rangle_\rho$

WLOG can assume $\rho > 0$.

Quasi-Adiabatic Evolution (aka "spectral flow")

$$D(s) = \int_{-\infty}^{\infty} dt\, \omega_\gamma(t) \int_0^t e^{iuH(s)}\, \partial_s H(s)\, e^{-iuH(s)}\, du$$

K

5th Ave
&
19th Str.

Family of local, gapped Hamiltonians $\{H_1(s)\}_{s=0}^{1}$

gap $\gamma_1(s) \geq \gamma_1 > 0$.

$P_0(s) \longleftarrow$ g.s. subspace

$P_0(0) \longrightarrow P_0(s) = U(s)\, P_0(0)\, U^*(s)$

We have $H(s)$ with $E_n(s) - E_0(s) \geq \gamma(s) \geq \gamma > 0$

e-values of $H(s)$

$$(H(s) - E_0(s))P(s) = 0$$

So $\frac{d}{ds}\{(H(s) - E_0(s))P(s)\} = 0$

$$\Rightarrow \frac{d}{ds}\{H(s) - E_0(s)\} \cdot P_0(s) = -(H(s) - E_0(s))\, \partial_s P_0(s)$$

$$(1 - P_0(s))\frac{d}{ds}P_0(s) = -(H(s) - E_0(s))^{-1}\left(\frac{d}{ds}H(s)\right)P_0(s)$$