CBMS: B'ham, AL
June 17, 2014

Lieb-Robinson Bounds: A Tutorial

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Quantum Spin Systems

We consider quantum spin systems defined over a countable set Γ.

In many examples, $\Gamma = \mathbb{Z}^\nu$ with $\nu \geq 1$ (or some finite subset thereof), but this is not necessary.

To each $x \in \Gamma$, associate a single site Hilbert space, $\mathcal{H}_x = \mathbb{C}^{n_x}$ with $n_x \geq 2$.

To each finite $\Lambda \subset \Gamma$, associate the Hilbert space of states in Λ:

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$$

and an algebra of observables in Λ:

$$\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$$

As $\mathcal{H}_\Lambda$ have finite dim., these $\mathcal{A}_\Lambda$ are just the matrices over $\mathcal{H}_\Lambda$.

Observables and Support

For finite $\Lambda_0 \subset \Lambda \subset \Gamma$, $\mathcal{A}_{\Lambda_0} \subset \mathcal{A}_\Lambda$ in the sense that each $A \in \mathcal{A}_{\Lambda_0}$ can be associated to $\tilde{A} = A \otimes \mathbb{1}_{\Lambda \setminus \Lambda_0} \in \mathcal{A}_\Lambda$.

We say that $A \in \mathcal{A}_\Lambda$ is supported on $X \subset \Lambda$ if A can be written as $A = \tilde{A} \otimes \mathbb{1}_{\Lambda \setminus X}$ for some $\tilde{A} \in \mathcal{A}_X$. The mimimal such set X is called the support of A; denoted by $\mathrm{supp}(A)$.

Note that spatially disjoint observables commute, i.e., if $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ and $X \cap Y = \emptyset$, then

$$[A, B] = 0$$

where, with some abuse of notation, we are regarding A and B as observables in some $\mathcal{A}_\Lambda$ for Λ with $X \cup Y \subset \Lambda$.

Models

A model is defined through an interaction, i.e., a mapping Φ from the set of finite subsets of Γ to the local observable algebra with: $\Phi(X)^* = \Phi(X) \in \mathcal{A}_X$ for each finite $X \subset \Gamma$.

Given an interaction, one can define local Hamiltonians i.e., for any finite $\Lambda \subset \Gamma$ set

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$$

and (using the spectral theorem) the corresponding Heisenberg dynamics, i.e.,

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} \quad \text{for any } A \in \mathcal{A}_\Lambda \, .$$

Example

Take $\Gamma = \mathbb{Z}^\nu$ for some $\nu \geq 1$.

For all $n \in \mathbb{Z}^\nu$, take $\mathcal{H}_n = \mathbb{C}^2$. In this case, with each finite $\Lambda \subset \mathbb{Z}^\nu$, the Hilbert space is

$$\mathcal{H}_\Lambda = \bigotimes_{n \in \Lambda} \mathbb{C}^2 = \mathbb{C}^{2^{|\Lambda|}}$$

Recall the Pauli matrices, i.e., σ^α with $\alpha \in \{x, y, z\}$ is given by

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For finite $\Lambda \subset \mathbb{Z}^\nu$ and any $n \in \Lambda$ denote by $\sigma_n^\alpha \in \mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$

$$\sigma_n^\alpha = \mathbb{1} \otimes \cdots \mathbb{1} \otimes \sigma^\alpha \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \quad \text{with } \sigma^\alpha \text{ in the } n\text{-th factor.}$$

Example (cont.)

Let Φ be defined by

$$\Phi(X) = \begin{cases} J\left(\sigma_n^x\sigma_m^x + \sigma_n^y\sigma_m^y + \sigma_n^z\sigma_m^z\right) & \text{if } X = \{n,m\} \text{ and } |n-m| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

where J is a real parameter.

Then, for each finite $\Lambda \subset \mathbb{Z}^\nu$,

$$\begin{aligned} H_\Lambda &= \sum_{X \subset \Lambda} \Phi(X) \\ &= J \sum_{<n,m> \in \Lambda} \left(\sigma_n^x\sigma_m^x + \sigma_n^y\sigma_m^y + \sigma_n^z\sigma_m^z\right) \end{aligned}$$

is a nearest-neighbor Heisenberg Hamiltonian.

Quasi-Locality of the Dynamics

The basic idea:

Fix Γ and Φ. Let $X, Y \subset \Gamma$ with $X \cap Y = \emptyset$.

Take $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $\Lambda \subset \Gamma$ finite with $X \cup Y \subset \Lambda$.

It is clear that

$$[\tau_0^\Lambda(A), B] = [A, B] = 0$$

Note, however, that for general Φ, $\mathrm{supp}(\tau_t^\Lambda(A)) = \Lambda$ for any $t \neq 0$, since this is a non-relativistic system.

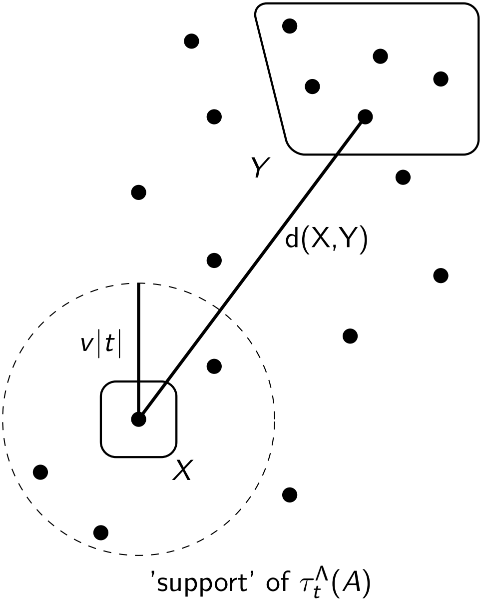
A typical Lieb-Robinson bound proves: for every $\mu > 0$, there exist C and v for which

$$\|[\tau_t^\Lambda(A), B]\| \leq C\|A\|\|B\|e^{-\mu(d(X,Y)-v|t|)}$$

In particular, these bounds show the commutator is still small for

$$|t| \leq \frac{d(X, Y)}{v}$$

Heuristic sketch of supports at time t.



On the Structure of Γ

Generally, Γ is a countable set equipped with a metric d.

If Γ is finite, no further assumptions are necessary. Otherwise:

We assume there is a non-increasing function $F : [0, \infty) \to (0, \infty)$ for which:

i) F is uniformly integrable

$$\|F\| = \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x, y)) < \infty$$

ii) F satisfies the convolution condition

$$C = \sup_{x,y \in \Gamma} \sum_{z \in \Gamma} \frac{F(d(x, z))F(d(z, y))}{F(d(x, y))} < \infty$$

On Γ continued

Given a set Γ and a function F satisfying i) and ii), it is easy to see that for any $a \geq 0$

$$F_a(x) = e^{-ax}F(x)$$

also satisfies i) and ii) with $\|F_a\| \leq \|F\|$ and $C_a \leq C$.

Example: Let $\Gamma = \mathbb{Z}^\nu$ for some $\nu \geq 1$. Then, for any $\epsilon > 0$, take

$$F(x) = \frac{1}{(1+x)^{\nu+\epsilon}}$$

then F satisfies i) and ii) with

$$C \leq 2^{\nu+\epsilon+1} \sum_{z \in \mathbb{Z}^\nu} F(|z|)$$

A Norm on Interactions

Fix Γ equipped with F as above. For any $a \geq 0$, let $\mathcal{B}_a(\Gamma)$ be the set of those Φ for which

$$\|\Phi\|_a = \sup_{x,y\in\Gamma} \sum_{\substack{X\subset\Gamma:\\ x,y\in X}} \frac{\|\Phi(X)\|}{F_a(d(x,y))} < \infty$$

This is a large class of interactions.

In fact, on $\mathbb{Z}^\nu$ with F as above:

General finite range, uniformly bounded interactions satisfy $\|\Phi\|_a < \infty$ for all $a > 0$.

A Lieb-Robinson Bound

Theorem (Lieb-Robinson Bound)

Let Γ *be equipped with* F *as above. Fix* $a > 0$ *and take* $\Phi \in \mathcal{B}_a(\Gamma)$. *There exist positive numbers* c *and* v_Φ *for which: Given any finite* $X, Y \subset \Gamma$ *with* $X \cap Y = \emptyset$, *any* $A \in \mathcal{A}_X$ *and* $B \in \mathcal{A}_Y$, *and finite* $\Lambda \subset \Gamma$ *with* $X \cup Y \subset \Lambda$, *then*

$$\|[\tau_t^\Lambda(A), B]\| \leq c\|A\|\|B\| \min[|X|, |Y|] e^{-a(d(X,Y) - v_\Phi |t|)}$$

for all $t \in \mathbb{R}$. *Here one can take*

$$c = \frac{2\|F_0\|}{C_a} \quad \textit{and} \quad v_\Phi = \frac{2\|\Phi\|_a C_a}{a}.$$

Some Comments

- Only useful for small times as

$$\|[\tau_t^\Lambda(A), B]\| \leq 2\|A\|\|B\|$$

- Still a bound if $X \cap Y \neq \emptyset$, but the above may be better . . .
- $\min[|X|, |Y|]$ can be replaced by boundaries; not volumes. . .
- E.g. on $\mathbb{Z}^\nu$, with finite range, uniformly bounded Φ, one can optimize v_Φ over $a > 0$. This produces a best possible estimate; often dubbed a Lieb-Robinson velocity.
- Our methods also apply in the case that $a = 0$ . . .

An Important Lemma

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{C}$ and denote by $\mathcal{B}(\mathcal{H})$ the bounded linear operators over $\mathcal{H}$. A mapping $A : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ is said to be strongly cont. (resp. strongly diff.) if: For all $\psi \in \mathcal{H}$, $A(t)\psi$ is cont. (resp. diff.) in t w.r.t. the norm-topology on $\mathcal{H}$.

Lemma

Let $A, B : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ be strongly continuous with A also being self-adjoint, i.e. $A(t)^* = A(t)$ for all t. The strong solution of

$$f'(t) = i[A(t), f(t)] + B(t) \quad \textit{with} \quad f(0) = f_0 \in \mathcal{B}(\mathcal{H}) \tag{1}$$

(which is unique) satisfies the estimate

$$\|f(t)\| \leq \|f_0\| + \int_{t_-}^{t_+} \|B(s)\| \, ds$$

for all $t \in \mathbb{R}$. Here $t_- = \min[0, t]$ and $t_+ = \max[0, t]$.

Proof of Lemma

The Dyson series

$$U_t = \mathbb{1} + \sum_{n=1}^{\infty} i^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} A(t_1) \cdots A(t_n) dt_n \cdots dt_1$$

can be used to construct a mapping $g : \mathbb{R} \to \mathcal{B}(\mathcal{H})$

$$g(t) = U_t g_0 U_t^*$$

which is a strong solution of

$$g'(t) = i[A(t), g(t)] \quad \text{with} \quad g(0) = g_0 \in \mathcal{B}(\mathcal{H})$$

As U_t is unitary, it is clear that g is norm-preserving, i.e.,

$$\|g(t)\| = \|g_0\| \quad \text{for all } t.$$

Proof of Lemma (cont.)

The mapping $f : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ given by

$$f(t) = U_t \left(f_0 + \int_0^t U_s^* B(s) U_s \, ds \right) U_t^*$$

is easily seen to be a strong solution of (1). As such, it satisfies

$$\|f(t)\| \leq \|f_0\| + \int_{t_-}^{t_+} \|B(s)\| \, ds$$

again, using unitarity of U_t. Uniqueness (in the context of both g and f) follows from an application of the Gronwall lemma.

Proof of the LRB

Consider the function $f : \mathbb{R} \to \mathcal{A}_\Lambda$ given by

$$f(t) = \left[\tau_t^\Lambda(A), B\right] = \left[e^{itH_\Lambda} A e^{-itH_\Lambda}, B\right]$$

It is clear that

$$\frac{d}{dt}\tau_t^\Lambda(A) = i\tau_t^\Lambda([H_\Lambda, A]) = i\tau_t^\Lambda\left([\tilde{H}_X, A]\right)$$

where

$$\tilde{H}_X = \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} \Phi(Z)$$

since A is supported in X. In this case,

$$\begin{aligned} f'(t) &= i\left[\tau_t^\Lambda\left([\tilde{H}_X, A]\right), B\right] \\ &= i\left[\left[\tau_t^\Lambda(\tilde{H}_X), \tau_t^\Lambda(A)\right], B\right] \\ &= -i\left[\left[\tau_t^\Lambda(A), B\right], \tau_t^\Lambda(\tilde{H}_X)\right] - i\left[\left[B, \tau_t^\Lambda(\tilde{H}_X)\right], \tau_t^\Lambda(A)\right] \end{aligned}$$

by Jacobi, i.e. $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$.

Proof of the LRB(cont.)

Re-writing things, we have shown that the function f satisfies

$$f'(t) = i[A(t), f(t)] + B(t) \quad \text{with} \quad f(0) = [A, B]$$

where we have set

$$A(t) = \tau_t^{\Lambda}(\tilde{H}_X) \quad \text{and} \quad B(t) = i\left[\tau_t^{\Lambda}(A), \left[B, \tau_t^{\Lambda}(\tilde{H}_X)\right]\right]$$

Using our lemma, we find that

$$\|[\tau_t^{\Lambda}(A), B]\| \leq \|[A, B]\| + 2\|A\| \int_{t_-}^{t_+} \|[\tau_s^{\Lambda}(\tilde{H}_X), B]\| \, ds$$

It is now convenient to define

$$C_B(X, t) = \sup_{\substack{A \in \mathcal{A}_X: \\ A \neq 0}} \frac{\|[\tau_t^{\Lambda}(A), B]\|}{\|A\|}$$

and observe that for $t > 0$ we have shown

$$C_B(X, t) \leq C_B(X, 0) + 2 \sum_{\substack{Z \subset \Lambda: \\ Z \cap X \neq \emptyset}} \|\Phi(Z)\| \int_0^t C_B(Z, s) \, ds$$

Proof of the LRB(cont.)

For any $Z \subset \Lambda$,

$$C_B(Z,0) \leq 2\|B\|\delta_Y(Z)$$

and so iteration yields

$$C_B(X,t) \leq 2\|B\| \sum_{n=0}^{\infty} \frac{(2|t|)^n}{n!} a_n$$

with

$$a_n = \sum_{\substack{Z_1 \subset \Lambda: \\ Z_1 \cap X \neq \emptyset}} \sum_{\substack{Z_2 \subset \Lambda: \\ Z_2 \cap Z_1 \neq \emptyset}} \cdots \sum_{\substack{Z_n \subset \Lambda: \\ Z_n \cap Z_{n-1} \neq \emptyset}} \delta_Y(Z_n) \prod_{i=1}^{n} \|\Phi(Z_i)\|$$

Now

$$a_1 \leq \sum_{x \in X} \sum_{y \in Y} \sum_{Z \ni x,y} \|\Phi(Z)\| \leq \|\Phi\|_a \sum_{x \in X} \sum_{y \in Y} F_a(d(x,y))$$

Proof of the LRB(cont.)

and

$$
\begin{aligned}
a_2 &\leq \sum_{x\in X}\sum_{y\in Y}\sum_{z\in\Lambda}\sum_{\substack{Z\subset\Lambda:\\ x,z\in Z_1}}\|\Phi(Z_1)\|\sum_{\substack{Z_2\subset\Lambda\\ z,y\in Z_2}}\|\Phi(Z_2)\| \\
&\leq \|\Phi\|_a^2\sum_{x\in X}\sum_{y\in Y}\sum_{z\in\Lambda}F_a(d(x,z))F_a(d(z,y)) \\
&\leq \|\Phi\|_a^2 C_a\sum_{x\in X}\sum_{y\in Y}F_a(d(x,y))
\end{aligned}
$$

and similarly

$$a_n \leq \|\Phi\|_a^n C_a^{n-1}\sum_{x\in X}\sum_{y\in Y}F_a(d(x,y))$$

Since

$$\sum_{x\in X}\sum_{y\in Y}F_a(d(x,y)) \leq e^{-ad(X,Y)}\min[|X|,|Y|]\|F\|$$

we have finished the proof.

An Application

These Lieb-Robinson bounds show that: For any $\Phi \in \mathcal{B}_a(\Gamma)$, the application of the finite-volume dynamics to a local observable, i.e. $\tau_t^\Lambda(A)$, remains essentially local for small times, in the sense that $\tau_t^\Lambda(A)$ almost commutes with any B supported far away from the support of A. Moreover, the estimates proven are uniform in the finite volume Λ.

As a result, one can prove that the finite-volume dynamics have a thermodynamic limit. In fact, for any $\Phi \in \mathcal{B}_a(\Gamma)$, the finite-volume dynamics $\tau_t^\Lambda(A)$ have a limit as $\Lambda \to \Gamma$.

On the Thermodynamic Limit

As we have discussed, for any finite sets $\Lambda_0 \subset \Lambda \subset \Gamma$, the algebras $\mathcal{A}_{\Lambda_0} \subset \mathcal{A}_\Lambda$. In this case, we define

$$\mathcal{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\Lambda \subset \Gamma} \mathcal{A}_\Lambda$$

the union taken over all finite subsets and take $\mathcal{A}_\Gamma$ to be the norm-completion of $\mathcal{A}_\Gamma^{\mathrm{loc}}$. $\mathcal{A}_\Gamma$ is a C^*-algebra.

Theorem

Let $a > 0$ and $\Phi \in \mathcal{B}_a(\Gamma)$. There exists a strongly continuous, one-parameter group of automorphisms $\tau_t^\Gamma(\cdot)$ on $\mathcal{A}_\Gamma$ and

$$\lim_{n \to \infty} \|\tau_t^{\Lambda_n}(A) - \tau_t^\Gamma(A)\| = 0$$

for any $A \in \mathcal{A}_\Gamma$ and any non-decreasing, exhaustive sequence of finite subsets $\{\Lambda_n\}$. The convergence is uniform on compact sets.

Proof of Thermo. Limit

Fix $A \in \mathcal{A}_X$. Take $n > m$ large enough so that $X \subset \Lambda_m \subset \Lambda_n$. It is easy to see that

$$\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A) = \int_0^t \frac{d}{ds}\left(\tau_s^{\Lambda_n} \circ \tau_{t-s}^{\Lambda_m}(A)\right)\, ds$$

and since

$$\frac{d}{ds}\left(\tau_s^{\Lambda_n} \circ \tau_{t-s}^{\Lambda_m}(A)\right) = i\tau_s^{\Lambda_n}\left(\left[H_{\Lambda_n} - H_{\Lambda_m}, \tau_{t-s}^{\Lambda_m}(A)\right]\right)$$

it is clear that for $t > 0$

$$\left\|\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A)\right\| \leq \sum_{z \in \Lambda_n \setminus \Lambda_m} \sum_{Z \ni z} \int_0^t \left\|[\Phi(Z), \tau_s^{\Lambda_m}(A)]\right\|\, ds$$

Proof of Thermo. Limit (cont.)

And from the proof of the Lieb-Robinson bound

$$
\begin{aligned}
\left\| \tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A) \right\| &\leq 2\|A\| C_a^{-1} \sum_{z \in \Lambda_n \setminus \Lambda_m} \sum_{Z \ni z} \|\Phi(Z)\| \times \\
&\qquad \times \int_0^t e^{2\|\Phi\|_a C_a s} ds \sum_{z' \in Z} \sum_{x \in X} F_a(d(z', x)) \\
&\leq 2\|A\| \|\Phi\|_a \int_0^t e^{2\|\Phi\|_a C_a s} ds \times \\
&\qquad \times |X| \sup_{x \in X} \sum_{z \in \Lambda_n \setminus \Lambda_m} F_a(d(x, z))
\end{aligned}
$$

This quantity clearly goes to 0 as $n, m \to \infty$. This proves that the sequence of finite volumes is Cauchy in norm (hence convergent) and the estimate is uniform on compact t-subsets.