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PART III : QUASI-ADIABATIC EVOLUTION

(Developed by M. B. Hastings in 2004)

Question: Assume that we are given a family of gapped Hamiltonian $H(s)$, that are sums of (quasi)-local interactions. Can we construct a corresponding family of unitaries $U(s)$, such that (i) $\partial_s U(s) = i \cdot D(s) \cdot U(s)$, with $U(0) = \mathbb{1}$ and $D(s)$ a sum of quasi-local interactions,

(ii) For $P_0(s)$ the groundstate projector of $H(s)$, we can write $P_0(s) = U(s)\, P_0(0)\, U^{\dagger}(s)$?

ANSWER: YES! The idea is to create an operator $D(s)$ (Hermitian) that simulates the action of the generator of the true adiabatic evolution on the groundstate subspace, but in a way that preserves locality of interactions as they change the Hamiltonian $H(s)$.

STEP 1 : Assume $H(s)$ is family of gapped Hamiltonians with gap $E_1(s) - E_0(s) = \gamma(s) \geq \gamma > 0$, and $H(s)$ is differentiable

Differentiating $(H(s) - E_0(s)) P_0(s) = 0$, w.r.t. s, we get:

$$\circledast \quad \{\partial_s (H(s) - E_0(s))\} P_0(s) = - (H(s) - E_0(s)) \partial_s P_0(s).$$

Since $P_0^2(s) = P_0(s)$, we have $\partial_s P_0^2(s) = \partial_s P_0(s) = (\partial_s P_0(s)) P_0(s) + P_0(s)(\partial_s P_0(s))$

$\Rightarrow P_0(s) \partial_s P_0(s) P_0(s) = 0$ and $(1 - P_0(s)) \partial_s P_0(s) (1 - P_0(s)) = 0$.

This implies that $\partial_s P_0(s) = (1 - P_0(s))(\partial_s P_0(s)) P_0(s) + P_0(s)(\partial_s P_0(s))(1 - P_0(s))$

Combined with $\circledast$, we get:

$$-(1 - P_0(s)) \frac{\partial_s (H(s) - E_0(s))}{H(s) - E_0(s)} P_0(s) = (1 - P_0(s))(\partial_s P_0(s)) P_0(s)$$

$$\Rightarrow \partial_s P_0(s) = - \left((1 - P_0(s)) \frac{\partial_s H(s)}{H(s) - E_0(s)} P_0(s) + P_0(s) \frac{\partial_s H(s)}{H(s) - E_0(s)} (1 - P_0(s)) \right)$$

where the inverse $(H(s) - E_0(s))^{-1}$ is well-defined in the $(1 - P_0(s))$ subspace.$^{\circledast\circledast}$ Denoting $G(s) = i (H(s) - E_0(s))^{-1} \partial_s (H(s) - E_0(s))$,

~~we have $(G^\dagger(s) = G(s))$ and~~ $\partial_s P_0(s) = i [\hat{G}(s), P_0(s)]$.

Let $\hat{G}(s) = G(s) + G^\dagger(s)$ $\quad = i\, G(s) P_0(s) - i P_0(s) G^\dagger(s)$

$\circledast\circledast \quad (H(s) - E_0(s))^{-1} \equiv (1 - P_0(s)) (H(s) - E_0(s))^{-1} (1 - P_0(s))$.

STEP 2: Recalling that $G(s) = i\,(1-P_0(s))\,(H(s)-E_0(s))^{-1}\,(1-P_0(s))\,\partial_s(H(s)-E_0$

we wish to find an operator $D(s) = D^{\dagger}(s)$, such that

$G(s)\,P_0(s) = D(s)\,P_0(s)$ and $D(s)$ is sum of quasi-local terms.

Recalling that $P_0(s)\,G(s)\,P_0(s) = 0$, we want $P_0(s)\,D(s)\,P_0(s) = 0$

and $(1-P_0(s))\,G(s)\,P_0(s) = (1-P_0(s))\,D(s)\,P_0(s)$, or equivalently,

$$\langle \Psi_n(s) |\, G(s)\, | \Psi_0(s) \rangle = \langle \Psi_n(s) |\, D(s)\, | \Psi_0(s) \rangle, \qquad (*)\,(*)$$

where $|\Psi_n(s)\rangle$ is an eigenstate of $H(s)$ with energy $E_n(s)$.

Now, recall that $E_n(s) - E_0(s) \geq \gamma(s) \geq \gamma$, for $n \geq 1$.

As in PART II (ENERGY FILTERING), we will make use of the "filter" function $w_\gamma(t)$, which satisfied:

(i) $w_\gamma(t) \geq 0$ and $\hat{\omega}_\gamma(\lambda) = 0$, for $|\lambda| \geq \gamma$

(ii) $\int_{-\infty}^{\infty} w_\gamma(t) = 1$ (normalized $\hat{\omega}_\gamma(0)$) and $w_\gamma(t) = w_\gamma(-t)$,

(iii) $w_\gamma(t) \leq \mathcal{O}\left(\exp\{-\frac{2}{7}\gamma|t|/\log^2(\gamma|t|)\}\right)$

order of, up to polynomial factors in $\gamma|t|$.

STEP 3 : We compute $\langle \Psi_n(s) | G(s) | \Psi_0(s) \rangle = \frac{i}{E_n(s)-E_0(s)} \cdot \langle \Psi_n(s) | \partial_s H(s) | \Psi_0(s) \rangle$

Define $D(s) = \int_{-\infty}^{\infty} dt \; w_\gamma(t) \int_0^t du \; e^{iuH(s)} \, \partial_s H(s) \, e^{-iuH(s)}$.

Calculating the term $\langle \Psi_n(s) | D(s) | \Psi_0(s) \rangle$, we get :

$$\langle \Psi_n(s) | D(s) | \Psi_0(s) \rangle = \int_{-\infty}^{\infty} dt \; w_\gamma(t) \int_0^t du \; e^{iu(E_n(s)-E_0(s))} \cdot \langle \Psi_n(s) | \partial_s H(s) | \Psi_0(s) \rangle$$

$$= \int_{-\infty}^{\infty} dt \; w_\gamma(t) \left(\frac{e^{it(E_n(s)-E_0(s))} - 1}{i(E_n(s)-E_0(s))} \right) \cdot \langle \Psi_n(s) | \partial_s H(s) | \Psi_0(s) \rangle$$

$$= \left[\frac{\hat{\omega}_\gamma(E_n(s)-E_0(s))}{i(E_n(s)-E_0(s))} + \frac{i}{E_n(s)-E_0(s)} \right] \langle \Psi_n(s) | \partial_s H(s) | \Psi_0(s) \rangle$$

$$= \langle \Psi_n(s) | G(s) | \Psi_0(s) \rangle, \text{ since } \hat{\omega}_\gamma(E_n(s)-E_0(s)) = 0 \text{ !}$$

Finally, since $w_\gamma(t)$ is an even function, we have :

$$\langle \Psi_0(s) | D(s) | \Psi_0(s) \rangle = \left(\int_{-\infty}^{\infty} dt \; w_\gamma(t) \cdot t \right) \cdot \langle \Psi_0(s) | \partial_s H(s) | \Psi_0(s) \rangle$$

$$= 0, \text{ as desired.}$$

At this point, we have shown that ⊛⊛ in page 3 is valid, which implies that $\partial_s P_0(s) = i\,[D(s), P_0(s)]$ and hence, $P_0(s) = U(s)\, P_0(0)\, U^\dagger(s)$, with $\partial_s U(s) = i\, D(s)\, U(s)$, $U(0) = \mathbb{1}$.

STEP 4: It remains to show that $D(s)$ is a sum of quasi-local terms. Recall that $H(s)$ is a sum of (quasi)-local terms such that $H(s) = \sum_{Z \subseteq \Lambda}' H_Z(s)$, where Z may be taken as a subset of sites within a ball of radius R_0 of site $z \in \Lambda \subseteq \mathbb{Z}^d$.

Then, $\partial_s H(s) = \sum_{Z \subseteq \Lambda}' \partial_s H_Z(s)$, which is also local, with $\partial_s H_Z(s)$ supported on sites that $H_Z(s)$ acted on, non-trivially.

Since we will be making use of Lieb-Robinson bounds (presented in PART I), we assume appropriate bounds on the locality and strength (norm) of $H_Z(s)$ and $\partial_s H_Z(s)$.

Now, we may write $D(s) = \sum_{Z \subseteq \Lambda}' \int_{-\infty}^{\infty} dt\, w_\gamma(t) \int_0^t du\, \tau_u^{H(s)}(\partial_s H_Z(s))$

where $\tau_u^{H(s)}(A) = e^{iuH(s)} A e^{-iuH(s)}$.

Setting $D_Z(s) = \int_{-\infty}^{\infty} dt\, w_\gamma(t) \int_0^t du\, \tau_u^{H(s)}(\partial_s H_Z(s))$, we have $D(s) = \sum_Z' D_Z(s)$. It remains to show that each $D_Z(s)$ can be decomposed into a sum of terms with ↑ support and ↓ strength

STEP 5: (Locality of $D_Z(s)$)

Back in PART II of the series, we proved a Lemma that used Lieb-Robinson Bounds to calculate the worst error we could make by approximating the evolution $T_t^{H(s)}(A_u)$, by a local version $T_t^{H_u(s,r)}(A_u)$, where $H_u(s,r)$ is the Hamiltonian $H(s)$ with interactions restricted within a ball of radius r, around the support of the operator A_u.

Let us now recall that Lemma and the statement of Lieb-Robinson bounds for $H(s) = \sum'_{Z\in\Lambda} H_Z(s)$, where the support of each $H_Z(s)$ is assumed to be $B_Z(1)$, the ball of radius 1, centered on $z \in \Lambda \subseteq \mathbb{Z}^d$. (see p. 3 of Part II)

But, first, we write $D_Z(s) = \sum'_{r\geq 0} D_Z(s,r) + \partial_s H_Z(s)$,

where $D_Z(s,r) = \int_{-\infty}^{\infty} dt\, w_\gamma(t) \int_0^t du \left[T_u^{H_u(s,r+1)}(\partial_s H_Z(s)) - T_u^{H_u(s,r)}(\partial_s H_Z(s)) \right]$

with $H_u(s,r) = \sum'_{\substack{Z: \\ B_Z(1)\subseteq B_u(r)}} H_Z(s)$ and $H_u(s,0) = 0$.

[LOCALITY ESTIMATES DETOUR]

① (Lieb-Robinson Bounds): Given a Hamiltonian $H = \sum_Z' H_Z$ and operators A_X and B_Y, supported on subsets X, Y of the lattice $\mathbb{Z}^d$, the following bound holds:

$$\| [T_t^H(A_X), B_Y] \| \leq 2 \min\{|X|, |Y|\} \, \|A_X\| \cdot \|B_Y\| \, e^{v_0 |t|} \, e^{-\mu_0 d(X,Y)},$$

where $T_t^H(A_X) = e^{itH} A_X e^{-itH}$, $d(X,Y) = \min_{\substack{x \in X \\ y \in Y}} d(x,y)$, $\mu_0 > 0$ and $v_0 = e^{\mu_0}(2 J B_1(d))$, where $J = \max_{Z \in \Lambda} \|H_Z\|$ and $B_1(d) =$ volume of unit ball in $\mathbb{Z}^d$.

② ("Localizing evolution" Lemma): Let $H_u(r) = \sum'_{\substack{Z: \\ B_Z(1) \subseteq B_u(r)}} H_Z$ and $T_t^{H_u(r)}(A_u)$, as above. Then, the following bound holds:

$$\| T_t^{H_u(r+1)}(A_u) - T_t^{H_u(r)}(A_u) \| \leq \|A_u\| \cdot (C_0 \cdot r^{d-1}) \, e^{-\mu_0 r} \cdot e^{v_0 |t|},$$

where C_0 = surface area of the unit ball in $\mathbb{Z}^d$.

NOTE: The proofs of ① and ② are given in PART I and PART II, respectively. In particular, ② follows from an application of ①.

STEP 5 (continued...)

Recall that $D_Z(s) = \sum_{r \geq 0} D_Z(s,r) + \partial_s H_Z(s)$.

It remains to show that $D_Z(s,r)$ has support on $B_u(r+1)$ and strength $\| D_Z(s,r) \| \leq \| \partial_s H_Z(s) \| \cdot f_0(r)$, where f_0 is a rapidly decaying function.

(i) By definition of $D_Z(s,r)$ and $T_u^{H_u(s,r+1)}(\partial_s H_Z(s))$, we see that each interaction term is supported non-trivially on $B_u(r+1)$, the support of $H_u(s,r+1)$.

(ii) Using a similar argument as the bound on p. 6 of PART II, we get: $\| D_Z(s,r) \| \leq \int_{-T}^{T} dt \;\; \omega_\gamma(t) \left(\| \partial_s H_Z(s) \| \cdot C_0 r^{d-1} e^{-\mu_0 r} \right) \int_0^{|t|} e^{v_0 \cdot |u|} du$

$+ \; 2 \, \| \partial_s H_Z(s) \| \int_{|t| \geq T} dt \;\; |t| \cdot \omega_\gamma(t)$

$$\leq \| \partial_s H_Z(s) \| \cdot f_0(r),$$

where $f_0(r) = 2 \left[\dfrac{C_0 \, r^{d-1} e^{-\left(\frac{\mu_0 r}{2}\right)}}{v_0^{\,2}} + \displaystyle\int_{|t| \geq \frac{\mu_0 r}{2 v_0}} dt \;\; |t| \, \omega_\gamma(t) \right]$, where $T = \dfrac{\mu_0 \cdot r}{2 v_0}$.

$\blacksquare$

Conclusions: Recalling the almost-exponential decay of $w_\gamma(t) \sim \exp\{-\frac{2}{7}\gamma|t|/\log^2(\gamma|t|)\}$, we immediately see that the function $f_0(r)$ has decay of the order $\exp\{-c_1 \cdot r/\log^2(c_1 r)\}$ for $c_1 = \frac{2}{7}\left(\frac{\gamma\mu_0}{2v_0}\right)$, which is super-polynomial in r. This implies that the quasi-adiabatic evolution $U(s)$ is generated by a quasi-local Hermitian operator, exactly what we need for Lieb-Robinson-type bounds on evolutions based on $U(s)$.

These Lieb-Robinson bounds are proven using techniques very similar to those in PART I and have applications in proving rigorous results about the properties of low-energy states of interacting Hamiltonians with a gap to the excited sector.

For some good examples, see recent results on the arXiv by Hastings, Nachtergaele, Ogata and Sims (some with co-authors and others individually.)