Tutorial on $\mathrm{SU}(2)$ and Spin Waves

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Reminders:

Tornado siren means go to lowest floor of a solid building, away from windows.

For receipts in a currency other than USD$, write the conversion to USD$ and write the conversion rate on the receipt.

Keep all meal receipts and turn them in.

Outline:

1. The Lie Group $\mathrm{SU}(2)$
2. Representations of $\mathrm{SU}(2)$
3. Tensor products of representations
4. Spin waves in the Heisenberg ferromagnet

Lie Group $\mathrm{SU}(2)$

$$\mathrm{SU}(2) = \{U \in M_2(\mathbb{C}) \,:\, U^*U = \mathbb{1}\,,\ \det(U) = 1\}$$

Define $\mathbb{i} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$, $\mathbb{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\mathbb{k} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$.

$$U = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \Rightarrow$$

$$U^*U = \begin{bmatrix} |\alpha|^2 + |\gamma|^2 & \overline{\alpha}\beta + \overline{\gamma}\delta \\ \overline{\beta}\alpha + \overline{\delta}\gamma & |\beta|^2 + |\delta|^2 \end{bmatrix} = \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\text{and} \qquad \det(U) = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \beta\gamma = 1$$

This implies $\delta = \overline{\alpha}$, $\gamma = -\overline{\beta}$ and $|\alpha|^2 + |\beta|^2 = 1$.

So $U = x\mathbb{i} + y\mathbb{j} + z\mathbb{k} + t\mathbb{1}$ with $(x, y, z, t) \in \mathbb{S}^3 \subset \mathbb{R}^4$.

Motivation from $\mathrm{SO}(3)$

Consider for example, the problem from QM

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{x},t) = \left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})\right)\Psi(\mathbf{x},t)\,, \tag{1}$$

for $V(\mathbf{x}) = v(|\mathbf{x}|)$ for some nice function $v : [0,\infty) \to \mathbb{R}$.

$\mathrm{SO}(3) = \{R \in M_3(\mathbb{R}) \ :\ R^T R = I\,,\ \det(R) = 1\}$

For any $R \in \mathrm{SO}(3)$,

- if Ψ solves (1),
- and if $\Phi(\mathbf{x},t) = \Psi(R\mathbf{x},t)$,
- then Φ also solves (1).

For any $R \in \mathrm{SO}(3)$ the operators H and U_R commute, where

$$H = -\frac{1}{2m}\Delta + V(\mathbf{x})$$

and

$$U_R \Psi(\mathbf{x}) = \Psi(R\mathbf{x})\,.$$

$$L_z \Psi(\mathbf{x}) = \frac{1}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\Psi(\mathbf{x}) = \frac{1}{i}\,\frac{d}{dt}\bigg|_{t=0} \Psi(R_t\mathbf{x})\,,$$

where $R_t = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

L_x and L_y defined symmetrically: derivatives of rotations around x and y.

Commutation relations

The commutator is $[A, B] = AB - BA$.

$$[L_x, L_y] = \left[\frac{1}{i}\left(y\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial y}\right)\,,\ \frac{1}{i}\left(z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z}\right)\right]$$

$$= -\left[y\frac{\partial}{\partial z}\,,\ z\,\frac{\partial}{\partial x}\right] - \left[z\frac{\partial}{\partial y}\,,\ x\,\frac{\partial}{\partial z}\right]$$

But $\frac{\partial}{\partial z}\big(z\Psi(\mathbf{x})\big) - z\frac{\partial}{\partial z}\Psi(\mathbf{x}) = \Psi(\mathbf{x})$. So

$$[L_x, L_y] = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} = iL_z\,.$$

And, similarly, $[L_y, L_z] = iL_x$ and $[L_z, L_x] = iL_y$.

Back to $\mathrm{SU}(2)$:

$$\mathbb{i} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \mathbb{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \text{ and } \mathbb{k} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}.$$

The multiplication formulas for the quaternions are

$$\mathbb{i}\mathbb{j} = \mathbb{k}\,,\ \mathbb{j}\mathbb{k} = \mathbb{i}\,,\ \mathbb{k}\mathbb{i} = \mathbb{j}\,,$$

and

$$\mathbb{j}\mathbb{i} = -\mathbb{k}\,,\ \mathbb{k}\mathbb{j} = -\mathbb{i}\,,\ \mathbb{i}\mathbb{k} = -\mathbb{j}\,,.$$

So $[\mathbb{i},\mathbb{j}] = 2\mathbb{k}$, etc.

$S^x = \frac{i}{2}\,\mathbb{i}$, $S^y = \frac{i}{2}\,\mathbb{j}$, $S^z = \frac{i}{2}\,\mathbb{k}$.

$$S^x = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}, \quad S^y = \begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix}, \quad S^z = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$$

Relation between $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$

$\mathrm{SU}(2)$ is the set of unit quaternions
$U = x\mathbb{i} + y\mathbb{j} + z\mathbb{k} + t\mathbb{1}$ with $x^2 + y^2 + z^2 + t^2 = 1$.

A "pure imaginary" quaternion is
$Q = a\mathbb{i} + b\mathbb{j} + c\mathbb{k}$, $(a, b, c) \in \mathbb{R}^3$.

Given a pure imaginary quaternion, UQU^* is also pure imaginary.

By multiplicativity of the ℓ^2-norm, $Q \mapsto UQU^*$ is an isometry.

If $U \neq \pm\mathbb{1}$, let $Q = (x\mathbb{i} + y\mathbb{j} + z\mathbb{k})/\sqrt{x^2 + y^2 + z^2}$.
Let R and S be pure imaginary unit quaternions such that $QR = S$.
Then $UQU^* = Q$ and setting $\cos\theta = t$,

$$URU^* = \cos(2\theta)R + \sin(2\theta)S\,, \quad USU^* = \cos(2\theta)S - \sin(2\theta)R\,.$$

If $U = \pm\mathbb{1}$ then $UQU^* = Q$ for all $Q = a\mathbb{i} + b\mathbb{j} + c\mathbb{k}$.

Irreducible representations

Representations of $\mathrm{SU}(2)$

A representation of $\mathrm{SU}(2)$ is a vector space V along with operators J^x, J^y and J^z on V such that

$$[J^x, J^y] = iJ^z\,, \quad [J^y, J^z] = iJ^x\,, \quad [J^z, J^x] = iJ^y\,.$$

The representation is finite dimensional if V is.

A finite dimensional representation is unitary if V is a Hilbert space and J^x, J^y and J^z are Hermitian.

Fact: Every finite dimensional representation may be equipped with an inner-product to make the representation unitary.

Let j be the largest eigenvalue of S^z. Let $\psi_j \in V$ be an eigenvector.

Define the spin-raising and lowering operators J^+ and J^- as $J^\pm = J^x \pm iJ^y$.

Then $[J^z, J^\pm] = [J^z, J^x] \pm i[J^z, J^y] = iJ^y \pm J^x = \pm J^\pm$.

Hence,

$$\begin{aligned} J^z J^+ \psi_j &= [J^z, J^+]\psi_j + J^+ J^z \psi_j \\ &= J^+\psi_j + J^+ j\psi_j \\ &= (j+1)J^+\psi_j \,. \end{aligned}$$

So $J^+\psi_j = 0$.

Similarly, for any $k \in \{1, 2, \dots\}$, we have $(J^-)^k\psi_j$ is an eigenvector of J^z with eigenvalue $j - k$, unless $(J^-)^k\psi_j = 0$.

Let k_* be the smallest k such that $(J^-)^k\psi_j = 0$.

For $k = 0, \ldots, k_* - 1$, let $\psi_{j-k} = (J^-)^k \psi_j / \|(J^-)^k \psi_j\|$.

We will prove by induction on $k \in \{0, \ldots, k_* - 1\}$ that for $m = j - k$

$$J^- \psi_m \;=\; \sqrt{j(j+1) - m(m-1)}\, \psi_{m-1}\,,$$

where $0\psi_{j-k_*}$ is interpreted as 0.

For $k = 0$, which is $m = j$, it is by definition since $(J^-)^0 \psi_j = I\psi_j = \psi_j = \psi_{j-0}$.

For the induction step suppose that

$$J^- \psi_{m+1} \;=\; \sqrt{j(j+1) - m(m+1)}\, \psi_m\,.$$

Then since $J^+ = (J^-)^*$, we know

$$J^- J^+ \psi_m \;=\; [j(j+1) - m(m+1)] \psi_m\,.$$

$$J^- J^+ \psi_m = [j(j+1) - m(m+1)]\psi_m \,.$$

But it is easy to check the commutation relation $[J^+, J^-] = 2J^z$.

So

$$\begin{aligned} J^+ J^- \psi_m &= [J^+, J^-]\psi_m + 2J^z\psi_m \\ &= [j(j+1) - m(m+1)]\psi_m + 2m\psi_m \\ &= [j(j+1) - m(m-1)]\psi_m \,. \end{aligned}$$

So, $\|J^-\psi_m\|^2 = j(j+1) - m(m-1)$.

So this proves the claim for the formula.
In particular we deduce $J^-\psi_{-j} = 0\psi_{-j-1} = 0$.
So $k_* = 2j$, meaning that we have $2j+1$ linearly independent eigenvectors of J^z, namely $\psi_j, \psi_{j-1}, \dots, \psi_{-j}$.
So $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$.

Irreducible representations of $\mathrm{SU}(2)$

Define $\mathcal{D}^j$ to be the finite dimensional Hilbert space $\mathbb{C}^{2j+1}$ with ortho-normal basis vectors $\psi_j, \psi_{j-1}, \dots, \psi_{-j}$.
Then the representation of $\mathrm{SU}(2)$ on this vector space is given by

$$S^z \psi_m = m\psi_m \,,$$

$$S^+ \psi_m = \sqrt{j(j+1) - m(m+1)}\,\psi_{m+1} \,,$$

$$S^- \psi_m = \sqrt{j(j+1) - m(m-1)}\,\psi_{m-1} \,,$$

and $S^x = (S^+ + S^-)/2$, $S^y = (S^+ - S^-)/(2i)$.

A finite dimensional representation of $\mathrm{SU}(2)$ is irreducible if it admits no non-trivial sub-representations.

We have proved that $\mathcal{D}^j$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ is the complete list of finite dimensional irreps, modulo equivalence.

Relation to the representations of $\mathrm{SO}(3)$

Since $\mathrm{SU}(2)$ maps homomorphically onto $\mathrm{SO}(3)$ (2-to-1) any representation of $\mathrm{SO}(3)$ may be considered to be a representation of $\mathrm{SU}(2)$ by precomposing with this homomorphism.

But not all representations of $\mathrm{SU}(2)$ are representations of $\mathrm{SO}(3)$. More specifically, for the spin-1/2 spin matrices

$$S^x = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}, \quad S^y = \begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix}, \quad S^z = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$$

we can see that $\exp(2\pi i S^z) = -\mathbb{1}$.
In $\mathcal{D}^j$, $S^z\psi_m = m\psi_m$ for $m \in \{j, j-1, \ldots, -j\}$.
So $\exp(2\pi i S^z)\psi_m = e^{2\pi i m}\psi_m$.
But since $\{\mathbb{1}, -\mathbb{1}\}$ are both mapped to the identity in $\mathrm{SO}(3)$, we need $e^{2\pi i m} = 1$. So $j \in \{0, 1, 2\}$ not in $\{\frac{1}{2}, \frac{3}{2}, \ldots\}$.

Tensor products

Just for motivation

We introduced a Schrödinger operator on $L^2(\mathbb{R}^3)$

$$H\Psi(\mathbf{x}) = \Big(-\frac{1}{2}\Delta + V(\mathbf{x})\Big)\Psi(\mathbf{x})$$

for a central potential $V(\mathbf{x}) = v(|\mathbf{x}|)$.

A rotationally invariant Hamiltonian for two particles, ostensibly on $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, is

$$H_{1,2}\Psi(\mathbf{x}_1, \mathbf{x}_2) = \sum_{i=1}^{2} \Big(-\frac{1}{2} \cdot \frac{\partial^2}{\partial \mathbf{x}_i^2} + V(\mathbf{x}_i)\Big)\Psi(\mathbf{x}_1, \mathbf{x}_2) \\ + W(\mathbf{x}_1 - \mathbf{x}_2)\Psi(\mathbf{x}_1, \mathbf{x}_2)\,.$$

If $W(\mathbf{x}) = w(|\mathbf{x}|)$ for some nice $w : [0, \infty) \to \mathbb{R}$,
then if $H_{1,2}\Psi = E\Psi$ then $H_{1,2}\Phi = E\Phi$ where
$\Phi(\mathbf{x}_1, \mathbf{x}_2) = \Psi(R\mathbf{x}_1, R\mathbf{x}_2)$ for $R \in \mathrm{SO}(3)$.

As Hilbert spaces $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ is equivalent to $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$.

If we consider the representation $R \in \mathrm{SO}(3)$ maps to U_R on $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ such that $U_R \Psi(\mathbf{x}_1, \mathbf{x}_2) = \Psi(R\mathbf{x}_1, R\mathbf{x}_2)$ then L^z_{tot} for example is given by

$$L^z_{\mathrm{tot}} \Psi(\mathbf{x}_1, \mathbf{x}_2) \;=\; \frac{1}{i}\,\frac{d}{dt}\bigg|_{t=0} \Psi(R_t\mathbf{x}_1, R_t\mathbf{x}_2)$$

where $R_t = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This leads to

$$L^z_{\mathrm{tot}} \;=\; \frac{1}{i}\left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1}\right) + \frac{1}{i}\left(x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2}\right).$$

On $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$ this is equivalent to $L^z \otimes I + I \otimes L^z$.

Tensor product of two fin. dim. irreps of $\mathrm{SU}(2)$

On $\mathcal{D}^j \otimes \mathcal{D}^{j'}$ consider

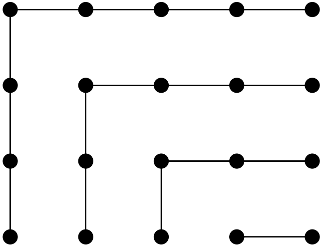
$$S^x_{\mathrm{tot}} = S^x_1 \otimes I_2 + I_1 \otimes S^x_2 \,, \ \dots$$

For $m \in \{j, \dots, -j\}$ let $\psi^{(1)}_m \in \mathcal{D}^j$ be the eigenvector of S^x_1 with eigenvalue m. Similarly for $m \in \{j', \dots, -j'\}$ and $\psi^{(2)}_m \in \mathcal{D}^{j'}$.

Then $S^z_{\mathrm{tot}} \psi^{(1)}_m \otimes \psi^{(2)}_{m'} = (m+m')\psi^{(1)}_m \otimes \psi^{(2)}_{m'}$.

Note

$$(2j+1)(2j'+1) = \sum_{\ell=|j-j'|}^{j+j'} (2\ell+1)\,.$$



$$5 \cdot 4 = 8 + 6 + 4 + 2$$

Let us prove the formula

$$\mathcal{D}^j \otimes \mathcal{D}^{j'} = \bigoplus_{\ell=|j-j'|}^{j+j'} \mathcal{D}^\ell .$$

Total spin

For a representation of $\mathrm{SU}(2)$ the Casimir operator is the total spin

$$\mathbf{S}^2 \,=\, (S^x)^2 + (S^y)^2 + (S^z)^2\,.$$

Note that, for example,

$$\begin{aligned}
[S^z, \mathbf{S}^2] \,&=\, [S^z, (S^x)^2 + (S^y)^2 + (S^z)^2] \\
&=\, [S^z, S^x S^x] + [S^z, S^y S^y] \\
&=\, [S^z, S^x]S^x + S^x[S^z, S^x] + [S^z, S^y]S^y + S^y[S^z, S^y] \\
&=\, iS^y S^x + S^x(iS^y) + (-iS^x)S^y + S^y(-iS^x) \\
&=\, 0\,.
\end{aligned}$$

Using the raising and lowering operators we can also write

$$\mathbf{S}^2 \,=\, (S^z)^2 + \frac{1}{2}\, S^+ S^- + \frac{1}{2}\, S^- S^+\,.$$

Consider $\mathcal{D}^j$. Let ψ_j be the highest weight vector,

$$S^z \psi_j = j\psi_j\,, \quad S^+\psi_j = 0\,, \quad \text{and}$$

$$S^-\psi_j = \sqrt{j(j+1) - j(j-1)}\,\psi_{j-1} = \sqrt{2j}\,\psi_{j-1}\,.$$

So

$$\begin{aligned}\mathbf{S}^2\psi_j &= \left[(S^z)^2 + \frac{1}{2}\,S^+S^- + \frac{1}{2}\,S^-S^+\right]\psi_j \\ &= j^2\psi^j + \frac{1}{2}(\sqrt{2j})^2\psi_j + 0 \\ &= j(j+1)\psi^j\end{aligned}$$

Note that for any other $m \in \{j, \dots, -j\}$ and $\psi_m \in \mathcal{D}^j$, we have $\psi_m = C_m(S^-)^{j-m}\psi_j$. So

$$\mathbf{S}^2\psi_m = C_m\mathbf{S}^2(S^-)^{j-m}\psi_j = C_m(S^-)^{j-m}\mathbf{S}^2\psi_j = j(j+1)\psi_m\,,$$

as well.

Back to tensor products

Suppose $j \geq j'$ and consider $\mathcal{D}^j \otimes \mathcal{D}^{j'}$.

Then $S^z_{\text{tot}} \psi^j \otimes \psi^{j'} = (S^z_1 + S^z_2)\psi^j \otimes \psi^{j'} = (j + j')\psi^j \otimes \psi^{j'}$.

Also $S^+_{\text{tot}} \psi^j \otimes \psi^{j'} = (S^+_1 + S^+_2)\psi^j \otimes \psi^{j'} = 0$.

Therefore $\mathbf{S}^2_{\text{tot}} \psi^j \otimes \psi^{j'} = (j + j')(j + j' + 1)\psi^j \otimes \psi^{j'}$.

Now $(S^-_{\text{tot}})^k \psi^j \otimes \psi^{j'}$ is non-zero for $k = 0, \dots, 2(j + j') - 1$.
Calling the normalized vector $\Psi^{j+j'}_m$, for $m = j + j' - k$,

$$S^z_{\text{tot}} \Psi^{j+j'}_m = m\Psi^{j+j'}_m\,, \qquad \mathbf{S}^2_{\text{tot}} \Psi^{j+j'}_m = \ell(\ell + 1)\Psi^{j+j'}_m\,,$$

for $\ell = j + j'$.

But the eigenspace of S^z_{tot} for eigenvalue $m = j + j' - 1$ has dimension 2 (assuming $j \geq j' > 0$) spanned by $\psi^{j-1} \otimes \psi^{j'}$ and $\psi^{j} \otimes \psi^{j'-1}$.

So there must be another vector Ψ orthogonal to $\Psi^{j+j'}_{j+j'-1}$,

$$\langle \Psi^{j+j'}_{j+j'}\,,\ S^+_{\mathrm{tot}}\Psi\rangle \ = \ \langle S^-_{\mathrm{tot}}\Psi^{j+j'}_{j+j'}\,,\ \Psi\rangle \ = \ 0$$

Since the eigenspace for S^z_{tot} with $m = j + j'$ is spanned by $\Psi^{j+j'}_{j+j'}$, this means $S^+_{\mathrm{tot}}\Psi = 0$.

So Ψ is a highest weight vector with $S^z\Psi = m\Psi$ for $m = j + j' - 1$.

So $\mathbf{S}^2_{\mathrm{tot}}\Psi = \ell(\ell + 1)\Psi$ for $\ell = j + j' - 1$.

Call it Ψ^ℓ_ℓ for $\ell = j + j' - 1$. For $m \in \{\ell, \dots, -\ell\}$ call Ψ^ℓ_m the normalized version of $(S^-_{\mathrm{tot}})^{\ell-m}\Psi^\ell_\ell$.

Proceed inductively.

Let $\mathcal{H}_m^{\mathrm{mag}}$ denote the eigenspace of S_{tot}^z with eigenvalue m.

Then, for example, as long as $j \geq j' > \frac{1}{2}$,

$$\mathcal{H}_{j+j'-2}^{\mathrm{mag}} = \mathrm{span}\{\psi^j \otimes \psi^{j'-2}\,,\ \psi^{j-1} \otimes \psi^{j'-1}\,,\ \psi^{j-2} \otimes \psi^{j'}\}\,.$$

So there is a (normalized) vector Ψ

$$\Psi \in \mathcal{H}_{j+j'-2}^{\mathrm{mag}} \cap \mathrm{span}\{\Psi_{j+j'-2}^{j+j'}, \Psi_{j+j'-2}^{j+j'-1}\}^{\perp}\,.$$

But $\mathrm{span}\{\Psi_{j+j'-1}^{j+j'}, \Psi_{j+j'-1}^{j+j'-1}\} = \mathcal{H}_{j+j'-1}^{\mathrm{mag}}$.

So $\Psi \in \mathcal{H}_{j+j'-2}^{\mathrm{mag}} \cap \mathrm{ran}(S_{\mathrm{tot}}^-)^{\perp}$.

So $S_{\mathrm{tot}}^+ \Psi = 0$.

So Ψ is a highest weight vector with $S_{\mathrm{tot}}^z \Psi = m\Psi$ for $m = j + j' - 2$.

So $\mathbf{S}_{\mathrm{tot}}^2 \Psi = \ell(\ell+1)\Psi$ for $\ell = j + j' - 2$.

This works to give a copy of $\mathcal{D}^\ell$ in $\mathcal{D}^j \otimes \mathcal{D}^{j'}$ for $\ell = j + j', j + j' - 1, \dots$ until you get to a ℓ with $\dim(\mathcal{H}_\ell^{\mathrm{mag}}) \leq \dim(\mathcal{H}_{\ell+1}^{\mathrm{mag}})$.

But, assuming $j \geq j'$, we have

$$\mathcal{H}_{j-j'}^{\mathrm{mag}} = \mathrm{span}\{\psi^{j-2j'} \otimes \psi^{j'}\,,\ \psi^{j-2j'+1} \otimes \psi^{j'}\,,\ \dots\,,\ \psi^{j} \otimes \psi^{-j'}\}$$

That is the biggest subspace because for $m = j - j' - 1$ we lose the last vector since there is no $\psi^{-j'-1}$ in $\mathcal{D}^{j'}$.

So

$$\mathcal{D}^j \otimes \mathcal{D}^{j'} \supseteq \bigotimes_{\ell=|j-j'|}^{j+j'} \mathcal{D}^\ell\,.$$

But since the dimensions are equal, the spaces are equal.

Spin Waves

The Heisenberg Ferromagnet

Suppose $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is a finite graph.

Let $\{\mathbf{i}_1, \dots, \mathbf{i}_N\}$ be an enumeration of $\mathscr{V}$.

Let $\mathcal{H}_{\mathscr{V}} = (\mathbb{C}^2)^{\otimes N} = (\mathcal{D}^{1/2})^{\otimes N}$.

For $a \in \{x, y, z\}$ and $k \in \{1, \dots, N\}$, let

$$S^a_{\mathbf{i}_k} = (\mathbb{1}_{\mathbb{C}^2})^{\otimes(k-1)} \otimes S^a \otimes (\mathbb{1}_{\mathbb{C}^2})^{n-k} .$$

Then the Heisenberg Hamiltonian $H_{\mathscr{G}}$ is the operator on $\mathcal{H}_{\mathscr{V}}$,

$$H_{\mathscr{G}} = \sum_{\{\mathbf{i},\mathbf{j}\} \in \mathscr{E}} h_{\mathbf{i},\mathbf{j}} ,$$

$$h_{\mathbf{i},\mathbf{j}} = \frac{1}{4}\mathbb{1} - S^x_{\mathbf{i}} S^x_{\mathbf{j}} - S^y_{\mathbf{i}} S^y_{\mathbf{j}} - S^z_{\mathbf{i}} S^z_{\mathbf{j}} .$$

Magnons

Restricting attention to $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$h_{1,2}|\uparrow\uparrow\rangle \;=\; h_{12}|\downarrow\downarrow\rangle \;=\; h_{12}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \;=\; 0\,,$$

$$h_{12}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \;=\; |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\,.$$

Going back to $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, given $X \subseteq \mathscr{V}$ let

$$\Psi_X \;=\; \prod_{\mathbf{i}\in X} S_{\mathbf{i}}^- |\uparrow \cdots \uparrow\rangle\,.$$

Then

$$H_{\mathscr{G}}\Psi_X \;=\; \frac{1}{2} \sum_{\substack{\mathbf{i}\in X\\ \mathbf{j}\in X^c\\ \{\mathbf{i},\mathbf{j}\}\in\mathscr{E}}} \left(\Psi_X - \Psi_{X\setminus\{\mathbf{i}\}\cup\{\mathbf{j}\}}\right).$$

Given $f_1, \dots, f_n : \mathscr{V} \to \mathbb{C}$, consider

$$\Psi_{f_1 \otimes \cdots \otimes f_n} = \sum_{\substack{X = \{\mathbf{i}_1, \dots, \mathbf{i}_n\} \\ |X| = n}} \sum_{\pi \in S_n} \prod_{k=1}^{n} f_{\pi(k)}(\mathbf{i}_k) \Psi_X$$

Then

$$H_{\mathscr{G}} \Psi_{f_1 \otimes \cdots \otimes f_n} = \sum_{k=1}^{n} \Psi_{f_1 \otimes \cdots \otimes (-\Delta_{\mathscr{G}} f_k) \otimes \cdots f_n}$$

$$- \frac{1}{2} \sum_{\substack{X = \{\mathbf{i}_1, \dots, \mathbf{i}_n\} \\ |X| = n}} \sum_{k=1}^{n} \sum_{\substack{1 \le j \le n \\ \{\mathbf{i}_j, \mathbf{i}_k\} \in \mathscr{E}}} \sum_{\pi \in S_n} \left(\prod_{\ell \neq k} f_{\pi(\ell)}(\mathbf{i}_\ell) \right) [f_{\pi(k)}(\mathbf{i}_k) - f_{\pi(k)}(\mathbf{i}_j)] \Psi_X \,.$$

If one were to ignore the zero-mode and the problem of neighboring particles,
then one might conclude that the model is comparable to the quantum harmonic oscillator on $\mathcal{G}$ with zero-mode removed.

That would give free energy density

$$-\frac{1}{\beta|\mathscr{V}|}\ln \mathrm{tr}[e^{-\beta H_{\mathscr{G}}}] \approx \frac{1}{\beta|\mathscr{V}|}\sum_{E\in\mathrm{spec}(-\Delta_{\mathscr{G}})\setminus\{0\}} \ln(1-e^{-\beta E})$$

In particular for $\mathscr{G}=\mathbb{T}_N^d$ the discrete torus

$$f_N(\beta) \approx \frac{1}{\beta N^d}\sum_{\xi\in\mathbb{T}_N^d\setminus\{0\}} \ln(1-e^{-2\beta\sum_{i=1}^d \sin^2(\pi\xi_i/N)})$$

$$f_N(\beta) \approx \frac{1}{\beta N^d} \sum_{\xi \in \mathbb{T}_N^d \setminus \{0\}} \ln(1 - e^{-2\beta \sum_{i=1}^d \sin^2(\pi \xi_i / N)})$$

$$\sim \frac{1}{\beta} \int_{[0,1]^d} \ln(1 - e^{-2\beta \sum_{i=1}^d \sin^2(\pi x_i)})\, dx_1 \cdots dx_d$$

$$= \frac{1}{(2\pi)^d \beta^{1+(d/2)}} \int_{[0,2\pi\beta^{1/2}]^d} \ln(1 - e^{-2\beta \sum_{i=1}^d \sin^2(\frac{y_i}{2\beta^{1/2}})})\, dy_1 \cdots dy_d$$

$$\sim \frac{1}{\pi^d \beta^{(2d+1)/2}} \int_{\mathbb{R}^d} \ln(1 - e^{-\|y\|^2/2})\, dy_1 \cdots dy_d$$

This is ignoring important issues. But it is easy to see that it leads to an upper bound in this limit: $N \to \infty$ first and then $\beta \to \infty$.

Correggi, Giuliani and Seiringer

Correggi, Giuliani and Seiringer recently showed how to obtain a matching lower bounds in the limit, $N \to \infty$ then $\beta \to \infty$.

One of their ideas is this: Suppose that Ψ is any eigenvector of $H_{\mathbb{T}_N^d}$ and simultaneously of $\mathbf{S}^2_{\mathrm{tot}}$. Then for eigenvalues E and $S(S+1)$, respectively,

$$E \geq \frac{C}{N^2} \left(\frac{1}{2} N^d - S\right).$$

The proof of this relies on three facts:
$h_{\mathbf{i},\mathbf{j}}$ is a positive semidefinite operator,
a simple discrete Sobolev type inequality (closely related to Poincaré's inequality),
and the formula for addition of angular momenta for $\mathrm{SU}(2)$.