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On the Random XY Spin Chain

Robert Sims
University of Arizona

The Isotropic XY-Spin Chain

Fix a real-valued sequence $\{\nu_j\}_{j\geq 1}$ and for each integer $n \geq 1$, set

$$H_n = \sum_{j=1}^{n-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \sum_{j=1}^{n} \nu_j \sigma_j^z,$$

acting on

$$\mathcal{H}_n = \bigotimes_{j=1}^{n} \mathbb{C}^2$$

Here, the Pauli matrices are

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and for $\alpha \in \{x, y, z\}$ and $1 \leq j \leq n$ we embed these into $\mathcal{B}(\mathcal{H}_n)$:

$$\sigma_j^\alpha = \mathbb{1} \otimes \cdots \mathbb{1} \otimes \sigma^\alpha \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \quad \text{with } \sigma^\alpha \text{ in the } j\text{-th factor.}$$

A Locality Review

Let us denote the Heisenberg dynamics associated to H_n by

$$\tau_t^n(A) = e^{itH_n} A e^{-itH_n} \quad \text{for all} \quad A \in \mathcal{B}(\mathcal{H}_n)$$

For simplicity, take $\mathcal{A}_k \subset \mathcal{B}(\mathcal{H}_n)$ to be all those observables supported at a single site $1 \leq k \leq n$.

For bounded sequences $\{\nu_j\}$, it is clear that Lieb-Robinson bounds apply and for any $\mu > 0$,

$$\|[\tau_t^n(A), B]\| \leq C\|A\|\|B\|e^{-\mu(|k-k'|-v|t|)}$$

for any $A \in \mathcal{A}_k$, $B \in \mathcal{A}_{k'}$, and $t \in \mathbb{R}$ with some $v > 0$ depending on μ and the sequence $\{\nu_j\}$.

Can one prove a stronger statement if the $\{\nu_j\}$ are random?

A Strong Form of Dynamical Localization

Assume $\{\nu_j\}$ is an i.i.d. random sequence with compactly supported bounded density ρ.

Theorem (Hamza-S-Stolz '11)

There are positive numbers C and η for which, given any integer $n \geq 1$ the bound

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|[\tau_t^n(A), B]\|\right) \leq C\|A\|\|B\|e^{-\eta|k-k'|}$$

holds for any $A \in \mathcal{A}_k$ and $B \in \mathcal{A}_{k'}$ with $1 \leq k < k' \leq n$.

One can think of this as a Lieb-Robinson bound with zero velocity.

After reviewing some basic properties of this model, the main goal of this talk is to prove this result.

Diagonalizing the Hamiltonian

As Gunter discussed, *diagonalizing* this Hamiltonian goes back to LSM '61:

First, one introduces raising and lowering operators:

$$a^* = \frac{1}{2}(\sigma^x + i\sigma^y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$a = \frac{1}{2}(\sigma^x - i\sigma^y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

It will also be useful to observe that:

$$a^* a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad aa^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e., these four operators form a basis for $\mathcal{B}(\mathbb{C}^2) = \mathbb{C}^{2\times 2}$.

Jordan-Wigner Transform

Then, one introduces the non-local Jordan-Wigner transform:

$$c_1 = a_1 \quad \text{and} \quad c_j = \sigma_1^z \ldots \sigma_{j-1}^z a_j$$

for each $j \geq 2$.

These operators are particularly useful because they satisfy the canonical anti-commutation relations (CAR):

$$\{c_j, c_k^*\} = \delta_{jk} \mathbb{1} \quad \text{and} \quad \{c_j, c_k\} = \{c_j^*, c_k^*\} = 0\,.$$

We often collect them as vectors:

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \text{or} \quad c^* = (c_1^*, c_2^*, \cdots, c_n^*)$$

Re-writing the Hamiltonian

A short calculation shows that

$$\begin{aligned} H_n &= \sum_{j=1}^{n-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \sum_{j=1}^{n} \nu_j \sigma_j^z \\ &= 2 \sum_{j=1}^{n-1} (a_j^* a_{j+1} + a_{j+1}^* a_j) + \sum_{j=1}^{n} \nu_j (2 a_j^* a_j - \mathbb{1}) \\ &= -2 \sum_{j=1}^{n-1} (c_j^* c_{j+1} + c_{j+1}^* c_j) + \sum_{j=1}^{n} \nu_j (2 c_j^* c_j - \mathbb{1}) \\ &= 2 c^* M_n c - E \mathbb{1} \end{aligned}$$

where

$$M_n = \begin{pmatrix} \nu_1 & -1 & & \\ -1 & \nu_2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & \nu_n \end{pmatrix} \quad \text{and} \quad E = \sum_{j=1}^{n} \nu_j$$

The Anderson model:

In the case that the $\{\nu_j\}$ are random, M_n corresponds to the well-studied Anderson model corresponding to a single quantum particle in a random environment.

With our assumptions on the $\{\nu_j\}$, one can prove

Theorem (Dynamical Localization)

There exist positive numbers C' and η' such that for all integers $n \geq 1$ and any $k, k' \in \{1, \dots, n\}$,

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left|(e^{-itM_n})_{kk'}\right|\right) \leq C' e^{-\eta'|k-k'|}.$$

For the purposes of this talk, we will assume this is well-known; see, for example, Kunz-Souillard or Aizenman-Molchanov.

Summary of findings so far:

We have seen that this many-body XY Hamiltonian can be expressed in terms of a single-particle Hamiltonian, i.e.,

$$H_n = 2c^* M_n c - E\mathbb{1}$$

In the case that the $\{\nu_j\}$ are *nice* random variables, we also know that the single-particle dynamics, i.e. e^{itM_n}, is dynamically localized.

We now show that this implies the desired result for the many-body dynamics, i.e. $\tau_t^n(\cdot)$.

Bogoliubov Transformations

As we saw, M_n is real symmetric. In this case, there is a real orthogonal U_n such that

$$U_n^t M_n U_n = \Lambda = \mathrm{diag}(\lambda_k)$$

Take

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = U_n^t \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \text{i.e.} \quad b = U_n^t c$$

Observe that these b-operators also satisfy the CAR and moreover

$$H_n = 2c^* M_n c - E\mathbb{1} = 2\sum_{k=1}^{n} \lambda_k b_k^* b_k - E\mathbb{1}$$

Thus, this last transform expresses H_n as a system of free Fermions.

Calculating the Many-Body Dynamics

As Gunter also discussed, a simple calculation shows that

$$\tau_t^n(b_k) = e^{-2it\lambda_k} b_k \quad \text{and} \quad \tau_t^n(b_k^*) = e^{2it\lambda_k} b_k^*$$

or

$$\tau_t^n(b) = e^{-2it\Lambda} b$$

A further calculation, using that $c = U_n b$, shows that

$$\tau_t^n(c) = e^{-2itM_n} c \quad \text{or} \quad \tau_t^n(c_j) = \sum_k \left(e^{-2itM_n} \right)_{jk} c_k$$

and so the many-body dynamics (of the c-operators) can be expressed explicitly in terms of the single-particle dynamics.

We can now begin the proof of the main result.

Recall the Main Result:

Theorem

Assume $\{\nu_j\}$ is an i.i.d. random sequence with compactly supported bounded density ρ. There are positive numbers C and η for which, given any integer $n \geq 1$ the bound

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|[\tau_t^n(A), B]\|\right) \leq C\|A\|\|B\|e^{-\eta|k-k'|}$$

holds for any $A \in \mathcal{A}_k$ and $B \in \mathcal{A}_{k'}$ with $1 \leq k < k' \leq n$.

Proof of Main Result:

Fix $1 \leq k < k' \leq n$ as indicated and take $B \in \mathcal{A}_{k'}$.

Consider first the *non-local* $A = c_k$, i.e.:

$$\begin{aligned} [\tau_t^n(c_k), B] &= \sum_{j=1}^{n} \left(e^{-2itM_n} \right)_{kj} [c_j, B] \\ &= \sum_{j=k'}^{n} \left(e^{-2itM_n} \right)_{kj} [c_j, B] \end{aligned}$$

Using the single-particle dynamical localization result, we find that

$$\mathbb{E}\left(\sup_t \| [\tau_t^n(c_k), B] \| \right) \leq 2C' \|B\| \sum_{j=k'}^{n} e^{-\eta'(j-k)} \leq \frac{2C'\|B\|}{1-e^{-\eta'}} e^{-\eta'(k'-k)}$$

This is an estimate of the type desired; excepting that it is for the non-local observable c_k. By taking adjoints, it is clear that a similar result holds for c_k^*.

Proof of Main Result (cont.):

Now take $A = a_k$. Recall $a_k = \sigma_1^z \cdots \sigma_{k-1}^z c_k$. Observe that

$$\begin{aligned} [\tau_t^n(a_k), B] &= \tau_t^n(\sigma_1^z) \cdots \tau_t^n(\sigma_{k-1}^z)[\tau_t^n(c_k), B] + \\ &\quad + [\tau_t^n(\sigma_1^z) \cdots \tau_t^n(\sigma_{k-1}^z), B]\tau_t^n(c_k) \end{aligned}$$

where we have used the automorphism property of $\tau_t^n(\cdot)$, i.e.

$$\tau_t^n(AB) = \tau_t^n(A)\tau_t^n(B)$$

and the Leibnitz rule:

$$[AB, C] = A[B, C] + [A, C]B$$

It is clear then that

$$\|[\tau_t^n(a_k), B]\| \leq \|[\tau_t^n(c_k), B]\| + \|[\tau_t^n(\sigma_1^z) \cdots \tau_t^n(\sigma_{k-1}^z), B]\|$$

Proof of Main Result (cont.):

Now, for any j, the quantity appearing above satisfies

$$\|[\tau_t^n(\sigma_1^z)\cdots\tau_t^n(\sigma_j^z), B]\| \leq \|[\tau_t^n(\sigma_j^z), B]\| + \|[\tau_t^n(\sigma_1^z)\cdots\tau_t^n(\sigma_{j-1}^z), B]\|$$

again by Leibnitz. Moreover,

$$\sigma_j^z = 2a_j^* a_j - \mathbb{1} = 2c_j^* c_j - \mathbb{1}$$

and so

$$[\tau_t^n(\sigma_j^z), B] = 2[\tau_t^n(c_j^*), B]\tau_t^n(c_j) + 2\tau_t^n(c_j^*)[\tau_t^n(c_j), B]$$

and, in fact:

$$\|[\tau_t^n(\sigma_j^z), B]\| \leq 2\|[\tau_t^n(c_j^*), B]\| + 2\|[\tau_t^n(c_j), B]\|$$

Consequently,

$$\|[\tau_t^n(a_k), B]\| \leq 2\sum_{j=1}^{k}\left(\|[\tau_t^n(c_j^*), B]\| + \|[\tau_t^n(c_j), B]\|\right)$$

Proof of Main Result (cont.):

Using our previous result, it is clear that

$$\begin{aligned} \mathbb{E}\left(\sup_t \|[\tau_t^n(a_k), B]\|\right) &\leq \frac{8C'\|B\|}{1-e^{-\eta'}} \sum_{j=1}^{k} e^{-\eta'(k'-j)} \\ &\leq \frac{8C'\|B\|}{(1-e^{-\eta'})^2} e^{-\eta'(k'-k)} \end{aligned}$$

which is the result for $A = a_k$.

By taking adjoints, a similar result holds for $A = a_k^*$.

Using Leibniz again, it is clear that a similar result holds for both $A = a_k^* a_k$ and $A = a_k a_k^*$.

Since these operators form a basis for $\mathcal{A}_k$, this completes the proof.

A Generalization

The isotropic XY model is not the only spin chain that reduces to a system of free Fermions. Consider e.g. the anisotropic XY Spin Hamiltonian

$$H_n = \sum_{j=1}^{n-1} \mu_j[(1+\gamma_j)\sigma_j^x\sigma_{j+1}^x + (1-\gamma_j)\sigma_j^y\sigma_{j+1}^y] + \sum_{j=1}^{n} \nu_j\sigma_j^z$$

with real parameters given by: interaction strengths $\{\mu_j\}$, anisotropy $\{\gamma_j\}$, and field strengths $\{\nu_j\}$. Introducing the same raising and lowering operators and then the Jordan-Wigner transform, this many-body operator can also be written in terms of an effective single-particle Hamiltonian.

Diagonalizing the Hamiltonian

As is discussed in the notes,

$$H_n = C^* M_n C$$

where

$$C = (c_1, c_2, \cdots, c_n, c_1^*, c_2^*, \cdots, c_n^*)^t$$

is a column vector and

$$C^* = (c_1^*, c_2^*, \cdots, c_n^*, c_1, c_2, \cdots, c_n)$$

In this case, the single particle Hamiltonian is a block-matrix

$$M_n = \begin{pmatrix} A_n & B_n \\ -B_n & -A_n \end{pmatrix}$$

Diagonalizing the Hamiltonian(cont.)

with

$$A_n = \begin{pmatrix} \nu_1 & -\mu_1 & & \\ -\mu_1 & \nu_2 & \ddots & \\ & \ddots & \ddots & -\mu_{n-1} \\ & & -\mu_{n-1} & \nu_n \end{pmatrix}$$

and

$$B_n = \begin{pmatrix} 0 & \gamma_1\mu_1 & & & \\ -\gamma_1\mu_1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \gamma_{n-1}\mu_{n-1} \\ & & & -\gamma_{n-1}\mu_{n-1} & 0 \end{pmatrix}$$

The More General Result (at least in words)

In our paper, we prove an analogous result:

If the single particle Hamiltonian M_n is dynamically localized, then the many body Hamiltonian satisfies dynamical localization as well, in the sense that we establish a zero-velocity Lieb-Robinson bound (in average).

What we do not quantify (and what remains an interesting open question) is: Under what conditions is this more general, random single particle system dynamically localized?

For the Experts:

By re-ordering the basis vectors, the block-matrix above is easily seen to be unitarily equivalent to

$$\tilde{M}_n = \begin{pmatrix} -\nu_1\sigma^z & \mu_1 S(\gamma_1) & & & \\ \mu_1 S(\gamma_1)^t & -\nu_2\sigma^z & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \mu_{n-1}S(\gamma_{n-1}) \\ & & & \mu_{n-1}S(\gamma_{n-1})^t & -\nu_n\sigma^z \end{pmatrix}$$

where

$$S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}$$

The question now becomes: If some of these coefficients are random, is such a one-dimensional, single-particle model dynamically localized?