The Spin-1 SU(2)-invariant model

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Spin Matrices

Many interesting observables are given in terms on *spin matrices* S^1, S^2, S^3, operators on $\mathcal{H}_x = \mathbb{C}^{2S+1}$ such that

$$\left[S^\alpha, S^\beta\right] = i \sum_\gamma \mathcal{E}_{\alpha\beta\gamma} S^\gamma, \qquad \mathbf{S} = (S^1, S^2, S^3).$$

For spin-1 we can pick the matrices

$$S^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, S^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We will also use a rectangular lattice:

$$\Lambda = \left\{-\frac{L_1}{2} + 1, ..., \frac{L_1}{2}\right\} \times ... \times \left\{-\frac{L_d}{2} + 1, ..., \frac{L_d}{2}\right\}.$$

The Heisenberg model

- The most general two-body SU(2) invariant spin-1 Hamiltonian can be written as

$$H_{\Lambda,\mathbf{0}}^{J_1,J_2} = -2 \sum_{\{x,y\}\in\mathcal{E}} \left(J_1 \left(\mathbf{S}_x \cdot \mathbf{S}_y \right) + J_2 \left(\mathbf{S}_x \cdot \mathbf{S}_y \right)^2 \right).$$

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- We define the associated *Gibbs state* as

$$\langle\cdot\rangle_{\beta,\Lambda,\mathbf{0}}^{J_1,J_2} = \frac{1}{Z_{\beta,\Lambda,\mathbf{0}}^{J_1,J_2}} Tr \cdot e^{-\beta H_{\Lambda,\mathbf{0}}^{J_1,J_2}}$$

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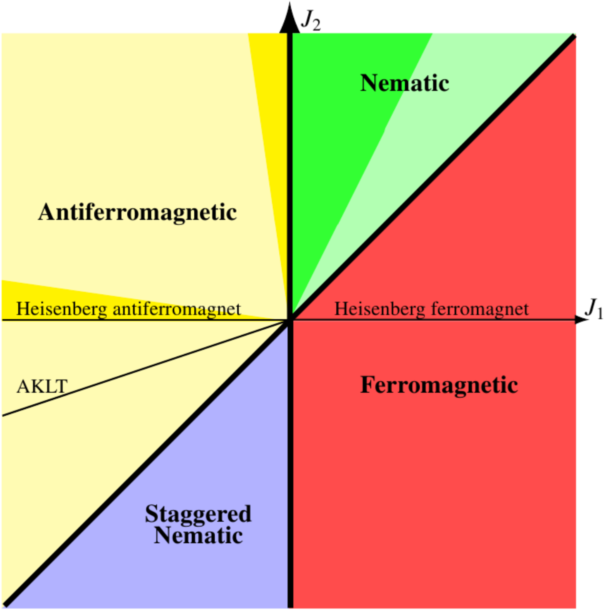
- In particular we will later be interested in the correlation

$$\rho(x) = \left\langle \left((S_0^3)^2 - \frac{2}{3} \right) \left((S_x^3)^2 - \frac{2}{3} \right) \right\rangle_{\beta,\Lambda,\mathbf{0}}^{J_1,J_2}.$$

The Phase Diagram



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- Fortunately there is a probabilistic representation that is valid on this region and *is* reflection positive, this representation is called the random loop model.
- Two loop models were introduced in the 90's, the model of Aizenman and Nachtergaele is valid on the line $J_1 = 0$ and the model of Tóth on the line $J_1 = J_2$. Recently the model was extended to the region inbetween by Ueltschi.

Long-range order for $J_1 = 0$

For the line $J_1 = 0$ we don't need the random loop model to prove long-range order, we can just use methods in quantum mechanics.

Theorem 1

Let $S = 1$. Assume $\mathbf{h} = 0$ and $L_1, ..., L_d$ are even. Then we have the bounds

$$\lim_{\beta\to\infty} \lim_{L_i\to\infty} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \rho(x) \geq \rho(e_1) - I_d \sqrt{\left\langle S_0^1 S_0^3 S_{e_1}^1 S_{e_1}^3 \right\rangle}.$$

$$I_d = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \sqrt{\frac{\varepsilon(k+\pi)}{\varepsilon(k)}} \left(\frac{1}{d} \sum_{i=1}^{d} \cos k_i \right)_+ \mathrm{d}k,$$

$$\varepsilon(k) = 2 \sum_{i=1}^{d} (1 - \cos k_i).$$

The case $J_1 < 0$

The method to prove the previous theorem uses the reflection positivity of the interaction. For $J_1 < 0$ the interaction $J_1(\mathbf{S}_x \cdot \mathbf{S}_y)$ is also reflection positive.

Theorem 2

Let $S = 1$, $J_2 > 0$ and $L_1, ..., L_d$ be even. Then there exists $J_1^0 < 0$, β_0 and $C = C(\beta, J_1) > 0$ such that if $J_1^0 < J_1 \leq 0$ and $\beta > \beta_0$ then

$$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \rho(x) \geq C$$

Proof outline

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- In order to apply these techniques we need a nice representation of the interaction.
- Once we have this representation we can quickly move from reflection positivity to obtain an infrared bound on the *Duhamel correlation function*.
- Transferring this infrared bound to the normal correlation is a big calculation.

Matrix representation of the interaction

- For our representation of the system we introduce the following matrix:

$$Q_x = \begin{pmatrix} (S_x^1)^2 - \frac{1}{3}S(S+1) & S_x^1 iS_x^2 & S_x^1 S_x^3 \\ S_x^1 iS_x^2 & (S_x^2)^2 - \frac{1}{3}S(S+1) & iS_x^2 S_x^3 \\ S_x^1 S_x^3 & iS_x^2 S_x^3 & (S_x^3)^2 - \frac{1}{3}S(S+1) \end{pmatrix}.$$

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- Now using a (slightly artificial) version of matrix multiplication we have the identity

$$\mathcal{TR}(Q_x Q_y) = (S_x^1 S_y^1 - S_x^2 S_y^2 + S_x^3 S_y^3)^2 - \frac{1}{3}S^2(S+1)^2 \mathbb{1}.$$

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- Also if we introduce the unitary operator $U = \prod_{x \in \Lambda_B} e^{i\pi S_x^2}$ then $U^{-1} \left(\mathbf{S}_x \cdot \mathbf{S}_y\right)^2 U = (S_x^1 S_y^1 - S_x^2 S_y^2 + S_x^3 S_y^3)^2.$

Infrared bound

- With this representation we can (with a bit of calculation and careful thought) show reflection positivity, and use this to give an infrared bound for the Duhamel correlation function.

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- We set our Fourier transform as

$$\mathcal{F}(f)(k) = \hat{f}(k) = \sum_{x \in \Lambda} e^{-ikx} f(x) \qquad k \in \Lambda^*,$$

$$\Lambda^* = \frac{2\pi}{L_1}\left\{-\frac{L_1}{2}+1, ..., \frac{L_1}{2}\right\} \times ... \times \frac{2\pi}{L_d}\left\{-\frac{L_d}{2}+1, .., \frac{L_d}{2}\right\}$$

Infrared bound

- Recall the infrared bound given in Dyson, Lieb, Simon:

$$\mathcal{F}(S_0^3 S_x^3)_{Duh}(k) \leq \frac{1}{2J_1\beta\varepsilon(k)}$$

where

$$(A,B)_{Duh} = \frac{1}{Z(0)}\frac{1}{\beta}\int_0^\beta \mathrm{d}s\, TrA^* e^{-sH} B e^{-(\beta-s)H}$$

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- here we have

$$\mathcal{F}\left((S_0^3)^2 - \frac{1}{3}S(S+1), (S_x^3)^2 - \frac{1}{3}S(S+1)\right)_{Duh}(k) \leq \frac{1}{2\beta\varepsilon(k)}$$

Infrared bound

- to get an infrared bound for $\rho(x) = \left\langle \left((S_0^3)^2 - \frac{2}{3}\right)\left((S_x^3)^2 - \frac{2}{3}\right)\right\rangle$ we us the Falk-Bruch inequality

$$\frac{1}{2}\langle A^*A + AA^*\rangle \leq (A, A)_{Duh} + \frac{1}{2}\sqrt{(A, A)_{Duh}\langle [A^*, [H, A]]\rangle}.$$

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- This is a difficult calculation for general spins because we take $A = \mathcal{F}\left((S_x^3)^2 - \frac{2}{3} \right)(k)$, because of this we specialise to $S = 1$ and take advantage of relations there (for example in spin-1 $S^i S^j S^i = 0$).

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- After some work we obtain the bound in theorem 1. Extending to $J_1 < 0$ is easy given what has already been done!