Automorphic equivalence

$$H_\Lambda(s) = \sum_{X \subset \Lambda} \Phi(X, s)$$

- Quasi-adiabatic evolution (Hastings)

Spectral flow (Bachmann, et. al.)

sa $H_0^* = H_0$, $\mathrm{Dom}(H_0) \subseteq \mathcal{H}$

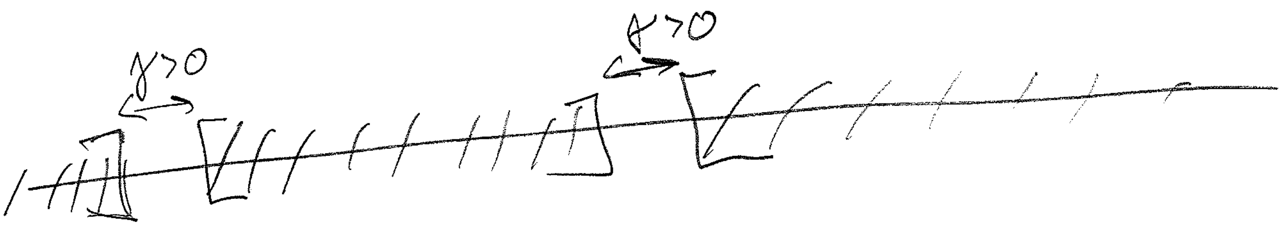
$H(s) = H_0 + \Phi(s)$, $s \in [0,1]$,

$\Phi(s) \in \mathcal{B}(\mathcal{H})$

$\frac{d}{ds}\Phi(s) = \Phi'(s) \in \mathcal{B}(\mathcal{H})$

$\sup_s \|\Phi'(s)\| < \infty$

$\mathrm{spec}(H(s)) = \Sigma(s)$



$$\Sigma(s) = \Sigma_1(s) \cup \Sigma_2(s) \cup \Sigma_3(s)$$

$P(s)$ = spectral projection of $H(s)$ corresponding to $\Sigma_1(s)$

$$P(s) = U(s)\, P(0)\, U(s)^*$$

a good $U(s)$ with this property is the solution of

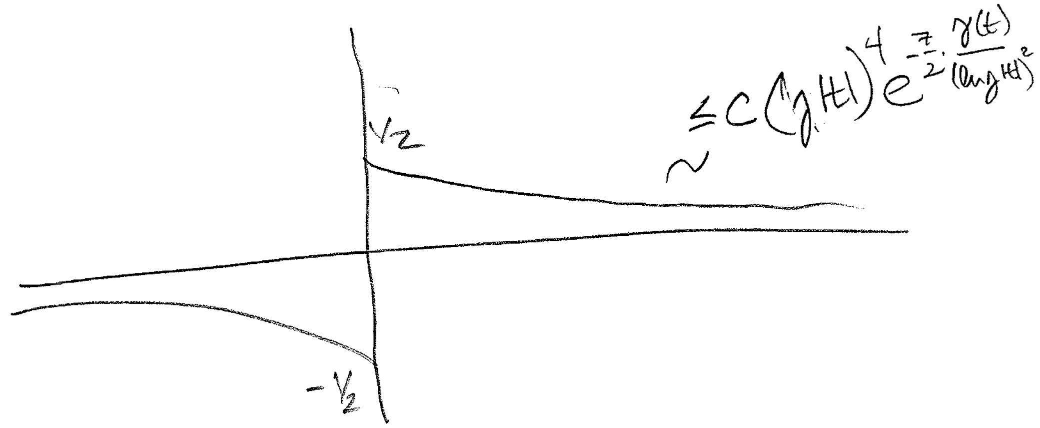
$$\frac{d}{ds} U(s) = i\, D(s)\, U(s), \quad U(0) = \mathbb{1}$$

$$D(s) = D(s)^* = \int_{-\infty}^{\infty} dt \; W_\gamma(t)\, e^{itH(s)}\, \Phi'(s)\, e^{-itH(s)}$$

$W_\gamma(t) \in L^1$

$\nearrow$

$\|W_\gamma\|_1 = \frac{k}{\gamma}$



"Local perturbations perturb locally"

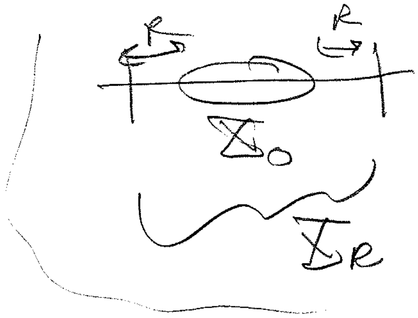
(Γ, d), $H_\Lambda(s) = \left(\sum_{X \subseteq \Lambda} \Phi(X)\right) + \Phi(s)$

$\Phi(s) \in \mathcal{A}_{X_0}$

$H_\Lambda(s)$ generates $\{\tau_t^{(s)}\}_{t\in\mathbb{R}}$

$$\|[\tau_t^{(s)}(A), B]\| \le C(A,B)\, e^{-a[d(\mathrm{supp}(A), \mathrm{supp}(B)) - v|t|]}$$

$X_R = \{ x \in \Gamma : d(X, x) \le R \}$



<u>Thm</u>: $\exists$ unitary $U_R(s) \in \mathcal{A}_{X_R}$ such that

$$\| U(s) - V_R(s) \| \le G\left(\frac{\gamma R}{2v}\right), \text{ where}$$

$$G(x) = C x^{10} e^{-\frac{7}{2} \cdot \frac{x}{(\log x)^2}}$$

(Γ, d), $\mathcal{A}_\Lambda$, $\Lambda \subseteq \Gamma$, $H_\Lambda(s) = \sum_{X \subseteq \Lambda} \Phi(X, s)$

F-function, $a > 0$, $F_a(r) = e^{-ar} F(r)$

- $$|||\Phi|||_{F_a} = \sup_{x,y \in \Gamma} \frac{1}{F_a(d(x,y))} \sum_{\substack{X \subset \Gamma \\ X \ni x,y}} \left\{ \sup_{s \in [0,1]} \| \Phi(X,s) \| + |X| \cdot \sup_{s \in [0,1]} \left\| \frac{d}{ds} \Phi(X,s) \right\| \right\}$$

- $\text{gap}(H_\Lambda(s)) \ge \gamma > 0, \quad \forall s \in [0,1]$

One common occurrence $\Phi(X, S) = \Phi(X) + s \Psi(X)$

$P(s) =$ ~~spectral projections of $H(s)$ corresponding to $\sum'(s)$~~

$P(s) = \cancel{U(s)\, P(0)\, U(s)^*}$

a go

finite $\Lambda \subseteq \Gamma$, $D_\Lambda(s) = \sum_{X \subseteq \Lambda} \Omega_\Lambda(X, s)$

$\Omega_\Lambda \to \Omega$, $\to$ F-function F_Ω for Ω

such that $\|\Omega\|_{F_\Omega} < \infty$

$\rightsquigarrow \alpha^\Lambda_{s,0}(A) = \mathcal{U}_\Lambda(s)^* A\, \mathcal{U}_\Lambda(s)$

have LR bounds for $\alpha^{(\Lambda)}_{s,0}$

$\tau^{(\Lambda)}_t$ $\exists$ thermodynamic limit

$\Rightarrow \alpha_{s,0}(\cdot)$ automorphism of $\mathcal{A}_\Gamma$

$\mathcal{S}_\Gamma(s) =$ thermodynamic limit of ground states of $H_\Lambda(s)$

$\mathcal{S}_\Gamma(s) = \{\omega \circ \alpha_{s,0} : \omega \in \mathcal{S}_\Gamma(0)\}$

SU(2) invariance

local symmetry

$\mathcal{H}_x = \mathbb{C}^d$, $\forall x \in \Gamma$, rep of SU(2)

$\mathcal{U}_g$ on $\mathbb{C}^d$, $g \in SU(2)$, $\mathcal{U}_g = e^{igS}$

Finite range interaction Φ is SU(2) invariant if $\forall X \subseteq \Gamma$

$$[\Phi(X), \bigotimes_{x \in X} \mathcal{U}_{g,x}] = 0$$

$$\Rightarrow [H_\Lambda, \bigotimes_{x \in \Lambda} \mathcal{U}_g] = 0$$

e.g. AKLT chain

$\Phi(\{x, x+1\}) = P^{(2)}$

limit $\Lambda \to \Gamma$ of the symmetries are automorphisms β_g on $\mathcal{A}_\Gamma$

$$\tau_t \circ \beta_g = \beta_g \circ \tau_t$$

$$\alpha_{s,0} \circ \beta_g = \beta_g \circ \alpha_{s,0}$$

β_g are represented on $\mathcal{S}_{\mathrm{pr}}(s)$ by a representation $\Pi_g^{(s)}$ and $\Pi_g^{(s)} = T\,\Pi_g^{(0)}\,T^{-1}$

Now set $\Gamma = \mathbb{Z}$ or $[1,\infty)$ or $(-\infty, 0]$

<u>Sven</u> $\mathcal{S}_{\mathbb{Z}} = \{\omega\}$, $\mathcal{S}_{[1,\infty)} \cong \mathcal{E}(\mathbb{C}^{d_L})$

$$\mathcal{S}_{(-\infty,0]} \cong \mathcal{E}(\mathbb{C}^{d_R})$$

Frustration-free SU(2)-invariant spin chains with a unique zero-energy ground state on $\mathbb{Z}$.

$\mathrm{Tr}\left[\rho\, \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n}(\mathbb{1})\right] = \omega(A_1 \otimes \cdots \otimes A_n)$, $A_i \in M_d$

$\mathbb{E}_A : M_k \to M_k$, $\mathbb{E}_A(B) = V^*(A \otimes B)V$, $V^*V = \mathbb{1}$

$V = \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix}$; SU(2) invariance implies existence of unitary representation u_g on $\mathbb{C}^k$

and $V u_g = (U_g \otimes u_g) V$ for all $g \in SU(2)$.

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(JFA 1994, Fannes-N-Werner)

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$\hat{\mathbb{E}} = \mathbb{E}_{\mathbb{1}}$ has TPS

$\Rightarrow \exists\, c>0,\ \lambda \in [0,1)$ s.t.

$$\| \hat{\mathbb{E}}^n - |\mathbb{1}\rangle\langle\rho| \| \le c\cdot\lambda^n$$

$[\rho, u_g] = 0$ as well

$\mathbb{E}_{U_g}$ is similar to $\mathbb{E}_{\mathbb{1}}$

$$\mathbb{E}_{U_g}(u_g) = V^*(U_g \otimes u_g)V$$
$$= V^* V u_g = u_g$$

$$\mathbb{E}^t_{U_g}(\rho u_g^*) = \rho u_g^*$$

$\Leftrightarrow \forall B \in M_d,\ \mathrm{Tr}\, \rho u_g^* \mathbb{E}_{U_g}(B)$

$\| \mathbb{E}^n_{U_g} - |u_g\rangle\langle\rho u_g^*| \| \le c\lambda^n$

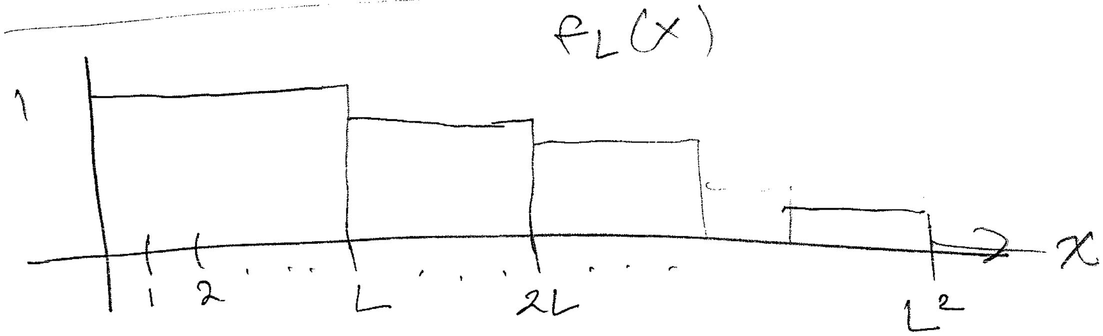
$$= \mathrm{Tr}\left[\rho u_g^* V^* (U_g \otimes B) V\right]$$
$$= \mathrm{Tr}\left[\rho u_g V^* (1 \otimes B u_g^*)(U_g \otimes u_g) V\right]$$
$$= ?$$

(too obscure to follow on the board)

<u>Claim:</u> There is a well-defined unitary representation of SU(2), on the GNS space of ω, of the form

$$U_g^{+} = \bigotimes_{x=1}^{\infty} U_g \qquad \text{on } \mathcal{H}$$

$$\Longleftrightarrow \quad \sum_{x=1}^{\infty} S_x$$



$$f_L(mL+n) = 1 - \frac{m}{L}, \qquad 0 \le m \le L \text{ and } n \le L-1 \; (? \, L \, ?)$$

$$S^{+}(L) = \sum_{x=1}^{L^2} f_L(x-1)\, S_x$$

$$U_g^{+}(L) = e^{i g S^{+}(L)}$$

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$$\omega(A_1 \otimes \cdots \otimes A_n) = \mathrm{Tr}\Big[\rho\, \underbrace{\mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n}}_{\mathbb{E}^{[1,n]}_{A_1 \otimes \cdots \otimes A_n}}(\mathbb{1})\Big]$$

Thm

$\forall N$, $A_1, A_2 \in \mathcal{A}_{[-N,N]}$

$$\lim_{L\to\infty} \omega\left(A_1^* \mathcal{U}_g^+(L) A_2\right)$$

$$\parallel$$

$$\langle \pi(A_1)\Omega, \pi(U_g^*(L))\, \pi(A_2)\Omega \rangle$$

$$= \mathrm{Tr}\left[\rho\, \mathbb{E}^{[-N,N]}_{A_1^* (\mathbb{1}^{\otimes N+1} \otimes U_g^{\otimes N}) A_2}(u_g)\right]$$

$$\left\| \mathbb{E}^{[1,L^2]}_{U_g^+(L)} - Q_g \right\| \le \frac{c}{L}$$

$$Q_g(B) = (\mathrm{Tr}\, \rho B)\, u_g\,, \qquad |u_g\rangle\langle \rho| = Q_g$$

$$\mathbb{E}^{[1,L^2]}_{U_g^+(L)} = \left(\mathbb{E}_{U_g}\right)^L \circ \left(\mathbb{E}_{U_{g(1-\frac{1}{L})}}\right)^L \circ \cdots \circ \left(\mathbb{E}_{U_{g\cdot\frac{1}{L}}}\right)^L$$

$$\xrightarrow{L\to\infty} P_g \circ P_{g(1-\frac{1}{L})} \cdots P_{g\cdot\frac{2}{L}} \circ P_{g\cdot\frac{1}{L}}$$

$\| \mathbb{E}^n_{U_g} - |u_g\rangle\langle \rho u_g^*| \| \le c\lambda^n$

call $|u_g\rangle\langle \rho u_g^*| = P_g$

$$|u_g\rangle\langle \rho u_g^* | u_{g\cdot(1-\frac{1}{L})}\rangle\langle \rho u^*_{g\cdot(1-\frac{1}{L})}| \cdots |u_{g\cdot\frac{1}{L}}\rangle\langle \rho u^*_{g\cdot\frac{1}{L}}|\mathbb{1}$$

$$\mathrm{Tr}\, \rho u_g^* u_{g(1-\frac{1}{L})} = \mathrm{Tr}\, \rho\, u^*_{g/L}$$

$\rightarrow \Theta_g \, |u_g\rangle\langle g|$

$$\lim_{L\to\infty} \mathbb{E}^{[1,L^2]}_{U_g^+(L)}(\mathbb{1}) = u_g$$

$A_1, A_2 \mid U_g^+$

$[-\infty, 0]$

$$\langle \pi(A_1)\Omega, U_g^+ \pi(A_2)\Omega \rangle$$

$$= \mathrm{Tr}\, \rho\, \mathbb{E}^{[-N,0)}_{A_1^* A_2}(u_g)$$

$\rightarrow$ ~~th~~ matrix elements of u_g

$\mathbb{E}^{[-N,0]}_{A_1^* A_2} \circ (\mathbb{E}_{u_g})^n (u_g)$

Conclusion:

On the left subspace $U_g^+ \cong \left(\bigoplus u_g\right)^\infty$