Jordan Wigner Transformation and zero-velocity Lieb-Robinson-Bound for a Random Gamma Matrix Model

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Gamma Matrix Model

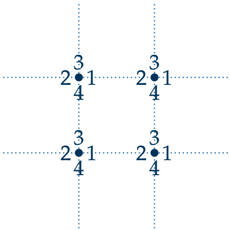
Wosiek '82, Szczerba '85, Yao *et al.* '09

Consider the operator on $\mathfrak{H} = \bigotimes_{x\in\Lambda} \mathbb{C}^4$

$$H_\Lambda^\Gamma = \mu \sum_{x\in\Lambda} \left(\Gamma_x^1 \Gamma_{x+e_1}^{25} + \Gamma_x^3 \Gamma_{x+e_2}^{45} - \Gamma_x^{15} \Gamma_{x+e_1}^{2} - \Gamma_x^{35} \Gamma_{x+e_2}^{4} \right) + \sum_{x\in\Lambda} \nu_x \Gamma_x^5$$

on a square lattice $\Lambda = [1, L]^2 \subset \mathbb{Z}^2$ with *periodic boundary conditions*

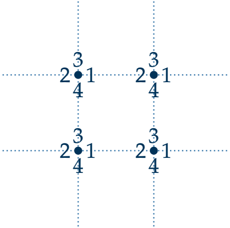
- $\mu > 0$, $\{\nu_x\}_{x\in\Lambda} \subset \mathbb{R}$
- Γ^a, $a = 1, \ldots, 4$, Dirac Gamma Matrices
- $\{\Gamma^a, \Gamma^b\} = \Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\delta_{ab}\mathbb{1}$
- $\Gamma^{ab} = \frac{1}{2\mathrm{i}}[\Gamma^a, \Gamma^b] = -\mathrm{i}\Gamma^a\Gamma^b$



Gamma Matrix Model

Conserved Fluxes

- plaquette operators/fluxes
  $W_P = -\Gamma^{13}_x \Gamma^{32}_{x+e_1} \Gamma^{24}_{x+e_1+e_2} \Gamma^{41}_{x+e_2}$
- global fluxes $W_X = \prod_{x_1=0}^{L-1} \Gamma^1_x \Gamma^2_{x+e_1}$,
  $W_Y = \prod_{x_2=0}^{L-1} \Gamma^3_x \Gamma^4_{x+e_2}$
- $W^*_\alpha = W_\alpha$, $W^2_\alpha = \mathbb{1}$ for all $\alpha = P, X, Y$
- $[W_P, W_Q] = 0$, and $[W_P, H^\Gamma] = 0$ for all P, Q elementary plaquettes
  - Hilbert space splits into sectors of fixed flux configurations $\{W_\alpha\} = \{w_\alpha\}$, $w_\alpha \in \{\pm 1\}$



Gamma Matrix Model

Auxiliary Hilbert Space

- for technical reasons: need to introduce auxiliary Hilbert space $\mathfrak{H}_0 \simeq \mathbb{C}^2$
- Γ_0 self-adjoint, $\Gamma_0^2 = \mathbb{1}$, $\mathrm{Tr}\Gamma_0 = 0$ on $\mathfrak{H}_0$

$$
\begin{aligned}
\widetilde{H}_\Lambda^\Gamma = \mathbb{1} \otimes \Bigg[\mu \sum_{x\in\Lambda} \left(\Gamma_x^1 \Gamma_{x+e_1}^{25} + \Gamma_x^3 \Gamma_{x+e_2}^{45} - \Gamma_x^{15} \Gamma_{x+e_1}^{2} - \Gamma_x^{35} \Gamma_{x+e_2}^{4} \right) + \sum_{x\in\Lambda} \nu_x \Gamma_x^5 \Bigg] \\
+ (\Gamma_0 - \mathbb{1}) \otimes \mu \Big(\Gamma_v^1 \Gamma_{v+e_1}^{25} + \Gamma_{v-e_1}^{1} \Gamma_v^{25} + \Gamma_v^3 \Gamma_{v+e_2}^{45} + \Gamma_{v-e_2}^{3} \Gamma_v^{45} \\
- \Gamma_v^{15} \Gamma_{v+e_1}^{2} - \Gamma_{v-e_1}^{15} \Gamma_v^{5} + \Gamma_v^{35} \Gamma_{v+e_2}^{4} + \Gamma_{v-e_2}^{35} \Gamma_v^{4} \Big) + (\Gamma_0 - \mathbb{1}) \otimes \nu_v \Gamma_v^5
\end{aligned}
$$

on $\mathfrak{H} = \mathbb{C}^2 \otimes \bigotimes_{x\in\Lambda} \mathbb{C}^4$, $v = (1,1)$.

- $\Gamma_0 \otimes \mathbb{1}$ commutes with $\widetilde{H}^\Gamma$, eigenstates ψ of $\widetilde{H}^\Gamma$ can be classified by $(\Gamma_0 \otimes \mathbb{1})\psi = \pm\psi$

Gamma Matrix Model

Jordan Wigner Transformation

The extended Gamma Matrix Model $\widetilde{H}^\Gamma$ with constraints

(i) $W_P = \mathbb{1}$ for all elementary plaquettes P and

(ii) $W_X = W_Y = \mathbb{1}$

is unitarily equivalent to

$$H = 2\mu \sum_{xy\in E} (c_x^* c_y + c_y^* c_x) + \sum_{x\in\Lambda} \nu_x (2c_x^* c_x - \mathbb{1}) \text{ on } \bigotimes_{x\in\Lambda} \mathbb{C}^2.$$

- constrained Hamiltonian $\widetilde{H}^\Gamma_\Xi = \Xi \widetilde{H}^\Gamma \Xi \big|_{\operatorname{ran}\Xi}$
- projection operator $\Xi = \left(\frac{\mathbb{1}+W_X}{2}\right)\left(\frac{\mathbb{1}+W_Y}{2}\right)\prod_P \left(\frac{\mathbb{1}+W_P}{2}\right)$
- $\exists$ unitary isomorphism U s.t. $\widetilde{H}^\Gamma_\Xi = UHU^*$

The Wosiek Szczerba Method

Majorana Operators

- c_x, c_x^* fermionic creation and annihilation operators on a finite, symmetric directed graph $\mathbb{L} = (\Lambda, E)$

$$H = 2\mu \sum_{xy\in E} \left(c_x^* c_y + c_y^* c_x\right) + \sum_{x\in\Lambda} \nu_x (2c_x^* c_x - \mathbb{1})$$

- introduce *Majorana operators* for each site: $\xi_x = c_x^* + c_x$, $\eta_x = -\mathrm{i}(c_x - c_x^*)$ with the properties

$$\begin{aligned} \zeta_x^* &= \zeta_x \\ \{\zeta_x, \varsigma_y\} &= 2\delta_{xy}\delta_{\zeta\varsigma} \quad \text{with} \quad \zeta, \varsigma \in \{\xi, \eta\} \end{aligned}$$

- $H = \mu \sum_{xy\in E} \left(\mathrm{i}\xi_x\eta_y - \mathrm{i}\eta_x\xi_y\right) + \sum_{x\in\Lambda} \nu_x \mathrm{i}\xi_x\eta_x$

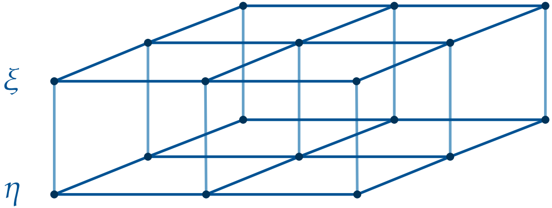
Double Lattice

$\widetilde{\mathbb{L}}$ directed graph with vertex set $\widetilde{\Lambda} = \Lambda \times \{\xi, \eta\}$ and edge set $\tilde{E}$, where $(x_\zeta, y_\varsigma) = ((x,\zeta),(y,\varsigma)) \in \tilde{E}$ if

(i) $xy \in E$, or

(ii) $x = y$ and $\zeta \neq \varsigma$

- Define standard link operators $S(w_\zeta, z_\varsigma) = \mathrm{i}\zeta_w \varsigma_z$



Link Operators

$\{S(\ell) : \ell \in \tilde{E}\}$ satisfy the *link algebra*, if

(i) $S(\ell)^* = S(\ell)$, $(S(\ell))^2 = \mathbb{1}$ for all $\ell \in \tilde{E}$

(ii) $\{S(\ell), S(\ell')\} = 0$ if the edges $\ell, \ell' \in E$ have one common vertex, and
$[S(\ell), S(\ell')] = 0$ otherwise

(iii) $\mathrm{Tr}\left(\prod_{x\in\Lambda} S(x_\xi, x_\eta)\right) = 0$

(iv) Let $\gamma = \ell_1 \circ \ell_2 \circ \cdots \circ \ell_N$ be a path in $\widetilde{\mathbb{L}}$ of length N, and define the *path operator* $S(\gamma) = (-\mathrm{i})^{N-1} S(\ell_1) \cdots S(\ell_N)$
If γ is closed, then $S(\gamma) = \mathrm{i}\mathbb{1}$.

Theorem (Szczerba)

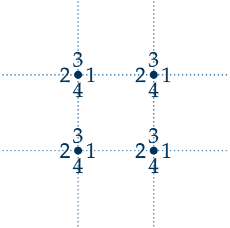
Let $\mathbb{L} = (\Lambda, E)$ be a directed graph and $\widetilde{\mathbb{L}} = (\tilde{\Lambda}, \tilde{E})$ its associated double graph. Then, given a set $\{S'(\ell) : \ell \in \tilde{E}\}$ of link operators on a finite-dimensional Hilbert space $\mathfrak{H}$, there exists a unitary transformation U, such that

$$S'(\ell) = US(\ell)U^*$$

where $\ell = (x_\zeta, y_\varsigma) \in \tilde{E}$, and $\zeta, \varsigma \in \{\xi, \eta\}$

Gamma Matrix Model

Jordan Wigner Transformation – Proof



Introduce the set of link operators

$$S^\Gamma(x_\xi,(x+e_1)_\xi) = -S^\Gamma((x+e_1)_\xi,x_\xi) = \Gamma_x^1\Gamma_{x+e_1}^2$$
$$S^\Gamma(x_\xi,(x+e_2)_\xi) = -S^\Gamma((x+e_2)_\xi,x_\xi) = \Gamma_x^3\Gamma_{x+e_2}^4$$
$$S^\Gamma(x_\eta,(x+e_1)_\eta) = -S^\Gamma((x+e_1)_\eta,x_\eta) = \Gamma_x^{15}\Gamma_{x+e_1}^{25}$$
$$S^\Gamma(x_\eta,(x+e_2)_\eta) = -S^\Gamma((x+e_2)_\eta,x_\eta) = \Gamma_x^{35}\Gamma_{x+e_2}^{45}$$
$$S^\Gamma(x_\xi,x_\eta) = -S^\Gamma(x_\eta,x_\xi) = \Gamma_x^5$$

Then the operators $\tilde{S}_\Xi^\Gamma = \Xi\tilde{S}^\Gamma\Xi\big|_{\mathrm{ran}\,\Xi}$ satisfy the link algebra, where

$$\tilde{S}^\Gamma(\ell) = \begin{cases} \mathbb{1}\otimes S^\Gamma(\ell) & \text{if } v_\xi \notin \ell \\ \Gamma_0\otimes S^\Gamma(\ell) & \text{if } v_\xi \in \ell \end{cases}$$

Dynamical Localisation

- $H_\Lambda = (c_1^*, \dots, c_{|\Lambda|}^*) \mathrm{A}^{(\Lambda)} (c_1, \dots, c_{|\Lambda|})^t$ with

$$\mathrm{A}^{(\Lambda)} = \mu \mathrm{Adj}(\Lambda) + \mathrm{diag}(\nu_1, \dots, \nu_{|\Lambda|})$$

- $\mathrm{A}^{(\Lambda)}$ dynamically localised, if there exist $C, \eta > 0$ such that for all $\Lambda \subset \mathbb{Z}^d$ and $x, y \in \Lambda$

$$\mathbb{E}\left[\sup_{t \in \mathbb{R}} \left| \mathrm{A}_{x,y}^{(\Lambda)}(t) \right| \right] \leq C e^{-\eta d(x,y)}$$

$\mathrm{A}_{x,y}^{(\Lambda)}(t) = (e^{\mathrm{i} t \mathrm{A}^{(\Lambda)}})_{x,y}$

- complete dynamical localisation of Anderson model for high disorder parameter (i.e. small μ) implies dynamical localisation of $\mathrm{A}^{(\Lambda)}$

Zero-velocity Lieb Robinson Bounds

in disorder average

Proposition

Assume that the constrained Gamma Matrix Model $\widetilde{H}_{\Xi}^{\Gamma}$ is dynamically localised. Then there exist constants $c, \eta > 0$ such that for any $\mathrm{I}, \Omega \subset \Lambda$ with $\mathrm{I} \cap \Omega = \emptyset$ one has

$$\mathbb{E}\Big[\big\|[\beta_t^{\Lambda}(X), Y]\big\|\Big] \leq c\, \min(1, |t|)\, \|X\|\, \|Y\|\, e^{-\eta \mathrm{dist}(\mathrm{I},\Omega)}$$

for all $X \in \mathfrak{S}_{\mathrm{I}}$ and $Y \in \mathfrak{S}_{\Omega}$.

- $\mathfrak{S}_{\Omega}$ C^*-algebra generated by the set $\{\tilde{S}_{\Xi}^{\Gamma}(\ell) : \ell \in \widetilde{\Omega}\}$ of link operators with edges ℓ in the (doubled) set $\widetilde{\Omega}$.
- β_t^{Λ} Heisenberg evolution associated with $\widetilde{H}_{\Xi}^{\Gamma}(\Lambda)$

Thank you for your attention!

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