Localization for Random Block Operators Related to the XY Spin Chain

Jacob W. Chapman

Division of Science
Southern Wesleyan University
Central, SC

Joint work with Günter Stolz

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The XY Spin Chain: The Model

- The anisotropic XY spin chain is given by the self-adjoint Hamiltonian

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$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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- $M_j := I \otimes \cdots \otimes I \otimes M \otimes I \otimes \cdots \otimes I$ (nontrivial in jth component)

Random Block Operators

- Via the Jordan-Wigner transform, one reduces the problem to proving dynamical localization for the random block operator

$$\hat{M}_n = \begin{pmatrix} A_n & B_n \\ -B_n & -A_n \end{pmatrix}$$

where

$$A_n = \begin{pmatrix} \nu_1 & -\mu_1 & & \\ -\mu_1 & \ddots & \ddots & \\ & \ddots & \ddots & -\mu_{n-1} \\ & & -\mu_{n-1} & \nu_n \end{pmatrix}$$

$$B_n = \begin{pmatrix} 0 & -\mu_1\gamma_1 & & \\ \mu_1\gamma_1 & \ddots & \ddots & \\ & \ddots & \ddots & -\mu_{n-1}\gamma_{n-1} \\ & & \mu_{n-1}\gamma_{n-1} & 0 \end{pmatrix}$$

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- How about small disorder? Since the 1-D Anderson model is dynamically localized at all energies at small disorder, is $\hat{M}_n$ as well?
- It is convenient to write $\hat{M}_n$ in the basis $(e_1, e_{n+1}, e_2, e_{n+2}, \ldots, e_n, e_{2n})$ as

$$M_n = \begin{pmatrix} \nu_1\sigma^z & -\mu_1 S(\gamma_1) & & \\ -\mu_1 S(\gamma_1)^t & \nu_2\sigma^z & \ddots & \\ & \ddots & \ddots & -\mu_{n-1}S(\gamma_{n-1}) \\ & & -\mu_{n-1}S(\gamma_{n-1})^t & \nu_n\sigma^z \end{pmatrix}$$

$$\text{where} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}$$

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- To study the effects of introducing randomness into our model, assume:
  - $\{\nu_j\}$ are i.i.d. with nontrivial distribution ρ of compact support
  - $\gamma_j \equiv \gamma \in (0,1) \cup (1,\infty)$ (0=isotropic, 1=Ising model)
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- Let $P_j : \ell^2([1,n];\mathbb{C}^2) \to \mathbb{C}^2$ be the projection $P_j u = u(j)$.
- With such coefficients we have

Theorem (Main Result)

For every compact interval $J \subset \mathbb{R} \setminus \{0\}$ and every $\zeta \in (0,1)$, there exist constants $C = C(J,\zeta) < \infty$ and $\eta = \eta(J,\zeta) > 0$ such that for all $n \in \mathbb{N}$ and $j,k \in [1,n]$,

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|P_j e^{-itM_n}\chi_J(M_n)P_k^*\|\right) \le Ce^{-\eta|j-k|^\zeta}.$$

Random Block Operators: Applications

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Theorem

If there exist $\zeta \in (0,1)$ *and* $C > 0$, $\eta > 0$ *such that for all* $n \in \mathbb{N}$ *and* $j, k \in [1, n]$,

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|P_j e^{-itM_n} P_k^*\|\right) \leq Ce^{-\eta|j-k|^\zeta}, \tag{1}$$

then for every $\varepsilon \in (0, \eta)$, *there exists* $C' = C'(\eta, \varepsilon, \zeta) > 0$ *such that*

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|[\tau_t^n(A), B]\|\right) \leq C'\|A\|\|B\|e^{-(\eta-\varepsilon)|j-k|^\zeta} \tag{2}$$

for all $1 \leq j < k$, $n \geq k$, $A \in \mathcal{A}_j$, *and* $B \in \mathcal{A}_{[k,n]}$. *Furthermore, if (1) holds with* $\zeta = 1$, *then (2) holds with* $\varepsilon = 0$.

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- Three (or more?) options to obtain an interesting result:
  - When $\gamma = 0$ (isotropic), use that (1) holds for 1-D Anderson model for arbitrary nontrivial distributions (of compact support) - the distributions do not need to be *nice*.

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  - Prove (1) from scratch, starting with regularity of Lyapunov exponents at 0.

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Proposition

Let

$$\hat{M} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}.$$

If there exists $\lambda > 0$ such that $A \geq \lambda$ or $-A \geq \lambda$, then

$$\sigma(\hat{M}) \cap (-\lambda, \lambda) = \emptyset.$$

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- For example, if $\operatorname{supp} \rho \subset (2, \infty)$, then the a.s. spectrum of the Anderson model is $\Sigma \subset [-2, 2] + (2, \infty)$. Thus $A_n \geq \lambda > 0$.

Proof of Main Result: A Thouless Formula

- It is natural to extend M_n to the infinite-volume, bounded, self-adjoint operator

$$M_{\nu,\gamma} := \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & \nu_{-1}\sigma^z & -S(\gamma) & & \\ & -S(\gamma)^t & \nu_0\sigma^z & -S(\gamma) & \\ & & -S(\gamma)^t & \nu_1\sigma^z & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

- Recall $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}$

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- Recall $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}$
- Proving dynamical localization for $M_{\nu,\gamma}$ is equivalent to proving it for M_n (finite vs. infinite volume): [Germinet/Klopp, 2012]

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- Lyapunov index $\gamma(E) := \frac{1}{\ell}[\gamma_1(E) + \cdots + \gamma_\ell(E)]$

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- Lyapunov index $\gamma(E) := \frac{1}{\ell}[\gamma_1(E) + \cdots + \gamma_\ell(E)]$
- Thouless formula for Anderson model on a strip:

$$\gamma(E) = \int_{\mathbb{R}} \log|E - E'| dN(E')$$

where $E \mapsto N(E)$ is the integrated density of states.

A Thouless Formula

- Non-standard hopping - modify [Craig/Simon, 1983] to get

$$\gamma(E) = -\frac{1}{\ell}\mathbb{E}(\log|\det g|) + \int_{\mathbb{R}} \log|E - E'|dN(E')$$

- It is valid for **ergodic block Jacobi matrices**:

$$M = M(\omega) = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & V_{-1} & -S_{-1} & & \\ & -S_{-1}^t & V_0 & -S_0 & \\ & & -S_0^t & V_1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

where $f, g : \Omega \to \mathbb{R}^{\ell\times\ell}$ measurable, $f(\omega)$ symmetric, $g(\omega)$ invertible, $\|f(\omega)\| + \|g(\omega)\| + \|g(\omega)^{-1}\| \leq D < \infty$ a.s., T ergodic bijection, $V_n(\omega) := f(T^n\omega)$, $S_n(\omega) := g(T^n\omega)$

Dynamical Localization: Assumption

- We prove dynamical localization for the class of models

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where $\{V_n\} \subset \mathbb{R}^{\ell \times \ell}$ are compactly supported i.i.d. symmetric,
$\{S_n\} \subset \mathbb{R}^{\ell \times \ell}$ are compactly supported i.i.d. invertible,
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  where $\{V_n\} \subset \mathbb{R}^{\ell \times \ell}$ are compactly supported i.i.d. symmetric, $\{S_n\} \subset \mathbb{R}^{\ell \times \ell}$ are compactly supported i.i.d. invertible, $\{V_n\}$ and $\{S_n\}$ independent from each other, $\|V_0\| + \|S_0\| + \|S_0^{-1}\| \le D < \infty$ a.s.
- Note: this covers the anisotropic XY spin chain with $\ell = 2$, $V_n = \nu_n \sigma^z$, $S_n = S(\gamma)$.

Dynamical Localization: Assumption

- Fürstenberg group $G_{\mu_E} := \overline{\langle \operatorname{supp} \mu_E \rangle}$ is the smallest closed subgroup of the symplectic $2\ell \times 2\ell$ matrices containing $\operatorname{supp} \mu_E$, where μ_E is the common distribution of the i.i.d. transfer matrices for the finite difference equation associated with M.

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- **Main Assumption (MA)**: G_{μ_E} is p-contracting and L_p-strongly irreducible for every $p = 1, ..., \ell$ and $E \in I$, where $I \subset \mathbb{R}$ is an open interval.

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- See [Bougerol/Lacroix, 1985] for definitions of these concepts. Suffice it to say, they generalize the notions of noncompactness and strong irreducibility required by Fürstenberg's theorem in the $\ell = 1$ case.

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- We proved

Theorem (Dynamical Localization)

If (MA) holds, then for every compact interval $J \subset I$ and every $\zeta \in (0,1)$, there exist $C < \infty$ and $\eta > 0$ such that for every $L \in \mathbb{N}$ and $j, k \in [-L, L]$,

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|P_j \chi_J(M_{[-L,L]}) e^{-itM_{[-L,L]}} P_k^*\|\right) \leq Ce^{-\eta|j-k|^\zeta}$$

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- This is what we did for the anisotropic XY chain, where $\ell = 2$. We found 10 lin. indep. elements in G_{μ_E} for all $E \neq 0$. This establishes our Main Result (for the XY chain).

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- This is what we did for the anisotropic XY chain, where $\ell = 2$. We found 10 lin. indep. elements in G_{μ_E} for all $E \neq 0$. This establishes our Main Result (for the XY chain).
- Our construction follows both [Gol'dsheid/Margulis, 1989] and [Boumaza/Stolz, 2007].

Proof of Theorem

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  - Representation formula for Green's function (pos. of Lyap. $\Rightarrow$ exp. decay of Green's function)
- We establish these estimates by adapting arguments from [Klein/Lacroix/Speis, 1990] for Anderson model on a strip.
- Appropriate care must be taken to account for non-standard hopping terms.

Open Questions

- How can we remove $\chi_J(M_n)$, $J \subset \mathbb{R} \setminus \{0\}$, from our Main Result (in cases where $0 \in \Sigma_{as}$)? What happens at $E = 0$?

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- The Fürstenberg group G_{μ_0} is no longer irreducible. Transfer matrices essentially reduce to a diagonal 2×2 block matrix.

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- How can we remove $\chi_J(M_n)$, $J \subset \mathbb{R} \setminus \{0\}$, from our Main Result (in cases where $0 \in \Sigma_{as}$)? What happens at $E = 0$?
- The Fürstenberg group G_{μ_0} is no longer irreducible. Transfer matrices essentially reduce to a diagonal 2×2 block matrix.
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- But to prove localization, suppose we assume $\gamma_1(0) > \gamma_2(0) > 0$ (which happens "generically"). Can one prove the required regularity of the Lyapunov exponents at 0?

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- A question such as this arises already for the Ising model, which arises from $\gamma = 1$:

$$\begin{pmatrix} \ddots & \ddots & & & \\ \ddots & V_\omega(-1) & -S(1) & & \\ & -S(1)^t & V_\omega(0) & -S(1) & \\ & & -S(1)^t & V_\omega(1) & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \cong \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 0 & -2 & & & & \\ & -2 & 0 & \omega_0 & & & \\ & & \omega_0 & \boxed{0} & -2 & & \\ & & & -2 & 0 & \omega_1 & \\ & & & & \omega_1 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

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- Standard arguments give non-compactness and strong irreducibility of G_{μ_E}, but again only for $E \neq 0$.

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  - Main difficulty is that lack of irreducibility at 0 implies a lack of uniqueness of an invariant measure associated with the Lyapunov exponent (used to prove Hölder continuity of Lyapunov exponents).
- If we can prove it, can we extend the proof to the anisotropic XY spin chain? This would involve understanding a more abstract, higher-order dynamical system.

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Thank you!