Spectral Gap of *d*-Dimensional PVBS Models

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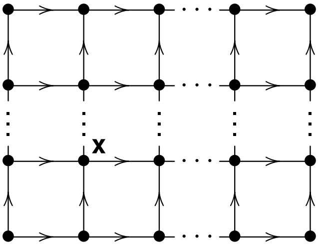
Hilbert Space

Let Λ be a finite connected subset of $\mathbb{Z}^d$.

$\Lambda =$ 

The one-site Hilbert space is give by $\mathcal{H}_{\mathbf{x}} = \mathbb{C}^2$. The Hilbert space for the whole system is given by

$$\mathcal{H}_\Lambda = \bigotimes_{\mathbf{x}\in\Lambda} \mathcal{H}_{\mathbf{x}}$$

PVBS Hamiltonian

For each dimension $k = 1, \ldots, d$, we assign a parameter $\lambda_k > 0$, $\lambda_k \neq 1$.

$$H_\Lambda = \sum_{k=1}^{d} \sum_{\mathbf{x},\, \mathbf{x}+e_k \in \Lambda} h_{\mathbf{x},\, \mathbf{x}+e_k}$$

$$h_{\mathbf{x},\, \mathbf{x}+e_k} = |1,\, 1\rangle\langle 1, 1| + |\phi_k\rangle\langle\phi_k|, \qquad \phi_k = \frac{1}{\sqrt{1+\lambda_k^2}}(|0, 1\rangle - \lambda_k|1, 0\rangle)$$

Ground State Space

Let $X \subseteq \Lambda$. Then $\mathcal{B} = \{\psi_X^\Lambda : X \subseteq \Lambda\}$ is an orthonormal basis for $\mathcal{H}_\Lambda$ where

$$\psi_X^\Lambda(\mathbf{x}) = \begin{cases} |1\rangle & \mathbf{x} \in X \\ |0\rangle & \mathbf{x} \notin X \end{cases}$$

Let $\lambda^{\mathbf{x}} = \prod_{k=1}^{d} \lambda_k^{x_k}$. Then ground state space for the one-species d-dimensional PVBS Hamiltonian is given by

$$\psi_\emptyset^\Lambda = \bigotimes_{\mathbf{x}\in\Lambda} |0\rangle, \qquad \psi_1^\Lambda = \frac{1}{\sqrt{C_\Lambda}} \sum_{\mathbf{x}\in\Lambda} \lambda^{\mathbf{x}} \psi_{\mathbf{x}}^\Lambda \tag{1}$$

When Λ is the rectangular lattice with end points $(0, 0, \ldots, 0)$ and $(n_1, n_2, \ldots, n_d)$ then

$$C_\Lambda = \prod_{k=1}^{d} c(\lambda_k, n_k) \quad \text{where} \quad c(\lambda_k, n_k) = \sum_{i=0}^{n_k} \lambda_k^{2i}.$$

The Spectral Gap

Definition

Let $\omega \in \mathcal{G}_\Gamma$ be an infinite volume ground state obtained as a weak-* limit of finite volume ground states. Then the spectral gap of the GNS Hamiltonian H_ω is

$$\gamma_\omega = \sup\{\delta > 0 \,:\, \mathrm{spec}(H_\omega) \cap (0,\delta) = \emptyset\}$$

if the RHS is well defined, or zero otherwise. We say that the spectrum is gapped if $\gamma_\omega > 0$.

Lower Bound for the Spectral Gap

To prove the existence of a spectral gap in the thermodynamic limit, we appeal to the following theorem.

Theorem (Spectral Gap Estimate)

Let H_{ω_0} be the GNS Hamiltonian of the ground state $\omega_0 \in \mathcal{G}_{\mathbb{Z}^d}$, and let $\gamma_{\mathbb{Z}^d}$ be the spectral gap of H_{ω_0}. Then

$$\gamma_{\mathbb{Z}^d} \geq \liminf_{n \geq 1} \lambda_1(n)$$

where $\lambda_1(n)$ is the smallest nonzero eigenvalue of the frustration-free Hamiltonians H_{Λ_n}, where Λ_n is an increasing and absorbing sequence of lattices $\Lambda_n \nearrow \mathbb{Z}^d$.

Conditions for the Martingale Method

The following conditions must hold for one and the same integer value $\ell > 0$.

(1) There exists a constant d_ℓ for which the local Hamiltonians satisfy

$$0 \leq \sum_{n=\ell}^{N} H_{\Lambda_n \setminus \Lambda_{n-\ell}} \leq d_\ell H_{\Lambda_N}.$$

(2) The local Hamiltonians H_{Λ_n} have a non-trivial kernel $\mathcal{G}_{\Lambda_n} \subseteq \mathcal{H}_{\Lambda_n}$. Furthermore, there is a nonvanishing spectral gap $\gamma_\ell > 0$ such that:

$$H_{\Lambda_n \setminus \Lambda_{n-\ell}} \geq \gamma_\ell (\mathbb{I} - G_{\Lambda_n \setminus \Lambda_{n-\ell}})$$

for all $n \geq n_\ell$ where G_{Λ_n} is the orthogonal projection onto $\mathcal{G}_{\Lambda_n}$.

(3) There exists a constant $\epsilon_\ell < \frac{1}{\sqrt{\ell+1}}$ and some n_ℓ such that for all $n \geq n_\ell$,

$$\left\| G_{\Lambda_{n+1} \setminus \Lambda_{n-\ell}} E_n \right\| \leq \epsilon_\ell$$

where $E_n = G_{\Lambda_n} - G_{\Lambda_{n+1}}$.

The Martingale Method

Theorem

Assume that conditions (1)-(3) are satisfied for the same integer ℓ. Then for any N and any $\psi \in \mathcal{H}_{\Lambda_N}$ such that $G_{\Lambda_N}\psi = 0$, one has

$$\langle \psi, H_{\Lambda_N}\psi \rangle \geq \frac{\gamma_{\ell+1}}{d_{\ell+1}}(1 - \epsilon_\ell \sqrt{\ell+1})^2 \|\psi\|^2 \tag{2}$$

Sequence of Increasing Lattices

We first pick a sequence of finite volumes increasing to $\mathbb{Z}^d$ to apply the spectral gap estimate theorem. We choose the sequence of hypercubic lattices

$$\Lambda_N = [0, N]^d \cap \mathbb{Z}^d.$$

For each term Λ_N of the sequence, we pick a finite sequence of lattices $\tilde{\Lambda}_m \nearrow \Lambda_N$ to apply the martingale method. For this we choose

$$\tilde{\Lambda}_m = \left([0, N]^{d-1} \times [0, m]\right) \cap \mathbb{Z}^d,$$

$m = 1, 2, \ldots, N$. The goal is to use the martingale method to obtain a *uniform* lower bound for the spectral gaps

$$\gamma(\Lambda_N) = \min\{\lambda \in \mathrm{spec}(H_{\Lambda_N}) \, : \, \lambda > 0\}.$$

Lower Bound Estimate for PVBS

Theorem (Bounds for the Spectral Gap)

For the PVBS model defined on $\mathbb{Z}^d$ with a single species of particle, the spectrum in the thermodynamic limit to $\mathbb{Z}^d$ is gapped if and only if $\lambda_k \neq 1$ for all $k = 1, \ldots, d$. Futhermore, the spectral gap, denoted $\gamma_{\mathbb{Z}^d}$, is bounded by

$$\frac{\gamma(B_d)}{2^d} \prod_{k=1}^{d} (1 - \epsilon(\lambda_k)\sqrt{2})^2 \leq \gamma_{\mathbb{Z}^d} \leq \min\left\{ \frac{(1-\lambda_k)^2}{1+\lambda_k^2} \; : \; k = 1, \ldots, d \right\} \quad (3)$$

where

$$\epsilon(\lambda_k) = \begin{cases} \frac{\lambda_k}{\sqrt{1+\lambda_k^2}} & \lambda_k < 1 \\ \frac{1}{\sqrt{1+\lambda_k^2}} & \lambda_k > 1 \end{cases} \quad (4)$$

and B_d is the d-dimensional unit hypercube.

Sketch of Proof:

Using the rectangular sequence $\tilde{\Lambda}_n \nearrow \Lambda_N$, and $\ell = 1$, we show that

$$\|G_{\tilde{\Lambda}_{n+1}\setminus\tilde{\Lambda}_{n-1}} E_n\| = \sup_{\psi \in \mathcal{G}_{\tilde{\Lambda}_n} \cap \mathcal{G}_{\tilde{\Lambda}_{n+1}}^{\perp}} \frac{\|G_{\tilde{\Lambda}_{n+1}\setminus\tilde{\Lambda}_{n-1}} \psi\|}{\|\psi\|} \leq \epsilon(\lambda_d)$$

where

$$\epsilon(\lambda_d) = \begin{cases} \frac{\lambda_d}{\sqrt{1+\lambda_d^2}} & \lambda_d < 1 \\ \frac{1}{\sqrt{1+\lambda_d^2}} & \lambda_d > 1 \end{cases}$$

This satisfies $\epsilon(\lambda_d) < \frac{1}{\sqrt{2}}$. So by the martingale method,

$$\gamma(\Lambda_N) \geq \frac{\gamma(\Lambda_N^{(1)})}{2}(1 - \epsilon(\lambda_d)\sqrt{2})^2,$$

where

$$\Lambda_N^{(1)} = ([0, N]^{d-1} \times [0, 1]) \cap \mathbb{Z}^d.$$

Sketch of Proof:

The sublattice $\Lambda_N^{(1)}$ still grows as $N \to \infty$, so the gap could close as $N \to \infty$. We recursively apply the martingale method, once for each direction the lattice grows to get the estimate:

$$\gamma(\Lambda_N^{(k)}) \geq \frac{\gamma(\Lambda_N^{(k+1)})}{2}(1 - \epsilon(\lambda_{d-k})\sqrt{2})^2$$

where $\Lambda_N^{(k)} = ([0, N]^{d-k} \times [0, 1]^k) \cap \mathbb{Z}^d$. Since $\Lambda_N^{(d)} = B_d$ the d-dimensional unit hypercube,

$$\gamma(\Lambda_N) \geq \frac{\gamma(B_d)}{2^d} \prod_{k=1}^{d} (1 - \epsilon(\lambda_k)\sqrt{2})^2$$

and the result follows from the spectral gap estimate theorem.