

Introduction to the Mathematics of the XY -Spin Chain

Günter Stolz

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Abstract

In the following we present an introduction to the mathematical theory of the XY spin chain. The importance of this model lies in the fact, first understood by Lieb, Schultz and Mattis in [4], that the XY spin chain is one of very few “exactly solvable” models in the theory of quantum many-body systems. Lieb, Schultz and Mattis considered the constant coefficient case. In the variable coefficient case considered here, “exactly solvable” should be understood as “reducible to an effective one-particle Hamiltonian”. The key method behind this is the Jordan-Wigner transform, which allows to map the XY chain to a free Fermion system.

1 The isotropic XY -spin chain

For a positive integer n , consider the Hamiltonian

$$H = - \sum_{j=1}^{n-1} \mu_j (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y) - \sum_{j=1}^n \nu_j \sigma_j^Z, \quad (1)$$

acting in $\mathcal{H} = \bigotimes_{j=1}^n \mathbb{C}^2$. The first sum represents a chain of n spins with next neighbors interacting through the X and Y -Pauli matrices. The second sum models the effect of an additional transverse magnetic field acting on the spins.

The parameters μ_j and ν_j are real. We also assume $\mu_j \neq 0$, as otherwise the chain breaks into separate pieces which could be analyzed individually. In most of the physics literature on the XY chain these parameters are chosen as constants $\mu \neq 0$ and ν , but we want to stress here that the methods used also work in the variable coefficient case. The word “isotropic” refers to the fact that the two interaction terms, in the X and Y -Pauli matrices, respectively, have equal weight. We start with this case, mostly because it is easier. The anisotropic XY chain will be considered in Section 5 below.

To be specific, let

$$\sigma_j^\bullet = I \otimes \dots \otimes I \otimes \sigma^\bullet \otimes I \otimes \dots \otimes I, \quad \bullet \in \{X, Y, Z\}, \quad (2)$$

with non-trivial entries in the j -th component of the tensor product, and choose the standard representation

$$\sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3)$$

of the Pauli matrices. We note that, using $\{A, B\} = AB + BA$ to denote anti-commutators,

$$(\sigma^X)^2 = (\sigma^Y)^2 = (\sigma^Z)^2 = I, \quad \{\sigma^X, \sigma^Y\} = \{\sigma^X, \sigma^Z\} = \{\sigma^Y, \sigma^Z\} = 0. \quad (4)$$

We will frequently work with the lowering and raising operators

$$a := \frac{1}{2}(\sigma^X - i\sigma^Y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (5)$$

and

$$a^* = \frac{1}{2}(\sigma^X + i\sigma^Y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (6)$$

Note that

$$a^*a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad aa^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (7)$$

are orthogonal projections. As above we write

$$a_j = I \otimes \dots \otimes I \otimes a \otimes I \otimes \dots \otimes I, \quad (8)$$

$$a_j^* = I \otimes \dots \otimes I \otimes a^* \otimes I \otimes \dots \otimes I. \quad (9)$$

This allows to express the Pauli matrices through

$$\sigma_j^X = a_j + a_j^*, \quad \sigma_j^Y = i(a_j - a_j^*), \quad \sigma_j^Z = 2a_j^*a_j - I. \quad (10)$$

The raising and lowering operators satisfy the mixed commutator and anti-commutator relations

$$\{a_j, a_j^*\} = I, \quad (a_j)^2 = (a_j^*)^2 = 0, \quad (11)$$

$$[a_j, a_k^*] = [a_j^*, a_k] = [a_j, a_k] = 0 \quad \text{for } j \neq k. \quad (12)$$

Using these identities one verifies

$$\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y = 2(a_j^* a_{j+1} + a_{j+1}^* a_j), \quad j = 1, \dots, n-1, \quad (13)$$

and therefore H can be expressed as

$$H = -2 \sum_{j=1}^{n-1} \mu_j (a_j^* a_{j+1} + a_{j+1}^* a_j) - \sum_{j=1}^n \nu_j (2a_j^* a_j - I). \quad (14)$$

2 The Jordan-Wigner transform

Lieb, Schultz and Mattis [4] were the first to realize that the Hamiltonian of the xy -spin chain is equivalent to an effective one-particle system, which, in cases where the Jacobi matrix M representing the one-particle system can be diagonalized (in particular for the constant coefficient case), makes the xy -spin chain explicitly solvable.

The first step in this diagonalization is a Jordan-Wigner transform, where a_j and a_j^* are replaced by a new set of so-called annihilation and creation operators (the meaning of which will become clearer in Section 3)

$$c_j := \sigma_1^Z \dots \sigma_{j-1}^Z a_j, \quad j = 1, \dots, n, \quad (15)$$

and

$$c_j^* = \sigma_1^Z \dots \sigma_{j-1}^Z a_j^*. \quad (16)$$

From this it follows readily that

$$c_j^2 = a_j^2 = 0, \quad (c_j^*)^2 = (a_j^*)^2 = 0, \quad c_j^* c_j = a_j^* a_j, \quad c_j c_j^* = a_j a_j^*, \quad (17)$$

and therefore $\{c_j, c_j^*\} = I$. In fact, the c_j and c_j^* satisfy the *canonical anti-commutation relations* (CAR),

$$\{c_j, c_k^*\} = \delta_{jk} I, \quad \{c_j, c_k\} = \{c_j^*, c_k^*\} = 0 \quad \text{for all } j, k = 1, \dots, n, \quad (18)$$

To verify this for $k \neq j$, use that $\{a_j, \sigma_j^Z\} = 0$.

To express H in terms of the operators c_j and c_k^* , note that

$$a_j^* a_{j+1} = c_j^* \sigma_j^Z c_{j+1} = c_j^* (2c_j^* c_j - I) c_{j+1} = -c_j^* c_{j+1} \quad (19)$$

and, taking adjoints,

$$a_{j+1}^* a_j = -c_{j+1}^* c_j. \quad (20)$$

This leads to

$$\begin{aligned} H &= 2 \sum_{j=1}^{n-1} \mu_j (c_j^* c_{j+1} + c_{j+1}^* c_j) - 2 \sum_{j=1}^n \nu_j c_j^* c_j + \sum_{j=1}^n \nu_j I \\ &= 2c^* M c + E_0 I. \end{aligned} \quad (21)$$

Here $c := (c_1, \dots, c_n)^t$, $c^* = (c_1^*, \dots, c_n^*)$, $E_0 := \sum_j \nu_j$ and M is the symmetric Jacobi matrix

$$M = (M_{jk})_{j,k=1}^n := \begin{pmatrix} -\nu_1 & \mu_1 & & & \\ \mu_1 & \ddots & \ddots & & \\ & \ddots & \ddots & \mu_{n-1} & \\ & & \mu_{n-1} & -\nu_n & \end{pmatrix}. \quad (22)$$

In the theory of the XY chain it turns out that M plays the role of an *effective one-particle Hamiltonian*. In principle, the many body Hamiltonian H (acting on a 2^n -dimensional Hilbert space) can be fully understood from properties of the one-particle Hamiltonian M (which acts on an n -dimensional Hilbert space).

3 Diagonalization of H

There is an orthogonal matrix U (with real entries) such that

$$UMU^t = \Lambda = \text{diag}(\lambda_j), \quad (23)$$

and therefore

$$e^{-iMt} = U^t e^{-i\Lambda t} U = U^t \text{diag}(e^{-i\lambda_j t}) U. \quad (24)$$

Define $b = (b_1, \dots, b_n)^t$ by

$$b = Uc. \quad (25)$$

By the bi-linearity of $\{\cdot, \cdot\}$ and orthogonality of U it follows readily that b_j , $j = 1, \dots, n$ satisfies the CAR. Moreover,

$$\begin{aligned} H &= 2c^* M c + E_0 I \\ &= 2c^* U^t \Lambda U c + E_0 I \\ &= 2b^* \Lambda b + E_0 I \\ &= \sum_{j=1}^n 2\lambda_j b_j^* b_j + E_0 I, \end{aligned} \quad (26)$$

i.e. in terms of the new creation and annihilation operators H takes the form of a Hamiltonian of a *free Fermion system*. Assuming knowledge of the λ_j , this operator can be explicitly diagonalized. This makes use of important algebraic properties of systems of operators satisfying the CAR. For a somewhat useful introduction or a reminder of these properties see [5]. For a brief treatment see Proposition II.6.2 in [7]. Some of them are:

- The operators $b_j^* b_j$, $j = 1, \dots, n$, are pairwise commuting orthogonal projections, meaning that they can be simultaneously diagonalized.
- The intersection of the kernels of $b_j^* b_j$ (which is the same as the kernel of b_j) contains at least one non-trivial normalized vector Ω , i.e. $b_j^* b_j \Omega = 0$, $j = 1, \dots, n$.
- For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0, 1\}^n$ use successive *creation operators* b_j^* to define

$$\psi_\alpha = (b^*)^\alpha \Omega := (b_1^*)^{\alpha_1} \dots (b_n^*)^{\alpha_n} \Omega, \quad (27)$$

which are an orthonormal system of common eigenvectors for the $b_j^* b_j$:

$$b_j^* b_j \psi_\alpha = \begin{cases} 0, & \text{if } \alpha_j = 0, \\ \psi_\alpha, & \text{if } \alpha_j = 1. \end{cases} \quad (28)$$

- In the given case $\dim \mathcal{H} = 2^n$, and thus the vectors ψ_α form an orthonormal basis. In particular, the normalized vector Ω is unique up to a trivial phase. It is frequently referred to as the “vacuum vector”.

The vectors ψ_α are eigenvectors for H ,

$$H\psi_\alpha = \left(2 \sum_{j:\alpha_j=1} \lambda_j + E_0 \right) \psi_\alpha, \quad (29)$$

which gives the spectrum of H and, in particular, the ground state energy

$$2 \sum_{j=1}^n \min\{0, \lambda_j\} + E_0 \quad (30)$$

and corresponding ground state $\varphi_0 = \psi_{\alpha^{(0)}}$, where

$$\alpha_j^{(0)} = \begin{cases} 1, & \text{if } \lambda_j < 0, \\ 0, & \text{else.} \end{cases} \quad (31)$$

which is non-degenerate if and only if $\lambda_j \neq 0$ for all j .

Note that $E_0 = \sum_j \nu_j = -\text{tr } M = -\sum_j \lambda_j$. Using this we can re-write (29) and describe the spectrum of H as

$$\sigma(H) = \{ 2 \sum_{j:\alpha_j=1} \lambda_j - \text{tr } M : \alpha \in \{0, 1\}^n \}. \quad (32)$$

We also note that the ψ_α are eigenvectors of the ‘‘total particle number’’ operator $\mathcal{N} := \sum_j b_j^* b_j$. In fact, by (28),

$$\mathcal{N}\psi_\alpha = N_\alpha \psi_\alpha, \quad N_\alpha := \#\{j : \alpha_j = 1\}. \quad (33)$$

The operator \mathcal{N} could alternatively be defined by using the operators c_j or a_j since we have

$$\mathcal{N} = \sum_j b_j^* b_j = \sum_j c_j^* c_j = \sum_j a_j^* a_j, \quad (34)$$

where (25) and orthogonality of U is used. The latter allows for a more concrete description of the eigenspaces of \mathcal{N} : Writing

$$e_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (35)$$

for the canonical basis vectors of \mathbb{C}^2 and

$$e_\alpha := e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n}, \quad \alpha \in \{0, 1\}^n \quad (36)$$

for the product basis of $\bigotimes^n \mathbb{C}^2$, it follows from $\mathcal{N} = \sum_j a_j^* a_j$ and (7) that $\mathcal{N}e_\alpha = N_\alpha e_\alpha$. Thus the eigenspace of \mathcal{N} to any integer $k \in \{0, 1, \dots, N\}$ is spanned by the set of all e_α with exactly k up-spins, and thus has dimension $\binom{n}{k}$.

The fact that H commutes with \mathcal{N} implies that all these eigenspaces are invariant under H , which could also be verified directly from (14). Thus H conserves the number of up-spins in each basis state, which is frequently referred to as *conservation of particle number* and provides the reason for calling \mathcal{N} the total particle number operator.

4 Dynamics of the creation and annihilation operators

For a bounded linear operator A on \mathcal{H} we will write

$$\tau_t(A) := e^{itH} A e^{-itH} \quad (37)$$

for the Heisenberg dynamics under H .

We first observe that, for all $k = 1, \dots, n$,

$$\tau_t(b_k) = e^{-2it\lambda_k} b_k \quad \tau_t(b_k^*) = e^{2it\lambda_k} b_k^*. \quad (38)$$

To see the first identity (the second follows by taking adjoints) check by Taylor expansion, using $b_k^2 = 0$, that

$$e^{2it\lambda_k b_k^* b_k} b_k = b_k, \quad (39)$$

and similarly, using $b_k b_k^* b_k = (-b_k^* b_k + I) b_k = b_k$,

$$b_k e^{-2it\lambda_k b_k^* b_k} = e^{-2it\lambda_k} b_k. \quad (40)$$

By combining this, we find (38) from

$$\begin{aligned} \tau_t(b_k) &= \prod_{\ell} e^{2it\lambda_{\ell} b_{\ell}^* b_{\ell}} b_k \prod_m e^{-2it\lambda_m b_m^* b_m} \\ &= e^{2it\lambda_k b_k^* b_k} b_k e^{-2it\lambda_k b_k^* b_k} \\ &= e^{-2it\lambda_k} b_k. \end{aligned} \quad (41)$$

Remark: A different proof of (38) uses that

$$\frac{d}{dt} \tau_t(b_k) = -i \tau_t([b_k, H]) \quad (42)$$

$$= -2i \sum_{\ell} \lambda_{\ell} \tau_t([b_k, b_{\ell}^* b_{\ell}]) \quad (43)$$

$$= -2i \lambda_k \tau_t([b_k, b_k^* b_k]) \quad (44)$$

$$= -2i \lambda_k \tau_t(b_k). \quad (45)$$

The unique solution of this differential equation with initial condition $\tau_0(b_k) = b_k$ is given by (38).

Using (25), (38) and (24), we find the dynamics of the c_k , working in convenient vector notation,

$$\begin{aligned} \tau_t(c) &= e^{itH} c e^{-itH} \\ &= e^{itH} U^t b e^{-itH} \\ &= U^t e^{itH} b e^{-itH} \\ &= U^t e^{-2it\Lambda} b \\ &= U^t e^{-2it\Lambda} U c \\ &= e^{-2iMt} c, \end{aligned} \quad (46)$$

or, in components,

$$\tau_t(c_j) = \sum_{\ell} v_{j\ell}(t)c_{\ell}, \quad (47)$$

where $v_{j\ell}(t) := (e^{-2iMt})_{j\ell}$. By taking adjoints we get

$$\tau_t(c_j^*) = \sum_{\ell} \overline{v_{j\ell}(t)}c_{\ell}^*. \quad (48)$$

What this means is that information on the time evolution $v_{j\ell}(t)$ of the effective one-particle Hamiltonian M can be turned into information on the Heisenberg evolution $\tau_t(\cdot)$ under the many-body Hamiltonian H . However, a difficulty arises from the fact that the Jordan Wigner transform is non-local, meaning that the operators c_j act non-locally (i.e. on many spins simultaneously). In order to get information on the Heisenberg evolution of local operators such as a_j and a_j^* from this, one needs to find ways to “undo” the Jordan-Wigner transform, which is generally a non-trivial task.

One example where undoing of Jordan-Wigner turned out to be possible is described in [2]. There the case of a random magnetic field, described by choosing the parameters ν_j as i.i.d. random variables, is considered. In this case the effective Hamiltonian M becomes the Anderson model. Known results on dynamical localization for the latter allow to deduce a so-called zero-velocity Lieb-Robinson bound for the XY chain in random field, a result which is interpreted as absence of information transport in the spin chain [2].

5 The anisotropic XY chain

We now consider an anisotropic generalization of (1), namely the Hamiltonian

$$H_{\gamma} = - \sum_{j=1}^{n-1} \mu_j [(1 + \gamma_j)\sigma_j^X \sigma_{j+1}^X + (1 - \gamma_j)\sigma_j^Y \sigma_{j+1}^Y] - \sum_{j=1}^n \nu_j \sigma_j^Z \quad (49)$$

in $\mathcal{H} = \bigotimes_{j=1}^n \mathbb{C}^2$. The additional parameters γ_j describe the anisotropy in the two interaction terms. We will generally assume $\gamma_j \in [0, 1]$, where the choice $\gamma_j = 0$ recovers the isotropic XY chain and $\gamma_j = 1$ gives the quantum Ising chain, where only the interaction terms proportional to $\sigma_j^X \sigma_{j+1}^X$ appear. We use the notation H_{γ} , with γ being short for $(\gamma_j)_{j=1}^n$, to better distinguish the anisotropic XY chain from the isotropic XY chain considered above.

Below we provide a mathematically streamlined version of a transformation described in [4] for the constant coefficient case (and without magnetic field), adapted to the case of variable coefficients μ_j , γ_j and ν_j . In addition to the Jordan-Wigner transform this uses a “Bogoliubov transformation” (of which (25) can be considered as a special case) to reduce the Hamiltonian to a free Fermion system.

Lieb, Schultz and Mattis [4] considered the model (49) without magnetic field, constant $\mu_j = 1$ and constant $\gamma_j = \gamma \in (0, 1)$. A constant transverse magnetic field was added in [3] and [1]. The Ising model case $\gamma = 1$ with constant transverse field was studied in [6].

Using facts from Section 1 as well as

$$\sigma_j^X \sigma_{j+1}^X - \sigma_j^Y \sigma_{j+1}^Y = 2(a_j a_{j+1} + a_{j+1}^* a_j^*) \quad (50)$$

we can express H_γ in terms of the operators a_j as

$$H_\gamma = -2 \sum_{j=1}^{n-1} \mu_j [a_j a_{j+1}^* + a_{j+1} a_j^* + \gamma_j (a_j a_{j+1} + a_{j+1}^* a_j^*)] - \sum_{j=1}^n \nu_j (2a_j^* a_j - I). \quad (51)$$

With c_j as above, using (17), (19), (20) as well as

$$a_j a_{j+1} = c_j c_{j+1}, \quad a_{j+1}^* a_j^* = c_{j+1}^* c_j^*, \quad (52)$$

we get

$$\begin{aligned} H_\gamma &= -2 \sum_{j=1}^{n-1} \mu_j [c_j c_{j+1}^* + c_{j+1} c_j^* + \gamma_j (c_j c_{j+1} + c_{j+1}^* c_j^*)] - \sum_{j=1}^n \nu_j (2c_j^* c_j - I) \\ &= - \sum_{j=1}^{n-1} \mu_j [c_j c_{j+1}^* - c_{j+1}^* c_j + c_{j+1} c_j^* - c_j^* c_{j+1} + \gamma_j (c_j c_{j+1} - c_{j+1} c_j + c_{j+1}^* c_j^* - c_j^* c_{j+1}^*)] \\ &\quad - \sum_{j=1}^n \nu_j (c_j^* c_j - c_j c_j^*) \\ &= \mathcal{C}^* \tilde{M} \mathcal{C}. \end{aligned} \quad (53)$$

where we have also used the anti-commutation properties (18). In the last line we have set

$$\mathcal{C} = (c_1, \dots, c_n, c_1^*, \dots, c_n^*)^t, \quad (54)$$

a ‘‘column vector’’, and interpret

$$\mathcal{C}^* = (c_1^*, \dots, c_n^*, c_1, \dots, c_n) \quad (55)$$

as a ‘‘row vector’’. We use the block matrix

$$\tilde{M} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}, \quad (56)$$

where $A = M$ from (22) and

$$B = \begin{pmatrix} 0 & \gamma_1 \mu_1 & & & & \\ -\gamma_1 \mu_1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \gamma_{n-1} \mu_{n-1} \\ & & & & -\gamma_{n-1} \mu_{n-1} & 0 \end{pmatrix}. \quad (57)$$

Note here that $A^* = A^t = A$ and $B^* = B^t = -B$, and thus $\tilde{M}^* = \tilde{M}^t = \tilde{M}$. Block matrices of the form (56) have many interesting properties (and their appearance in physics is not restricted to the theory of quantum spin chains). One of these properties is that

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tilde{M} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = -\tilde{M}, \quad (58)$$

i.e. \tilde{M} is unitarily equivalent to $-\tilde{M}$ and, in particular, $\sigma(\tilde{M}) = -\sigma(\tilde{M})$.

Let $S = A + B$ and $0 \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_n$ be the singular values of S , i.e. the eigenvalues of $(S^*S)^{1/2}$, counted with multiplicity. Let $\tilde{\Lambda} := \text{diag}(\tilde{\lambda}_1, \dots, \tilde{\lambda}_n)$. The singular value decomposition of S gives orthogonal matrices U and V such that

$$USV^t = U(A + B)V^t = \tilde{\Lambda}. \quad (59)$$

This implies

$$\tilde{\Lambda} = \tilde{\Lambda}^t = VS^tU^t = V(A - B)U^t. \quad (60)$$

Let

$$W := \frac{1}{2} \begin{pmatrix} V + U & V - U \\ V - U & V + U \end{pmatrix}. \quad (61)$$

Then W is an orthogonal $2n \times 2n$ -matrix. This can be checked by directly verifying that $WW^t = I$, or, alternatively, by noting that

$$SW S^{-1} = \begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix}, \quad (62)$$

with the orthogonal matrix

$$S := \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix}. \quad (63)$$

A calculation shows that W diagonalizes \tilde{M} via

$$W \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} W^t = \begin{pmatrix} \tilde{\Lambda} & 0 \\ 0 & -\tilde{\Lambda} \end{pmatrix}. \quad (64)$$

This leads to another interesting property of block matrices of the form (56), namely that

$$S = A + B \text{ is invertible} \iff \text{All } \lambda_j \neq 0 \iff \tilde{M} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} \text{ is invertible.} \quad (65)$$

It is easily checked that

$$\mathcal{B} := WC \quad (66)$$

is of the form

$$\mathcal{B} = (b_1, \dots, b_n, b_1^*, \dots, b_n^*)^t. \quad (67)$$

That the c_j , $j = 1, \dots, n$, satisfy CAR is equivalent to

$$\mathcal{C}\mathcal{C}^* + J(\mathcal{C}\mathcal{C}^*)^t J = I_{2n}, \quad (68)$$

the $2n \times 2n$ -identity matrix (where the 1's and 0's which appear are understood as the operators I and 0 on \mathcal{H}). Here

$$J := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} = J^t.$$

Note that $JW = WJ$ and $JW^t = W^tJ$. By (66),

$$\begin{aligned}
\mathcal{B}\mathcal{B}^* + J(\mathcal{B}\mathcal{B}^*)^t J &= WCC^*W^t + J(WCC^*W^t)^t J \\
&= W(CC^* + J(CC^*)^t J)W^t \\
&= WI_{2n}W^t \\
&= I_{2n}.
\end{aligned} \tag{69}$$

Thus the b_j , $j = 1, \dots, n$, satisfy CAR as well. Essentially, what we have shown here is the fact that (66) is a *Bogoliubov transformation*.

This Bogoliubov transformation diagonalizes H_γ : By (53) and (64) we have

$$\begin{aligned}
H_\gamma &= C^* \tilde{M} C = \mathcal{B}^* W \tilde{M} W^t \mathcal{B} = \mathcal{B}^* \begin{pmatrix} \tilde{\Lambda} & 0 \\ 0 & -\tilde{\Lambda} \end{pmatrix} \mathcal{B} \\
&= \sum_{j=1}^n \tilde{\lambda}_j (b_j^* b_j - b_j b_j^*) \\
&= 2 \sum_{j=1}^n \tilde{\lambda}_j b_j^* b_j - \tilde{E}_0 I,
\end{aligned} \tag{70}$$

where $\tilde{E}_0 = \sum_{j=1}^n \tilde{\lambda}_j$. Thus H_γ has been written in the form of a free fermion system.

As in Section 3 we can argue that the intersections of the kernels of the b_j is one-dimensional, i.e. that they contain an essentially unique vacuum vector $\tilde{\Omega}$ from which an ONB of eigenvectors of H_γ is found as

$$\psi_\alpha = (b_1^*)^{\alpha_1} \dots (b_n^*)^{\alpha_n} \tilde{\Omega}, \quad \alpha \in \{0, 1\}^n. \tag{71}$$

The corresponding eigenvalues are $2 \sum_{j:\alpha_j=1} \tilde{\lambda}_j - \tilde{E}_0$, so that one gets

$$\sigma(H_\gamma) = \left\{ 2 \sum_{j:\alpha_j=1} \tilde{\lambda}_j - \tilde{E}_0 : \alpha \in \{0, 1\}^n \right\}. \tag{72}$$

All $\tilde{\lambda}_j$ are non-negative, which leads to expressions for the ground state and ground state energy which, at least formally, look simpler than what was obtained for the isotropic case in Section 3 above: The ground state energy of H_γ is

$$E_0 = -\tilde{E}_0 = -\text{tr}(S^* S)^{1/2}. \tag{73}$$

If $S = A + B$ (or, equivalently by (65), \tilde{M}) is invertible, then the vacuum vector $\tilde{\Omega}$ is the unique ground state of H_γ .

For theoretical arguments it is a useful property that the ground state coincides with the vacuum (which was not the case in Section 3 above). From this point of view, the diagonalization procedure described in the current section is superior to the one for the isotropic case described earlier. However, in concrete examples there is also a price to pay, which comes in the form of having to find a the singular value decomposition (59) of the non-self-adjoint matrix $A + B$ instead of just the eigenvalues and eigenvectors of the self-adjoint A .

Remark: Of course, the isotropic XY -chain should arise as a special case of the anisotropic XY -chain. Indeed, the Hamiltonians in (1) and (51) coincide in the case where all $\gamma_j = 0$, which is expressed by the fact that M from (22) reappears as A in (56), while the off-diagonal blocks $\pm B$ in (56) vanish in the isotropic case. The reason that (70) takes a form different from (26) is that the λ_j and $\tilde{\lambda}_j$ are chosen differently. In (26) the λ_j are chosen as eigenvalues of M , while when using (70) in the isotropic case the $\tilde{\lambda}_j$ are eigenvalues of $|A| = |M|$. Essentially, treating the isotropic xy -chain as a special case of the methods introduced for the anisotropic xy -chain amounts to block-diagonalizing $\begin{pmatrix} |M| & 0 \\ 0 & -|M| \end{pmatrix}$ instead of $\begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}$, which are, of course, unitarily equivalent.

We end this section by describing how the arguments on dynamics from Section 4 can be extended to the anisotropic case. For the Heisenberg dynamics of b_k and b_k^* one finds as before

$$\tau_t(b_k) = e^{-2it\lambda_k} b_k \quad \tau_t(b_k^*) = e^{2it\lambda_k} b_k^*, \quad (74)$$

i.e.

$$\tau_t(\mathcal{B}) = \begin{pmatrix} e^{-2it\Lambda} & 0 \\ 0 & e^{2it\Lambda} \end{pmatrix} \mathcal{B}. \quad (75)$$

Furthermore,

$$\begin{aligned} \tau_t(\mathcal{C}) &= e^{itH_\gamma} \mathcal{C} e^{-itH_\gamma} \\ &= e^{itH_\gamma} W^t \mathcal{B} e^{-itH_\gamma} \\ &= W^t e^{itH_\gamma} \mathcal{B} e^{-itH_\gamma} \\ &= W^t \begin{pmatrix} e^{-2it\Lambda} & 0 \\ 0 & e^{2it\Lambda} \end{pmatrix} \mathcal{B} \\ &= W^t \begin{pmatrix} e^{-2it\Lambda} & 0 \\ 0 & e^{2it\Lambda} \end{pmatrix} W \mathcal{C} \\ &= e^{-2it\tilde{M}} \mathcal{C}, \end{aligned} \quad (76)$$

or, writing out the first n components,

$$\tau_t(c_j) = \sum_{\ell=1}^n v_{j,\ell}(2t) c_\ell + \sum_{\ell=1}^n v_{j,n+\ell}(2t) c_\ell^*, \quad j = 1, \dots, n, \quad (77)$$

where

$$v_{j,\ell}(t) := \left(e^{-i\tilde{M}t} \right)_{j,\ell}. \quad (78)$$

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