University of California
Los Angeles

On Invariant Measures of the Exclusion Process and Related Processes

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

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2003

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2003

To my Lord and Saviour, Jesus Christ…

Thank you for taking my place on the cross.

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Acknowledgments

I'd like to thank my advisor, Professor Thomas M. Liggett, for his continual support and encouragement during the undertaking of this thesis. I'm indebted to him for the countless hours he has spent teaching me mathematics, and more importantly, teaching me to be a mathematician.

I'd also like to thank my parents and my best friend and brother, Peter. Without them, I wouldn't be where I am today.

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The Noisy Voter-Exclusion Process. To appear in Stochastic Processes and Applications, 2003.

Abstract of the Dissertation

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Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2003

Professor Thomas M. Liggett, Chair

This thesis studies the invariant measures $\mathcal{I}$ of the exclusion process and other closely related interacting particle systems.

For the exclusion process with symmetric kernel $p(x,y) = p(y,x)$, $\mathcal{I}$ has been completely studied by analyzing the dual process. We give a brief overview of those results and then prove some new results concerning cases for which $p(x,y) = p(y,x)$ except for finitely many $x, y \in \mathcal{S}$ where $p(x,y)$ corresponds to a transient Markov chain on a countable graph $\mathcal{S}$. The two techniques used in proving the new results include an approximation to the dual process and a certain coupling known as the infinitesimal coupling.

Next, we consider asymmetric exclusion processes where in general we do not have that $p(x,y) = p(y,x)$. The characterization of $\mathcal{I}$ in these cases is typically much more difficult. We will characterize $\mathcal{I}$ for exclusion processes on $\mathbb{Z}$ with certain reversible transition kernels. Some examples for which $\mathcal{I}$ is given include all reversible finite-range kernels that are asymptotically equal to $p(x,x+1) = p(x,x-1) = 1/2$. One tool used in the proofs gives a necessary and sufficient condition for reversible measures to be extremal in the set of invariant

measures, which is an interesting result in its own right.

Finally, we will study $\mathcal{I}$ for a hybrid of the symmetric exclusion process and the voter model. The reason such a hybrid is interesting is that the dual processes for the two systems are closely related. The dual processes can thus be combined to analyze the hybrid process. In fact, the dual of the hybrid process allows one to also add spontaneous births and deaths of particles at no cost to the techniques used. Also, an ergodic theorem for a process related to the hybrid process is proved using certain coupling methods.

CHAPTER 1

Introduction

1.1 Interacting particle systems

An *interacting particle system* is a stochastic process η_t for which particles live on the vertices of some countably infinite graph $\mathcal{S}$ and behave according to dynamics which depend on the configuration of particles in some surrounding neighborhood. The study of these processes began in the late 1960's and in a large part was fuelled by the field of statistical dynamics. The motivation came from the idea that (interacting) particle systems model certain processes in the natural and physical world.

The first question to ask about any stochastic process is, to what distributions does the process converge? In other words, what are the invariant (or equivalently, stationary) measures? Thus, a fundamental issue concerning particle systems is classifying the invariant measures $\mathcal{I}$ and giving properties of these measures. A closely related problem is to characterize the set of all initial measures which converge in distribution to a given invariant measure. The set of initial measures which converge in distribution to $\mu \in \mathcal{I}$ is called the *domain of attraction* of μ.

A large class of particle systems consists of systems for which at most one particle can occupy a vertex or site at any given time. The state space for these particle systems is given by $\mathbb{X} = \{0,1\}^{\mathcal{S}}$. Here a 1 at $x \in \mathcal{S}$ represents a particle occupying site x whereas a 0 indicates that x is vacant. A *configuration* is an element $\eta \in \mathbb{X}$ where either $\eta(x) = 0$ or $\eta(x) = 1$ for each $x \in \mathcal{S}$. All of the particle systems that are studied in this thesis will have $\mathbb{X}$ as the state space. Therefore we will from now on assume that $\mathcal{P}$ is the set of all probability measures on $\mathbb{X}$.

Some examples of particle systems for which the state space is $\mathbb{X}$ are *spin systems*. In particular, a particle system is known as a spin system if the state space is $\mathbb{X}$ and the value of η_t switches at only one site for any given time $t \in [0, \infty)$ (i.e. the transition from one state to another is given by $\eta \mapsto \eta_x$ where

$$\eta_x(u) = \begin{cases} \eta(u) & \text{if } u \neq x \\ 1 - \eta(u) & \text{if } u = x). \end{cases}$$

The interpretation is that a particle is born when $\eta(x)$ goes from 0 to 1 and a particle dies when $\eta(x)$ goes from 1 to 0. The rate at which η goes to η_x is given by a nonnegative function $c(x, \eta)$.

Let $\mathcal{D}(\mathbb{X})$ denote the set of all functions on $\mathbb{X}$ that depend on finitely many coordinates. The generator for a spin system is given by the closure of the operator Ω defined on $\mathcal{D}(\mathbb{X})$:

$$\Omega f(\eta) = \sum_x c(x, \eta)[f(\eta_x) - f(\eta)].$$

Under appropriate conditions on the flip rates $c(x, \eta)$, the generator Ω uniquely determines a Feller process on $\mathbb{X}$. For a detailed account of the construction of these processes we refer the reader to Chapter I of *Interacting Particle Systems* (Liggett(1985)) which we will henceforth abbreviate as IPS.

For the most part, the processes studied here are not spin systems, but we have introduced them here because they are related to the exclusion process which we will study in detail. In fact, in Chapter 4 we will consider a generalization of the exclusion process for which a certain spin system called the *voter model* is a special case.

The *exclusion process* is another example of a particle system with state space $\mathbb{X}$. Again, we think of 1's as particles and 0's as empty sites. A particle at site $x \in \mathcal{S}$ waits an exponential time with parameter $q_x = \sum_y q(x,y)$ at which time it chooses a $y \in \mathcal{S}$ with probability $q(x,y)/q_x$. If y is empty then the particle at x goes to y, while if y is occupied the particle at x does not move. The exclusion process will be the focus of study in the next two chapters. As such, the remainder of this chapter provides a brief introduction to this process.

1.2 The exclusion process

The exclusion process is one of the most well-known interacting particle systems. This is in part due to its many applications. It is used in biology as a model for the particle motion of ribosomes (Macdonald, Gibbs, and Pipkin(1968)), in physics as a model for a lattice gas at infinite temperature (Spitzer(1970)), and in ecology as a model in which two opposing species swap territory (Clifford and Sudbury(1973)).

An intuitive description of the process is given at the end of the previous section. For a technical description, let

$$\sup_y \sum_x q(x,y) < \infty \text{ and } \sup_x \sum_y q(x,y) < \infty \text{ for } q(x,y) \geq 0.$$

Similar to spin systems, described previously, the generator for the exclusion process is given by the closure of Ω defined on $\mathcal{D}(\mathbb{X})$. If

$$\eta_{xy}(u) = \begin{cases} \eta(y) & \text{if } u = x \\ \eta(x) & \text{if } u = y \\ \eta(u) & \text{if } u \neq x, y \end{cases}$$

then

$$\Omega f(\eta) = \sum_{x,y} q(x,y)\eta(x)(1-\eta(y))[f(\eta_{xy}) - f(\eta)]. \tag{1.1}$$

The corresponding semigroup will be denoted by $S(t)$.

The construction of the exclusion process is fully described in IPS. It is assumed there that the transition kernel satisfies $\sum_y p(x,y) = 1$ (here $p(x,y)$ is used instead of $q(x,y)$ since they can be thought of as probabilities), however, this is just a normalization of the process we have just described. To see this, simply add self-jump rates to the process we have described above:

$$q(x,x) = \sup_z \sum_y q(z,y) - \sum_y q(x,y).$$

Dividing all transition rates by $\sup_z \sum_y q(z,y)$ gives us the process constructed in IPS.

In the sequel, we will write $p(x,y)$ instead of $q(x,y)$ to indicate that the rates have been normalized– with one exception: in Chapter 2 the rates $\bar{p}(x,y)$ will not in general be normalized, but we write $\bar{p}(x,y)$ nonetheless since these rates are derived from a normalized kernel $p(x,y)$.

Many of the results in the following chapters require the analysis of the finite exclusion process (exclusion processes where $\eta_t(x) = 1$ for finitely many $x \in \mathcal{S}$, for all $t \geq 0$). Since the state space for a finite exclusion process starting with n particles is the set of all subsets of $\mathcal{S}$ containing n elements, we introduce the notation

$$\mathcal{S}_n = \mathcal{S}^n \backslash \{\vec{x} : x_i = x_j \text{ for some } i < j\}$$

as the state space of such a process. Also, let $\mathbb{Y} = \cup_{n\geq 1}\mathcal{S}_n$ denote all the finite subsets of $\mathcal{S}$.

1.3 Results for the symmetric process

We will assume throughout the rest of this chapter that we are dealing with the symmetric exclusion process where $p(x,y) = p(y,x)$, and $p(x,y)$ is irreducible. Here the problems of characterizing $\mathcal{I}$ and the domains of attraction for $\mu \in \mathcal{I}$ have been completely solved. Theorems 1.3.2 and 1.3.3 (proved by Liggett and Spitzer in the 1970's) give a summary of these results.

For preliminaries, we describe in more detail the finite exclusion process A_t. The process A_t is just the exclusion process with the added condition that its initial state A_0 has finitely many sites where $\eta(x) = 1$. We write $|A_t| = n$ to denote the number of sites that are 1's. In particular A_t is a countable state Markov chain that acts like n independent particles having transition rates $p(x,y)$, except that when a particle tries to move to an occupied site its motion is suppressed.

Although we do not prove Theorems 1.3.2 and 1.3.3, the following proposition which is the main tool used in the two proofs, is worth stating. This next proposition is the reason why symmetric exclusion processes are fundamentally different from asymmetric exclusion processes.

Proposition 1.3.1 (Spitzer)**.** *Suppose $p(x,y) = p(y,x)$. If $A \in \mathbb{Y}$ and $\eta \in \mathbb{X}$ then*

$$P^{\eta}[\eta_t(x) = 1 \text{ for all } x \in A] = P^{A}[\eta(x) = 1 \text{ for all } x \in A_t]$$

for all $t \geq 0$.

The dual nature of η_t and A_t in the above proposition is the reason why A_t is known in the literature as the *dual process.* We do not give the proof of the above lemma since we later prove a generalization of it in Proposition 4.2.1.

Before stating the two theorems we need some definitions. Denote the set of harmonic functions on $\mathcal{S}$ taking values between 0 and 1 as

$$\mathcal{H} = \left\{ \alpha : \mathcal{S} \to [0,1] \text{ such that } \sum_y p(x,y)\alpha(y) = \alpha(x) \text{ for all } x \right\}.$$

Define ν_α to be the product measure on $\mathbb{X}$ with marginals $\nu_\alpha\{\eta : \eta(x) = 1, x \in \mathcal{S}\} = \alpha(x)$. In the sequel we will abbreviate by writing $\nu_\alpha\{\eta(x) = 1\}$.

Theorem 1.3.2 (Liggett, Spitzer)**.** *If $p(x,y)$ is a symmetric, irreducible transition kernel on $\mathcal{S}$ then $\lim_{t\to\infty} \nu_\alpha S(t) = \mu_\alpha$ exists for each $\alpha(x) \in \mathcal{H}$, and*

$$\mathcal{I}_e = \{\mu_\alpha : \alpha(x) \in \mathcal{H}\}.$$

One should note that by the Krein-Milman theorem $\mathcal{I}$ is the closed, convex hull of its extreme points. Therefore characterizing $\mathcal{I}_e$ is equivalent to characterizing $\mathcal{I}$.

Let $\{X_1(t), \ldots, X_n(t)\}$ be independent Markov chains on $\mathcal{S}$ with transition probabilities

$$p_t(x,y) = e^{-t} \sum_{n\geq 0} \frac{t^n}{n!} p^{(n)}(x,y)$$

where $p^{(n)}(x,y)$ are the n-step transition probabilities corresponding to $p(x,y)$. Define the following function on $\mathcal{S}^2$,

$$g(x,y) = P^{(x,y)}[X_1(t) = X_2(t) \text{ for some } t > 0].$$

Theorem 1.3.3 (Liggett, Spitzer)**.** *Suppose that $p(x,y)$ is symmetric and irreducible on $\mathcal{S}$. Suppose also that $\alpha(x) \in \mathcal{H}$ and $\mu \in \mathcal{P}$.*

(a) When $g \not\equiv 1$, $\lim_{t\to\infty} \mu S(t) = \mu_\alpha$ if and only if

$$\lim_{t\to\infty} \sum_y p_t(x,y)\mu\{\eta(y) = 1\} = \alpha(x) \text{ for all } x \in \mathcal{S}, \text{ and} \tag{1.2}$$

$$\lim_{t\to\infty} \sum_{u,v} p_t(x,u)p_t(y,v)\mu\{\eta(u) = \eta(v) = 1\} = \alpha(x)\alpha(y) \text{ for all } x,y \in \mathcal{S}.$$

(b) When $g \equiv 1$, the only bounded harmonic functions are constant and $\lim_{t\to\infty} \mu S(t) = \mu_\alpha$ *if and only if (1.2) holds and*

$$\lim_{t\to\infty} E^{\{x,y\}} \mu\{\eta(u) = 1 \text{ for all } u \in A_t\} = \alpha^2 \text{ for all } x \neq y \in \mathcal{S}.$$

The fact that all bounded harmonic functions are constant when $g \equiv 1$ will be proved in the next section.

If $g(x,y) = 1$ for some $(x,y) \in \mathcal{S}_2$ then $g(x,y) \equiv 1$. To see this suppose $g(x,y) < 1$ for some $(x,y) \in \mathcal{S}_2$. If $z \neq x, y$, then by irreducibility either a particle can go from z to y without passing through x or a particle can go from x to y without passing through z and from z to x without passing through y. In either case, $\{X_1(t), X_2(t)\}$ can go from (x,z) to (x,y) without passing through $\mathcal{S}^2 \backslash \mathcal{S}_2$ so that $g(x,z) < 1$.

Also, it should be noted that $g \equiv 1$ implies that $X(t)$ is recurrent but not conversely. To prove the recurrence use the Chapman-Kolmogorov equation to get

$$\begin{aligned} p_{2t}(x,x) &= \sum_y p_t(x,y) p_t(y,x) \\ &= \sum_y [p_t(x,y)]^2 = P^{(x,x)}[X_1(t) = X_2(t)]. \end{aligned}$$

So if $X(t)$ is transient then $g(x,y) < 1$ for some $x, y \in \mathcal{S}$ since

$$\int_0^\infty P^{(x,x)}[X_1(t) = X_2(t)]\, dt < \infty$$

(this argument will be made more explicit in Section 4.3). An example of a symmetric, irreducible recurrent Markov chain for which $g \not\equiv 1$ is given in Liggett(1974).

As stated earlier, we will not prove either result; however, the next section will be devoted to proving a special case of Theorem 1.3.2 so as to give the reader a taste of the techniques used in the proofs of the general theorems. For a complete treatment we direct the reader to Chapter VIII of IPS.

1.4 Coupling

Coupling is arguably the most important technique when dealing with interacting particle systems. The coupling of two processes with state spaces S_1 and S_2 is a joint process defined on a common probability space, usually having $S_1 \times S_2$ as its state space. The two marginal processes of the joint process are exactly the original two processes. Normally, the coupling is useful only if the two marginal processes are not independent.

As an example we now give the transition kernel of a coupling of the Markov chains $X_1(t)$ and $X_2(t)$ defined in the previous section (here we will not assume they are independent). If $x \neq z$ then

$$p((x,z),(y,z)) = p((z,x),(z,y)) = p((x,x),(y,y)) = p(x,y)$$
$$\text{and } p((a,b),(c,d)) = 0 \text{ elsewise.}$$

This coupling can be described by saying that $X_1(t)$ and $X_2(t)$ evolve independently until the first time they meet after which point they move together. If the two processes eventually meet with probability one, then the coupling is said to be *successful*.

As a second example, we will describe the dual process, A_t with k particles, as a coupling of the processes $X_1(t), \ldots, X_k(t)$. The particles $X_i(t)$ and $X_j(t)$ move independently except that when $X_i(t)$ at x tries to move to a site y which is occupied by $X_j(t)$, the two particles switch places. In other words, the coupling is such that the particles move independently except that $X_i(t)$ at x goes to y at exactly the same time that $X_j(t)$ at y goes to x. A_t is just the set of sites occupied by the k particles and can be thought of as an element of $\mathcal{S}_k$. Since $p(x,y) = p(y,x)$, each marginal process can be taken to be a Markov chain on $\mathcal{S}$ with transition kernel $p(x,y)$.

Proposition 1.4.1. *Suppose $g \equiv 1$. If f is a bounded harmonic function for the dual process, A_t with k particles, then f is constant on $\mathcal{S}_k$.*

Proof. The crux of the proof is constructing a successful coupling of two finite exclusion processes A_t and A'_t starting from $A, A' \in \mathcal{S}_k$ respectively. If we are able to do this, then for all bounded harmonic f on $\mathcal{S}_k$

$$|f(A) - f(A')| = |Ef(A_t) - Ef(A'_t)| \leq E|f(A_t) - f(A'_t)| \leq 2\|f\|P(A_t \neq A'_t) \to 0$$

showing that f is constant.

To construct the successful coupling of A_t and A'_t, we start by writing $A_t = \{X_1(t), \ldots, X_k(t)\}$ and $A'_t = \{X'_1(t), \ldots, X'_k(t)\}$ using the coupling described right before the statement of the proposition. We can then couple the $X_i(t)$'s and $X'_j(t)$'s so that $X_i(t)$ moves independently of $X'_j(t)$ until the first time it meets $X'_j(t)$ for some j. From that point on, $X_i(t)$ and $X'_j(t)$ move together. Since $g \equiv 1$, this coupling of A_t and A'_t is successful. □

Corollary 1.4.2. *If $g \equiv 1$, then all bounded harmonic functions on $\mathcal{S}$ are constant.*

Let $\nu_\rho, \rho \in [0,1]$ be the product measure on $\mathbb{X}$ with marginals $\nu_\rho\{\eta(x) = 1\} = \rho$. We are now ready to prove the following special case of Theorem 1.3.2.

Theorem 1.4.3. *Let $p(x,y)$ be a symmetric, irreducible transition kernel on $\mathcal{S}$. If $g(x,y) \equiv 1$, then $\mathcal{I}_e = \{\nu_\rho : \rho \in [0,1]\}$.*

Proof. By De Finetti's Theorem the statement that $\mathcal{I}_e = \{\nu_\rho : \rho \in [0,1]\}$ is equivalent to saying that μ is invariant if and only if μ is exchangeable.

By Proposition 1.3.1, $\mu \in \mathcal{I}$ if and only if $\mu\{\eta = 1 \text{ for all } x \in A\}$ is harmonic for the chain A_t for all initial states $A \in \mathbb{Y}$. Proposition 1.4.1 tells us that such harmonic functions are constant, so it must be that $\mu \in \mathcal{I}$ if and only if $\mu\{\eta = 1 \text{ for all } x \in A\}$ depends only on the cardinality of A. But this last statement just says that μ is exchangeable completing the proof. $\square$

CHAPTER 2

The Quasi-symmetric Exclusion Process

In this chapter we consider exclusion processes which have symmetric transition kernels outside of a finite set. In particular, if $p(x, y) = p(y, x)$ for all $x, y \in \mathcal{S}$ and $p(x, y)$ is irreducible then suppose that $\bar{p}(x, y) = p(x, y)$ for all (x, y) except for n ordered pairs $\{(x_1, y_1), \ldots, (x_n, y_n)\}$. At (x_i, y_i) we have the perturbation $\bar{p}(x_i, y_i) = p(x_i, y_i) + \epsilon_i$ for $\epsilon_i \geq -p(x_i, y_i)$. Note that the x_i's and y_i's are not necessarily distinct. We will say that transition kernels $\bar{p}(x, y)$ satisfying the above requirement are *quasi-symmetric*. Note that in general $\bar{p}(x, y)$ is not normalized thus it is possible that $\sum_y \bar{p}(x, y) \neq 1$. In order to avoid complications we will also assume throughout this chapter that $\bar{p}(x, y)$ is irreducible. As for notation in this chapter, $S(t)$ and $\mathcal{I}$ will denote the semigroup and invariant measures of the symmetric process and $\bar{S}(t)$ and $\bar{\mathcal{I}}$ the semigroup and invariant measures of the quasi-symmetric process.

An analog of the dual process discussed in Section 1.3 does not exist for quasi-symmetric processes which are not symmetric. However, an approximation to the dual is available which makes the problem of characterizing $\mathcal{I}$ for quasi-symmetric processes much more tenable than processes with no symmetry whatsoever. In Section 2.1 we will state Theorem 2.1.1 which describes the invariant measures for a quasi-symmetric process, and in Section 2.2 we will use an approximation to the dual process to prove Theorem 2.1.1.

The fact that quasi-symmetric kernels are mostly symmetric also allows us to take advantage of a certain coupling technique in order to show that the family of invariant measures $\{\mu_\rho : \rho \in [0,1]\}$ exists (where $\mu_\rho = \lim_{t\to\infty} \nu_\rho S(t)$). This last statement will be proven in Section 2.3.

2.1 The invariant measures

Let $\bar{X}(t)$ be a continuous-time Markov chain on $\mathcal{S}$ with respect to $\bar{p}(x,y)$. Note that $\bar{X}(t)$ is transient with respect to $\bar{p}(x,y)$ if and only if the Markov chain with respect to $p(x,y)$, $X(t)$, is transient.

Theorem 2.1.1. *Suppose $\bar{p}(x,y)$ is quasi-symmetric and irreducible. Also, assume that for $\{A^n\}$ a sequence in $\mathcal{S}_k$, each $x \in \mathcal{S}$ is only in finitely many A^n. If $\bar{X}(t)$ is transient then*

(a) for each $\bar{\mu} \in \bar{\mathcal{I}}$ there exists $\mu \in \mathcal{I}$ such that

$$\lim_{n\to\infty} |\bar{\mu}\{\eta(x) = 1 \text{ for all } x \in A^n\} - \mu\{\eta(x) = 1 \text{ for all } x \in A^n\}| = 0 \qquad (2.1)$$

for all sequences $\{A^n\}$ satisfying the above, and

(b) for each $\mu \in \mathcal{I}$ there exists a measure $\bar{\mu} \in \bar{\mathcal{I}}$ satisfying (2.1).

Since we have a characterization of $\mathcal{I}$ given by Theorem 1.3.2, the measure $\mu \in \mathcal{I}$ in part (a) must be unique. It would be interesting if one could somehow show that $\bar{\mu} \in \bar{\mathcal{I}}$ in part (b) is unique as well, for if this were so then we would have a one-to-one correspondence between $\mathcal{I}$ and $\bar{\mathcal{I}}$ thereby giving us a characterization of $\bar{\mathcal{I}}$.

From the point of view of practicality, Theorem 2.1.1 gives us as good of a characterization of $\bar{\mathcal{I}}$ as one could hope for. The reason for this is that even if one were to show that $\bar{\mu}$ in part (b) was unique, one would not expect to be able to calculate

$$\bar{\mu}\{\eta(x) = 1 \text{ for all } x \in A\} \tag{2.2}$$

explicitly for each $A \in \mathbb{Y}$. The best one could hope for is to know the asymptotics of (2.2) for some sequence $\{A^n\}$ in $\mathcal{S}_k$. But Theorem 2.1.1 already gives this to us.

Besides giving information about $\bar{\mathcal{I}}$, the theorem has an interesting consequence. One can ask the following question: Does a local perturbation of the dynamics of a process have global consequences on the evolution?

If we think of the quasi-symmetric exclusion process as a perturbation of the symmetric exclusion process then the answer is affirmative when $\mathcal{S} = \mathbb{Z}$ and there exists a reversible measure $\pi(x) > 0$ with respect to the transition kernel (i.e. a measure satisfying $\pi(x)\bar{p}(x,y) = \pi(y)\bar{p}(y,x)$). To see this, consider the simple case where

$$\bar{p}(x,y) = 1/2 \text{ for all } (x,y) \neq (0,1) \text{ and } \bar{p}(0,1) = 1/2 + \epsilon, \epsilon > 0.$$

It will be seen later (by Theorem 3.1.1) that the only extremal invariant measures are the product measures $\{\nu^c : c \in [0,\infty]\}$ with marginals

$$\nu^c\{\eta : \eta(x) = 1\} = \begin{cases} \frac{c}{1+c} & \text{for } x \leq 0 \\ \\ \frac{c+2c\epsilon}{1+c+2c\epsilon} & \text{for } x > 0. \end{cases}$$

If we choose a sequence of times $\{T_n\}$ going to infinity so that

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \nu_\rho \bar{S}(t)dt = \bar{\mu}_\rho$$

exists, then for any continuous f on $\mathbb{X}$,

$$\begin{aligned} \int \bar{S}(s) f \, d\bar{\mu}_\rho &= \lim_{n\to\infty} \int \bar{S}(s) f \, d \left[\frac{1}{T_n} \int_0^{T_n} \nu_\rho \bar{S}(t)\, dt \right] \\ &= \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \left[\int \bar{S}(s+t) f \, d\nu_\rho \right] dt \\ &= \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n+s} \left[\int \bar{S}(t) f \, d\nu_\rho \right] dt \\ &= \lim_{n\to\infty} \int f \, d \left[\frac{1}{T_n} \int_0^{T_n} \nu_\rho \bar{S}(t)\, dt \right] \\ &= \int f \, d\bar{\mu}_\rho \end{aligned} \tag{2.3}$$

so that $\bar{\mu}_\rho$ is invariant. Therefore $\bar{\mu}_\rho$ must be a mixture of the measures $\{\nu^c : c \in [0, \infty]\}$. Consequently

$$\lim_{x\to\infty} \bar{\mu}_\rho\{\eta(x) = 1\} > \lim_{x\to-\infty} \bar{\mu}_\rho\{\eta(x) = 1\},$$

however, this clearly shows that the perturbation at the origin affects the evolution of the process globally.

On the other hand, we will see from the proof of Theorem 2.1.1 that if $\mu \in \mathcal{I}$ and

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \mu \bar{S}(t) dt = \bar{\mu}$$

exists, then $\bar{\mu}$ is asymptotically equal to μ. Thus we have a negative answer to the above question on local perturbations having a global effect.

2.2 Approximating the dual

In order to prove Theorem 2.1.1 we will need to think of the symmetric exclusion process in a different way so that we can couple η_t and A_t. Using a symmetric transition kernel, assign to the subset $\{x, y\} \in \mathcal{S}_2$ an exponential clock with rate $p(x, y)$. Since $p(x, y) = p(y, x)$, this assignment is well-defined. When the exponential clock for $\{x, y\}$ goes off, the values for $\eta(x)$ and $\eta(y)$ will switch. This motion describes the symmetric exclusion process.

We can now couple A_t with η_t using this new description. The process A_t is equal to A_0 until the first time that an exponential clock for $\{x, y\}$ with $x \in A_0$ and $y \notin A_0$ goes off. At that time A_t becomes $(A_0 \backslash x) \cup y$. Let A_t^T be the dual process running backwards in time starting from time T so that $A_t^T = A_{T-t}$. Since the exponential times for $\{x, y\}$ are uniformly distributed on $[0, T]$, we can use the same clocks for both A_t and A_t^T. We then have that

$$\{\eta_T(x) = 1 \text{ for all } x \in A_0^T\} = \{\eta_0(x) = 1 \text{ for all } x \in A_T^T\}. \tag{2.4}$$

One might recognize the similarity between (2.4) and Proposition 1.3.1.

Notice that when $\eta(x) = \eta(y) = 1$, switching values is the same as not switching values. For the symmetric exclusion process, we can reinterpret this statement in the following way. When a particle tries to move to an occupied site, instead of its motion being suppressed, the two particles switch places. We therefore see that this alternate way of thinking of the exclusion process is the reason that we are able to couple A_t with $\{X_1(t), \ldots, X_k(t)\}$ in Section 1.4. We will need the following lemma which is a consequence of the coupling of A_t and $\{X_1(t), \ldots, X_k(t)\}$.

Lemma 2.2.1. *Suppose $\{A_0^n\}$ is a sequence in $\mathcal{S}_k$. If each $x \in \mathcal{S}$ belongs to only finitely many A_0^n and the symmetric kernel $p(x, y)$ corresponds to a transient Markov chain on $\mathcal{S}$, then for each fixed $z \in \mathcal{S}$*

$$\lim_{n\to\infty} P(z \in A_t^n \text{ for some } t \geq 0) = 0.$$

Proof. To remind the reader, we shall describe the coupling again. The particles $X_i(t)$ and $X_j(t)$ move independently following the motions of a Markov chain on $\mathcal{S}$, except that when $X_i(t)$ at x tries to move to a site y which is occupied by $X_j(t)$, the two particles switch places. Since $p(x,y) = p(y,x)$, this is just the coupling of the two processes where $X_i(t)$ at x goes to y at the same time that $X_j(t)$ at y goes to x. If $A_0^n = \{X_1^n(0), \ldots, X_k^n(0)\}$, then using this coupling $A_t^n = \{X_1^n(t), \ldots, X_k^n(t)\}$. Therefore

$$\lim_{n\to\infty} P^{A_0^n}(z \in A_t^n \text{ for some } t \geq 0) \leq \lim_{n\to\infty} \sum_{i=1}^{k} P^{X_i^n(0)}(X_i^n(t) = z \text{ for some } t \geq 0) = 0.$$

□

Let

$$B = \{x \in \mathcal{S} : \bar{p}(x,y) \neq \bar{p}(y,x) \text{ for some } y \in \mathcal{S}\}.$$

We will now describe a process $\bar{A}_t$ which approximates the process A_t. The process $\bar{A}_t$ can be thought of as a family of $\mathcal{S}_n$-valued functions $\bar{A}_t(\bar{A}_0, \bar{\omega})$ indexed by time t. The two arguments of $\bar{A}_t$ are the set $\bar{A}_0 \in \mathcal{S}_n$ such that $\bar{A}_0 \cap B = \emptyset$ and $\bar{\omega}$ an element of the path space associated with the quasi-symmetric process. Let $\bar{P}_\nu$ be the probability measure on the path space of the quasi-symmetric process having ν as its initial distribution (likewise, let P_ν be the probability measure on the path space of the symmetric process with ν as its initial distribution).

If $x \in \bar{A}_t, y \notin \bar{A}_t \cup B$ then $\bar{A}_t$ goes to $(\bar{A}_t \backslash x) \cup y$ at rate $p(x,y)$ according to the exponential clock of $\{x,y\}$. If $x \in \bar{A}_t, y \notin \bar{A}_t \cup B^c$ and the exponential clock for $\{x,y\}$ goes off then $\bar{A}_t$ goes to either $\bar{A}_t \backslash x$ if $\eta_t(x) = 1$ or the cemetery state Δ if $\eta_t(x) = 0$. Since the values of $\eta_t(x)$ and $\eta_t(y)$ switch when the clock for $\{x,y\}$ goes off, we will assume that the evaluation of $\eta_t(x)$ is taken before the switch.

For a fixed $T > 0$, the process $\bar{A}_t^T$ follows the motion described above except that it runs backwards in time from T to 0 while η_s runs forward in time; when the exponential clock for $\{x,y\}$ goes off, the evaluation of $\eta_s(x)$ takes place after the switching of $\eta_s(x)$ and $\eta_s(y)$ at time $s = T - t$ takes place. Setting $\eta(\Delta) \equiv 0$, we then have following analog of (2.4) for the quasi-symmetric process η_t:

$$\{\eta_T(x) = 1 \text{ for all } x \in \bar{A}_0^T\} = \{\eta_0(x) = 1 \text{ for all } x \in \bar{A}_T^T\}. \tag{2.5}$$

The processes A_t and $\bar{A}_t$ are coupled so that they start from the same $A \in \mathbb{Y}$ and move together as much as possible (after the first time they are different, they move independently); likewise for the processes A_t^T and $\bar{A}_t^T$. Therefore denote

$$\mathcal{N}_A = \{\bar{A}_t \text{ starting from } A \text{ equals } A_t \text{ for all } t \geq 0\}$$

and

$$\mathcal{N}_A^T = \{\bar{A}_t^T \text{ starting from } A \text{ equals } A_t^T \text{ for all } t \in [0, T]\}.$$

Proof of Theorem 2.1.1. Fix $A \in \mathbb{Y}$ and suppose that both $\bar{A}_t$ and A_t start from A. Let

$$f_{\bar{A}_t}(\bar{\omega}) = \begin{cases} 1 & \text{if } \eta_0(x) = 1 \text{ for all } x \in \bar{A}_t(\bar{\omega}) \\ 0 & \text{otherwise} \end{cases}$$

and define f_{A_t} similarly. Also if $\bar{A}_t^T$ and A_t^T both start from A, let

$$f_{\bar{A}_T^T}(\bar{\omega}) = \begin{cases} 1 & \text{if } \eta_T(x) = 1 \text{ for all } x \in \bar{A}_T^T(\bar{\omega}) \\ 0 & \text{otherwise} \end{cases}$$

and define $f_{A_T^T}$ similarly.

Take $\bar{\mu} \in \bar{\mathcal{I}}$. Since $A_t = \bar{A}_t$ on $\mathcal{N}_A$ we have that

$$\int f_{A_t}\, dP_{\bar{\mu}} - P(\mathcal{N}_A^c) \leq \int f_{\bar{A}_t} 1_{\mathcal{N}_A}\, d\bar{P}_{\bar{\mu}}. \tag{2.6}$$

Recall that $S(t)$ is the semigroup of the symmetric process. By Proposition 1.3.1 (or equivalently by (2.4))

$$\int f_{A_t}\, dP_{\bar{\mu}} = E^A \int 1_{\{\eta(x)=1 \forall x \in A_t\}}\, d\bar{\mu} = \int 1_{\{\eta(x)=1 \forall x \in A\}}\, d\bar{\mu} S(t).$$

By the argument given in (2.3) we can choose a sequence T_n so that

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \bar{\mu} S(t) dt$$

converges to $\mu \in \mathcal{I}$. By the fact that $\bar{\mu} \in \bar{\mathcal{I}}$ and by (2.5), we have

$$\int f_{\bar{A}_T} 1_{\mathcal{N}_A} \, d\bar{P}_{\bar{\mu}} \leq \int f_{\bar{A}_T^T} 1_{\mathcal{N}_A^T} \, d\bar{P}_{\bar{\mu}} \leq \int 1_{\{\eta_0(x)=1 \forall x \in A_T^T(\omega)\}} \, d\bar{P}_{\bar{\mu}} = \bar{\mu}\{\eta(x) = 1 \text{ for all } x \in A\}$$

for all $T \geq 0$ so that (2.6) yields

$$\mu\{\eta(x) = 1 \text{ for all } x \in A\} - P(\mathcal{N}_A^c) \leq \bar{\mu}\{\eta(x) = 1 \text{ for all } x \in A\}. \tag{2.7}$$

Using $\bar{\mu} \in \bar{\mathcal{I}}$ once more, we also have

$$\begin{aligned} & \int 1_{\{\eta(x)=1 \forall x \in A\}} \, d\bar{\mu} - \frac{1}{T_n} \int_0^{T_n} \int 1_{\{\eta_t(x)=1 \forall x \in A\}} [1 - 1_{\mathcal{N}_A}] \, d\bar{P}_{\bar{\mu}} \, dt \\ = & \frac{1}{T_n} \int_0^{T_n} \int 1_{\{\eta_t(x)=1 \forall x \in A\}} \, d\bar{P}_{\bar{\mu}} \, dt - \frac{1}{T_n} \int_0^{T_n} \int 1_{\{\eta_t(x)=1 \forall x \in A\}} [1 - 1_{\mathcal{N}_A}] \, d\bar{P}_{\bar{\mu}} \, dt \\ = & \frac{1}{T_n} \int_0^{T_n} \int 1_{\{\eta_t(x)=1 \forall x \in A\}} 1_{\mathcal{N}_A} \, d\bar{P}_{\bar{\mu}} \, dt \end{aligned}$$

so that

$$\bar{\mu}\{\eta(x) = 1 \text{ for all } x \in A\} - P(\mathcal{N}_A^c) \leq \mu\{\eta(x) = 1 \text{ for all } x \in A\}.$$

Combining this with (2.7) gives us

$$|\bar{\mu}\{\eta(x) = 1 \text{ for all } x \in A\} - \mu\{\eta(x) = 1 \text{ for all } x \in A\}| \leq P(\mathcal{N}_A^c). \tag{2.8}$$

We complete the proof of part (a) by noting that Lemma 2.2.1 tells us $\lim_{n\to\infty} P(\mathcal{N}_{A^n}^c) = 0$.

The proof of part (b) is similar. Choose $\mu \in \mathcal{I}$. Again, we can choose a subsequence T_m so that

$$\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \mu \bar{S}(t) dt$$

converges to $\bar{\mu} \in \bar{\mathcal{I}}$. Then

$$\begin{aligned}
\mu\{\eta(x) = 1 \text{ for all } x \in A\} - P(\mathcal{N}_A^c) &\leq \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \int f_{A_T} 1_{\mathcal{N}_A} \, dP_\mu \, dT \\
&\leq \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \int 1_{\{\eta_0(x)=1 \forall x \in \bar{A}_T^T(\omega)\}} 1_{\mathcal{N}_A^T} \, d\bar{P}_\mu \, dT \\
&= \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \int 1_{\{\eta_T(x)=1 \forall x \in A\}} 1_{\mathcal{N}_A^T} \, d\bar{P}_\mu \, dT \\
&\leq \bar{\mu}\{\eta(x) = 1 \text{ for all } x \in A\}.
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mu}\{\eta(x) = 1 \text{ for all } x \in A\} - P(\mathcal{N}_A^c) &\leq \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \int 1_{\{\eta_0(x)=1 \forall x \in \bar{A}_T^T(\omega)\}} 1_{\mathcal{N}_A^T} \, d\bar{P}_\mu \, dT \\
&\leq \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} E^A \int 1_{\{\eta(x)=1 \forall x \in A_T\}} \, d\mu \, dT \\
&= \mu\{\eta(x) = 1 \text{ for all } x \in A\}
\end{aligned}$$

so that we again obtain (2.8). □

2.3 The infinitesimal coupling

Theorem 2.1.1 gives us information concerning the invariant measures, but tells us little about the domains of attraction for those measures. In this section we will prove the following theorem which concerns convergence starting from the product measure ν_ρ with marginals $\nu_\rho\{\eta(x) = 1\} = \rho$.

Theorem 2.3.1. *Suppose $\bar{p}(x, y)$ is quasi-symmetric and irreducible. Also, assume that for $\{A^n\}$ a sequence in $\mathcal{S}_k$, each $x \in \mathcal{S}$ is only in finitely many A^n. If $\bar{p}(x, y) > 0$ whenever $p(x, y) > 0$ and $X(t)$ is transient then*

$$\lim_{t\to\infty} \nu_\rho \bar{S}(t) = \mu_\rho \in \bar{\mathcal{I}}$$

exists for each $\rho \in [0, 1]$ and

$$\lim_{n\to\infty} \mu_\rho\{\eta(x) = 1 \text{ for all } x \in A^n\} = \rho^k. \tag{2.9}$$

The motivation behind Theorem 2.3.1 lies in the idea that the measures ν_ρ are natural initial measures for the process to start off with. So a natural question to ask is whether or not $\lim_{t\to\infty} \nu_\rho \bar{S}(t)$ exists. As we will see, (2.9) then follows immediately from the proof of Theorem 2.1.1 (b).

Also, Theorem 2.3.1 strengthens the argument given in Section 2.1 concerning local perturbations of the transition kernel having global effects on the evolution of the process. In particular, the theorem gives us conditions under which $\lim_{t\to\infty} \nu_\rho \bar{S}(t)$ is not very different from ν_ρ.

The main tool used to prove Theorem 2.3.1 is the so called *infinitesimal coupling* of the process η_t introduced by Andjel, Bramson, and Liggett(1988). In this section we will describe the infinitesimal coupling and present two lemmas concerning this coupling.

The infinitesimal coupling of the process η_t follows the motion of the *basic coupling* (defined below) for the two processes η_t and ξ_t^s having joint initial measure $\tilde{\nu}$ (also defined below). The marginal process ξ_t^s can be thought of as an approximation of η_{t+s} for small values of s.

Let us now define the basic coupling of two exclusion processes η_t and ξ_t having the same generator. Essentially, it is the coupling which allows η_t and ξ_t to move together as much as possible. The generator for the basic coupling is the closure of the operator $\tilde{\Omega}$ defined on $\mathcal{D}(\mathbb{X} \times \mathbb{X})$:

$$\begin{aligned}\tilde{\Omega} f(\eta, \xi) = & \sum_{\eta(x)=\xi(x)=1, \eta(y)=\xi(y)=0} \bar{p}(x,y)[f(\eta_{xy}, \xi_{xy}) - f(\eta, \xi)] \qquad (2.10) \\ + & \sum_{\eta(x)=1, \eta(y)=0 \text{ and } (\xi(y)=1 \text{ or } \xi(x)=0)} \bar{p}(x,y)[f(\eta_{xy}, \xi) - f(\eta, \xi)] \\ + & \sum_{\xi(x)=1, \xi(y)=0 \text{ and } (\eta(y)=1 \text{ or } \eta(x)=0)} \bar{p}(x,y)[f(\eta, \xi_{xy}) - f(\eta, \xi)].\end{aligned}$$

The initial measure $\tilde{\nu}$ depends on the transition kernel of the process. To describe $\tilde{\nu}$, we will consider the following simple kernel: Start with a symmetric irreducible transition kernel $p(x,y)$ on $\mathcal{S}$. Pick some site to be the origin, 0, and label one of its neighbors 1. Choosing $\epsilon > 0$, we can define $\bar{p}(x,y)$ by

$$\bar{p}(0,1) = p(0,1) + \epsilon, \quad \bar{p}(x,y) = p(x,y) \text{ elsewise.} \tag{2.11}$$

In order to simplify the description of $\tilde{\nu}$, we will assume throughout most of this section that our transition kernel is given by (2.11). It is under this assumption that we will explicitly describe $\tilde{\nu}$ and prove the lemmas. At the end of the section we will give an argument that extends the results to the general case.

We are now ready to describe $\tilde{\nu}$ under the assumption of (2.11). Following Andjel, Bramson, and Liggett(1988), the basic idea is to couple the measures ν_ρ and $\nu_\rho \bar{S}(s)$ together for small values of s (in particular, we impose the restriction $s < \frac{1}{\epsilon}$). The problem is that one cannot explicitly write out the distribution of $\nu_\rho \bar{S}(s)$; however, it turns out that a first order approximation to $\nu_\rho \bar{S}(s)$ is good enough. Therefore, for the time being we will think of μ^s as some measure $\nu_\rho \bar{S}(s) + o(s)$ as $s \to 0$.

Our goal now is to define μ^s and $\tilde{\nu}$ in such a way that $\tilde{\nu}$ has a small number of discrepancies (a *discrepancy* occurs when $\eta(x) \neq \xi(x)$). Then letting the coupled process run according to the basic coupling, we can analyze the behavior of the discrepancies to prove that the measure $\mu_\rho = \lim_{t\to\infty} \nu_\rho \bar{S}(t)$ exists.

Let us now explicitly describe the initial coupling measure $\tilde{\nu}$ which will carry out our program. In doing so, we will also define the marginal measure μ^s which approximates $\nu_\rho \bar{S}(s)$. As a first step, let the coupled process (η_t, ξ_t^s) have initial marginal measures ν_ρ and μ^s respectively. The measure $\tilde{\nu}$ is then a product of a marginal measure for

$$\begin{pmatrix} \xi_0^s(0) & \xi_0^s(1) \\ \eta_0(0) & \eta_0(1) \end{pmatrix} \tag{2.12}$$

and independent Bernoulli random variables for all $x \neq 0, 1$. The Bernoulli random variables give the probability ρ to $\eta_0(x) = \xi_0^s(x) = 1$ and $1 - \rho$ to $\eta_0(x) = \xi_0^s(x) = 0$ for all $x \neq 0, 1$. The distribution of (2.12) is as follows:

Value	Probability
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	ρ^2
$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\rho(1-\rho) - s[\rho(1-\rho)\epsilon]$
$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\rho(1-\rho)$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$(1-\rho)^2$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$s[\rho(1-\rho)\epsilon]$.

Note that this measure is well-defined for $s < \frac{1}{\epsilon}$.

The probabilities above give the marginal distribution ν_ρ to η_0. Define the measure μ^s as the marginal distribution for ξ_0^s; it is a product of independent Bernoulli random variables with parameter ρ at sites $x \neq 0, 1$ with the following distribution for $(\xi_0^s(0), \xi_0^s(1))$:

Value	Probability
$(1,1)$	ρ^2
$(1,0)$	$\rho(1-\rho) - s[\rho(1-\rho)\epsilon]$
$(0,1)$	$\rho(1-\rho) + s[\rho(1-\rho)\epsilon]$
$(0,0)$	$(1-\rho)^2$

As desired, up to first order in s, $(\xi_0^s(0), \xi_0^s(1))$ has the same distribution as $(\eta_s(0), \eta_s(1))$ under ν_ρ. This is what lies behind the next lemma whose proof is similar to that of Lemma 3.3 in Andjel, Bramson, and Liggett(1988).

Lemma 2.3.2. *For any $f \in \mathcal{D}(\mathbb{X})$,*

$$\lim_{s \to 0} \frac{Ef(\xi_0^s) - \int f \, d\nu_\rho \bar{S}(s)}{s} = 0.$$

Proof. Because ν_ρ is invariant under a permutation of the coordinates $\xi(x)$ and $\xi(y)$,

$$\begin{aligned} &\int f(\xi_{xy})\{\bar{p}(x,y)\xi(x)[1-\xi(y)] + \bar{p}(y,x)\xi(y)[1-\xi(x)]\} \, d\nu_\rho \\ = &\int f(\xi)\{\bar{p}(x,y)\xi(y)[1-\xi(x)] + \bar{p}(y,x)\xi(x)[1-\xi(y)]\} \, d\nu_\rho. \end{aligned}$$

Since $\bar{p}(x,y) = \bar{p}(y,x)$ for all $(x,y) \neq (0,1), (1,0)$, we can subtract the right-hand side of the above equation from the left-hand side to get

$$\int \bar{p}(x,y)[f(\xi_{xy}) - f(\xi)]\{\xi(x)[1-\xi(y)] + \xi(y)[1-\xi(x)]\} \, d\nu_\rho = 0$$

for all $(x,y) \neq (0,1), (1,0)$.

Using (1.1), we therefore have that

$$\int \Omega f \, d\nu_\rho = \int \sum_{x,y} \bar{p}(x,y)\xi(x)(1-\xi(y))[f(\xi_{xy}) - f(\xi)] \, d\nu_\rho$$

$$= \int \{\bar{p}(0,1)\xi(0)(1-\xi(1))[f(\xi_{01}) - f(\xi)] + \bar{p}(1,0)\xi(1)(1-\xi(0))[f(\xi_{01}) - f(\xi)]\} \, d\nu_\rho$$

$$= \int \{\bar{p}(0,1)\xi(0)(1-\xi(1))[f(\xi_{01}) - f(\xi)] + \bar{p}(1,0)\xi(0)(1-\xi(1))[f(\xi) - f(\xi_{01})]\} \, d\nu_\rho$$

$$= \int \epsilon \, \xi(0)(1-\xi(1)[f(\xi_{01}) - f(\xi)] \, d\nu_\rho$$

$$= \int \epsilon(\xi(1) - \xi(0)) f \, d\nu_\rho.$$

But now, using the explicit expression for the distribution of ξ_0^s, we also get for $s > 0$ that

$$\frac{Ef(\xi_0^s) - \int f \, d\nu_\rho}{s} = \int \epsilon(\xi(1) - \xi(0)) f \, d\nu_\rho = \int \Omega f \, d\nu_\rho.$$

By the definition of the generator

$$\int \Omega f \, d\nu_\rho = \lim_{s\to 0} \frac{\int f \, d\nu_\rho \bar{S}(s) - \int f \, d\nu_\rho}{s}.$$

Combining the last two equations gives us

$$\lim_{s\to 0} \frac{Ef(\xi_0^s) - \int f \, d\nu_\rho \bar{S}(s)}{s} = 0.$$

□

Let $(\eta_t^{(u)}, \xi_t^{(u)})$ be a process that runs according to the basic coupling. Its initial distribution is a product of independent Bernoulli random variables giving probability ρ to $\eta_0^{(u)}(x) = \xi_0^{(u)}(x) = 1$ and $1-\rho$ to $\eta_0^{(u)}(x) = \xi_0^{(u)}(x) = 0$ for all $x \neq u$ and probability 1 to $\xi_0^{(u)}(u) = 1, \eta_0^{(u)}(u) = 0$.

Also, define $(\hat{\eta}_t, \hat{\xi}_t^s)$ by conditioning (η_t, ξ_t^s) on the event that

$$\begin{pmatrix} \xi_0^s(0) & \xi_0^s(1) \\ \eta_0(0) & \eta_0(1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is the only event for which η_0 and ξ_0^s differ. Note that after conditioning, the distribution of the coupling no longer depends on s.

The proof of the next lemma follows that of Lemma 3.4 in Andjel, Bramson, and Liggett(1988).

Lemma 2.3.3. *If $A \in \mathbb{Y}$ then*

$$|\frac{d}{dt}\nu_\rho \bar{S}(t)\{\eta(x) = 1 \textit{ for all } x \in A\}| \leq \epsilon\rho(1-\rho)\sum_{u=0,1}\sum_{x\in A} E[\xi_t^{(u)}(x) - \eta_t^{(u)}(x)].$$

Proof. Let

$$g(\eta) = \prod_{x\in A}\eta(x).$$

Then $g \in \mathcal{D}(\mathbb{X})$, so $f = \bar{S}(t)g$ is also in $\mathcal{D}(\mathbb{X})$ by Theorem I.3.9 of IPS. Letting $\nu_\rho^t = \nu_\rho \bar{S}(t)$, we compute

$$\begin{aligned}
& \frac{d}{dt}\nu_\rho \bar{S}(t)\{\eta(x) = 1 \text{ for all } x \in A\} \\
= \ & \lim_{s\to 0}\frac{1}{s}[\nu_\rho^{t+s}\{\eta(x) = 1 \text{ for all } x \in A\} - \nu_\rho^t\{\eta(x) = 1 \text{ for all } x \in A\}] \\
= \ & \lim_{s\to 0}\frac{1}{s}[\int g\, d\nu_\rho^{t+s} - \int g\, d\nu_\rho^t] \\
= \ & \lim_{s\to 0}\frac{1}{s}[\int f\, d\nu_\rho^s - \int f\, d\nu_\rho] \\
= \ & \lim_{s\to 0}\frac{Ef(\xi_0^s) - \int f\, d\nu_\rho}{s}
\end{aligned}$$

where the last equality follows from Lemma 2.3.2. This in turn equals

$$\begin{aligned}
\lim_{s\to 0}\frac{Ef(\xi_0^s) - Ef(\eta_0)}{s} &= \lim_{s\to 0}\frac{Eg(\xi_t^s) - Eg(\eta_t)}{s} \\
&= \lim_{s\to 0}\frac{1}{s}E[\prod_{x\in A}\xi_t^s(x) - \prod_{x\in A}\eta_t(x)].
\end{aligned}$$

The proof is completed by the inequalities

$$
\begin{aligned}
\lim_{s\to 0}\frac{1}{s}|E\prod_{x\in A}\xi_t^s(x)-\prod_{x\in A}\eta_t(x)| &\leq \lim_{s\to 0}\frac{1}{s}E|\prod_{x\in A}\xi_t^s(x)-\prod_{x\in A}\eta_t(x)| \\
&\leq \lim_{s\to 0}\frac{1}{s}P(\xi_t^s(x)\neq\eta_t(x)\text{ for some }x\in A) \\
&\leq \lim_{s\to 0}\frac{1}{s}\sum_{x\in A}P(\xi_t^s(x)\neq\eta_t(x)) \\
&= \epsilon\rho(1-\rho)\sum_{x\in A}P(\hat{\xi}_t(x)\neq\hat{\eta}_t(x)) \\
&\leq \epsilon\rho(1-\rho)\sum_{u=0,1}\sum_{x\in A}P(\xi_t^{(u)}(x)-\eta_t^{(u)}(x)).
\end{aligned}
$$

The last inequality is due to a property given by the basic coupling: when the two discrepancies

$$
\begin{pmatrix} \xi_T^s(x)=1 \\ \eta_T(x)=0 \end{pmatrix} \text{ and } \begin{pmatrix} \xi_T^s(x)=0 \\ \eta_T(x)=1 \end{pmatrix}
$$

meet, they cancel each other out to result in no discrepancies for all $t\geq T$. □

We now give an argument that extends the infinitesimal coupling and the two lemmas to a general quasi-symmetric kernel. The first thing is to realize that if ϵ is negative, we can obtain analogs of the two lemmas if we make the following changes to the distribution of (2.12):

Value	Probability
$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\rho(1-\rho)$
$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\rho(1-\rho) - s[\rho(1-\rho)\lvert\epsilon\rvert]$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$s[\rho(1-\rho)\lvert\epsilon\rvert]$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	0.

Next we see that if there are multiple differences between $p(x, y)$ and $\bar{p}(x, y)$, we can superimpose the changes to the distribution of $\tilde{\nu}$ to get analogs of the two lemmas. For instance if

$$\bar{p}(w, y) = p(w, y) + \epsilon_1 \text{ and } \bar{p}(w, z) = p(w, z) + \epsilon_2 \text{ where } \epsilon_i > 0,$$

then when $s < \frac{1}{\epsilon_1+\epsilon_2}$, the distribution of the coupling at (w, y, z) at time 0 is identical to the marginal measures for $(\eta_0(w), \eta_0(y), \eta_0(z))$ and for $(\xi_0^s(w), \xi_0^s(y), \xi_0^s(z))$, except at the values in the table below:

Value	Probability
$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\rho(1-\rho)^2 - s[\rho(1-\rho)^2(\epsilon_1+\epsilon_2)]$
$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\rho^2(1-\rho) - s[\rho^2(1-\rho)\epsilon_1]$
$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\rho^2(1-\rho) - s[\rho^2(1-\rho)\epsilon_2]$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$s[\rho(1-\rho)^2\epsilon_1]$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$s[\rho^2(1-\rho)\epsilon_1]$
$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$s[\rho(1-\rho)^2\epsilon_2]$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$s[\rho^2(1-\rho)\epsilon_2]$.

We have the following analog of Lemma 2.3.3:

Corollary 2.3.4. *If A is any finite subset of $\mathcal{S}$ then there exists $C < \infty$ such that*

$$|\frac{d}{dt}\nu_\rho \bar{S}(t)\{\eta(x) = 1 \textit{ for all } x \in A\}| \leq C \sum_{u\in B}\sum_{x\in A} E[\xi_t^{(u)}(x) - \eta_t^{(u)}(x)].$$

The proof of the corollary is essentially the same as that of Lemma 2.3.3 so we only make the following remark. It is important to note that a pair of discrepancies of opposite type $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ occur together, but any two pairs do not occur at the same time. Therefore, we still have that the only interaction between discrepancies is when two discrepancies of opposite type cancel each other out.

We no longer assume that the transition kernel is given by (2.11). Instead, we will prove Lemma 2.3.5 and Theorem 2.3.1 for a general quasi-symmetric transition kernel.

Given the process $(\eta_t^{(u)}, \xi_t^{(u)})$ described in the previous section, let $X^*(t)$ mark the position at time t of the discrepancy that starts at u. Notice that while the process $X^*(t)$ is not a Markov process, the joint process $(X^*(t), \eta_t)$ is a Markov process. Let

$$G^*(u,x) = E^u \int_0^\infty P(X^*(t) = x)\, dt$$

be the expected time that the discrepancy starting at u spends at x given that the initial distribution of $(X^*(t), \eta_t)$ for sites not equal to u are independent Bernoulli random variables with parameter ρ. If X_n^* is the embedded discrete-time process for $X^*(t)$, define

$$H^*(u,x) = \sup_\eta P^{(u,\eta)}(X_n^* = x \text{ for some } n \geq 1).$$

Lemma 2.3.5. *If $X(t)$ is transient and $\bar{p}(x,y) > 0$ whenever $p(x,y) > 0$ then $G^*(u,x) < \infty$ for all $u, x \in \mathcal{S}$.*

Proof. Recall that

$$B = \{x \in \mathcal{S} : \bar{p}(x,y) \neq \bar{p}(y,x) \text{ for some } y \in \mathcal{S}\}.$$

If the discrepancy is at site x, it goes to y at rate $\bar{p}(x,y)$ when $\xi^{(u)}(y) = \eta^{(u)}(y) = 0$ and at rate $\bar{p}(y,x)$ when $\xi^{(u)}(y) = \eta^{(u)}(y) = 1$. But when $x \notin B$, $\bar{p}(x,y) = \bar{p}(y,x)$.

Therefore when $X^*(t) \notin B$, $X^*(t)$ moves according to the same transition rates as $X(t)$.

Couple $X^*(t)$ and $X(t)$ starting from u so that they move together as much as possible and let

$$E = \{\omega : X^*(t)(\omega) = X(t)(\omega) \text{ for all } t \geq 0, X_n \neq u \text{ for all } n \geq 1\}$$

where $X_n, n \geq 0$ is the embedded discrete-time chain for $X(t)$. Since B is finite and $X(t)$ is transient, and since $\bar{p}(x,y) > 0$ whenever $p(x,y) > 0$, we see from the argument above that $\inf_\eta P^{(u,\eta)}(E) > 0$.

For each x we have

$$\begin{aligned} H^*(x,x) &= \sup_\eta P^{(x,\eta)}\left[\{X_n^* = x \text{ for some } n \geq 1\} \cap (E \cup E^c)\right] \\ &= \sup_\eta P^{(x,\eta)}(\{X_n^* = x \text{ for some } n \geq 1\} \cap E^c) \leq 1 - \inf_\eta P^{(x,\eta)}(E). \end{aligned}$$

Using Proposition 4-20 in Kemeny, Snell, and Knapp(1976) we get that for some constant C,

$$G^*(u,x) \leq C \sum_{k \geq 0} (H^*(x,x))^k < \infty.$$

□

Proof of Theorem 2.3.1. We first prove that $\lim_{t\to\infty} \nu_\rho \bar{S}(t)$ exists. By the inclusion-exclusion principle we need only show that for each $A \in \mathbb{Y}$,

$$\lim_{t\to\infty} \nu_\rho \bar{S}(t)\{\eta(x) = 1 \text{ for all } x \in A\} \tag{2.13}$$

exists.

Suppose to the contrary that there exists some A for which (2.13) does not exist. Then there exists a sequence $\{t_n\}$ going to infinity such that the set

$$\{\nu_\rho \bar{S}(t_n)\{\eta(x) = 1 \text{ for all } x \in A\}\}$$

must have at least two different limit points. Therefore it must be that

$$\int_0^\infty |\frac{d}{dt}\nu_\rho \bar{S}(t)\{\eta(x) = 1 \text{ for all } x \in A\}|dt = \infty.$$

On the other hand, by Corollary 2.3.4 and Lemma 2.3.5,

$$\begin{aligned}\int_0^\infty |\frac{d}{dt}\nu_\rho \bar{S}(t)\{\eta(x) = 1 \text{ for all } x \in A\}|dt &\leq C\int_0^\infty \sum_{u\in B}\sum_{x\in A} E[\xi_t^{(u)}(x) - \eta_t^{(u)}(x)]dt \\ &\leq C\sum_{u\in B}\sum_{x\in A} G^*(u,x) < \infty,\end{aligned}$$

a contradiction. Therefore (2.13) exists for all finite A.

Now by the proof of Theorem 2.1.1 (b) it must be that

$$\lim_{n\to\infty} |\mu_\rho\{\eta(x) = 1 \text{ for all } x \in A^n\} - \nu_\rho\{\eta(x) = 1 \text{ for all } x \in A^n\}| = 0$$

giving us (2.9). □

CHAPTER 3

The Asymmetric Exclusion Process on $\mathbb{Z}$

As mentioned in the abstract, the study of the asymmetric exclusion process has been much more elusive than that of the symmetric process. General classes of invariant measures are known in the two cases where $p(x,y)$ is doubly stochastic (i.e. $\sum_{x\in\mathcal{S}} p(x,y) = 1$ for all $y \in \mathcal{S}$) or when there exists a reversible measure $\pi(x) > 0$ on $\mathcal{S}$ (i.e. $\pi(x)p(x,y) = \pi(y)p(y,x)$). However, a complete description of $\mathcal{I}$ is known only in the three cases when either

(a) $p(x,y)$ is reversible and positive recurrent for either the particles or holes (1's or 0's) (Liggett(1976))

(b) $p(x,y)$ corresponds to certain random walks on $\mathbb{Z}$ (Liggett(1976) and Bramson, Liggett, and Mountford(2002)) or

(c) $p(x,y)$ corresponds to a birth and death chain on $\mathbb{Z}^+$ (Liggett(1976)).

Almost nothing is known about the domains of attraction concerning invariant measures in the asymmetric case, although we note here that there are some nice theorems concerning the case where $p(x,y)$ is an asymmetric simple random walk on $\mathbb{Z}$ (see Liggett(1999)).

Our purpose in this chapter is to shed some more light on the problem of classifying $\mathcal{I}$ and its respective domains of attraction for the asymmetric exclusion process when a reversible measure $\pi(x)$ exists for $p(x,y)$. In order to describe the results of this chapter we must first discuss case (a) and state a special case of (b) above.

We start by stating what is known for the mean zero case of (b). As defined in the introductory chapter, ν_ρ is the product measure on $\mathbb{X}$ with marginals $\nu_\rho\{\eta(x) = 1\} = \rho$. Liggett(1976) uses a coupling of two exclusion processes to show that when $p(x,y) = p(0, y-x), \sum_x |x|p(0,x) < \infty$, and $\sum_x xp(0,x) = 0$ on $\mathbb{Z}$, the set of extremal invariant measures is

$$\mathcal{I}_e = \{\nu_\rho : \rho \in [0,1]\}. \tag{3.1}$$

Before describing the invariant measures for case (a), we define some extremal reversible invariant measures $\{\nu^{(n)}\}$ when a reversible measure $\pi(x)$ satisfying

$$\sum_x \pi(x)/[1+\pi(x)]^2 < \infty \tag{3.2}$$

exists. This family of extremal reversible measures was first discovered by Liggett. In particular, he breaks down (3.2) into three cases and writes

1. If $\sum_x \pi(x) < \infty$, let $A_n = \{\eta : \sum_x \eta(x) = n\}$ for nonnegative integers n.

2. If $\sum_x 1/\pi(x) < \infty$, let $A_n = \{\eta : \sum_x [1 - \eta(x)] = n\}$ for nonnegative integers n.

3. If $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$, $\sum_x \pi(x) = \infty$, and $\sum_x 1/\pi(x) = \infty$, there exists a $T \subset \mathcal{S}$ for which $\sum_{x \in T} \pi(x) < \infty$ and $\sum_{x \notin T} 1/\pi(x) < \infty$. In this case, let

$$A_n = \{\eta \in A : \sum_{x \in T} \eta(x) - \sum_{x \notin T} [1 - \eta(x)] = n\}$$

for integers n.

To define $\{\nu^{(n)}\}$, let ν^c be the product measure with marginals $\nu^c\{\eta : \eta(x) = 1\} = \frac{c\pi(x)}{1+c\pi(x)}$. Liggett shows that the measures

$$\begin{aligned} \nu^{(n)}(\cdot) &= \nu^c(\cdot|A_n) \text{ for } n \in \mathbb{Z}, \\ \nu^{(\infty)} &= \text{ the pointmass on } \eta(x) \equiv 1, \\ \nu^{(-\infty)} &= \text{ the pointmass on } \eta(x) \equiv 0 \end{aligned} \tag{3.3}$$

are the unique stationary distributions for the positive recurrent Markov chains on A_n (for the first two cases of the trichotomy, assume $\nu^{(n)} = \nu^{(-\infty)}$ for all $n \leq 0$). Note that changing T in the third case of the trichotomy above amounts to a relabeling of the sequence $\{\nu^{(n)} : n \in \mathbb{Z}\}$.

A simple consequence of Theorem B52 in Liggett(1999) is that the reversible measures $\{\nu^{(n)}\}$ are extremal in the set of invariant measures. For the first two cases in the trichotomy of (3.2) above, these are the only extremal invariant measures:

Theorem 3.0.6. *If either $\sum_x \pi(x) < \infty$ or $\sum_x 1/\pi(x) < \infty$, then*

$$\mathcal{I}_e = \{\nu^{(n)} : 0 \leq n \leq \infty\}.$$

For a proof of this theorem we refer the reader to Theorem VIII.2.17 in IPS. Note that this theorem corresponds exactly to case (a) at the very beginning of this chapter.

Whenever a reversible measure $\pi(x)$ on $\mathcal{S}$ exists, the product measures $\{\nu^c : c \in [0, \infty]\}$ are well-defined. A simple generator computation shows that these measures are invariant for the exclusion process (see Theorem VIII.2.1 in IPS for details). Applying Kakutani's Dichotomy (e.g. page 244 of Durrett(1996)) we have that $\sum_x \pi(x)/[1 + \pi(x)]^2 = \infty$ is a necessary and sufficient condition for the measures $\{\nu^c : c \in [0, \infty]\}$ to be mutually singular. Since all the results in this chapter concern the reversible measures $\{\nu^c : c \in [0,\infty]\}$, we will assume throughout the chapter that a reversible $\pi(x)$ exists.

3.1 The results

In Section 3.2 we prove Theorem 3.2.1 which states that $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ is exactly the situation in which the measure ν^c is extremal invariant. Not only does this result have some nice applications, but knowing that an invariant measure is extremal in the set of invariant measures has always been an interesting question concerning particle systems. Examples of such results are Theorem III.1.17 in Liggett(1999) and Theorem 1.4 in Sethuraman(2001). The main reason extremality of invariant measures is interesting is its close connection with ergodicity. In particular, if the initial measure for a process is an extremal invariant measure then the process evolution is *ergodic* with respect to time shifts. This means that

(i) for all finite collections of times $t_1, \ldots, t_n$, the joint distributions of

$$(\eta_{t_1+t}, \ldots, \eta_{t_n+t})$$

are independent of t and

(ii) if E is an event in the path space that is invariant under time shifts, $P(E) = 0$ or 1.

Sections 3.3 and 3.4 use Theorem 3.2.1 to extract information about the invariant measures of the process on $\mathbb{Z}$. In particular, Section 3.3 modifies Liggett's original proof of the result stated above equation (3.1) to obtain the following result:

Theorem 3.1.1. *Let $\mathbb{Z}$ be irreducible with respect to a transition kernel $p(x, y)$*

for which there exists a reversible measure $\pi(x)$. Suppose $q_i(z)$ is a transition kernel such that $\sum_z zq_i(z) = 0$ and $\sum_z |z|q_i(z) < \infty$ for $i = 1, 2$, and suppose that

$$\lim_{K\to\infty} \sum_{x\geq 0} \sum_{|z|\geq|x-K|} |p(x,x+z)-q_1(z)| = 0 \text{ and } \lim_{K\to\infty} \sum_{x\leq 0} \sum_{|z|\geq|x+K|} |p(x,x+z)-q_2(z)| = 0. \tag{3.4}$$

(a) If $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ then $\mathcal{I}_e = \{\nu^c : c \in [0,\infty]\}$.

(b) If $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$ then $\mathcal{I}_e = \{\nu^{(n)} : -\infty \leq n \leq \infty\}$.

In essence the above theorem says that when the transition probabilities are asymptotically translation invariant and have an asymptotic mean of zero, the reversible measures are the only invariant measures. Theorem 3.1.1 is merely an extension (in the case where $\pi(x)$ exists) of the result stated above equation (3.1).

Condition (3.4) may seem somewhat daunting, but note that if $\lim_{x\to\infty} p(x,x+z) = q_1(z)$, $\lim_{x\to-\infty} p(x,x+z) = q_2(z)$, and $p(x,y)$ has finite range (i.e. there exists an n such that $p(x,y) = 0$ for $|x-y| > n$), then (3.4) and $\sum_z |z|q_1(z) < \infty$ are both automatically satisfied. Also, the below condition which is somewhat easier to grasp than (3.4) implies (3.4):

$$\sum_{x\geq 0} \sum_{z} |p(x,x+z) - q_1(z)| < \infty \text{ and } \sum_{x\leq 0} \sum_{z} |p(x,x+z) - q_2(z)| < \infty.$$

A typical situation for which the theorem holds is when the transition rates are nearest-neighbor ($p(x,y) = 0$ when $|x-y| > 1$) and are given by $p(x,x+1) = p(x,x-1) = 1/2$ except for finitely many x.

Note that the premises of the theorem together with the assumption that a reversible $\pi(x)$ exists imply that $q_i(z)$ must be symmetric. To see this suppose $q_1(z)$ is not symmetric. Also, assume that $q_1(z_1) > q_1(-z_1) > 0$ for some $z_1 \in \mathbb{N}$.

We can do this without loss of generality since $q_1(z) > 0$ implies $q_1(-z) > 0$ by the reversibility of $\pi(x)$. The mean zero assumption tells us there exists $z_2 \in \mathbb{N}$ such that $q_1(z_2) < q_1(-z_2)$. If z_3 is a multiple of both z_1 and z_2 then since $p(x, x+z) \to q_1(z)$ we can find x_1 so that for $x > x_1$, $\pi(x) < \pi(x+z_3)$. But we can also find x_2 so that for $x > x_2$, $\pi(x) > \pi(x+z_3)$, a contradiction. So $q_1(z)$ must be symmetric. The proof that $q_2(z)$ is symmetric follows similarly.

The proof of the above theorem follows Liggett's original outline and does not actually require Theorem 3.2.1. However, the usefulness of Theorem 3.2.1 is seen in the simplification of one part of Liggett's original proof.

In Section 3.4 we prove a theorem concerning the nearest-neighbor exclusion process on $\mathbb{Z}$. For the statement of the theorem we will need the following definitions.

Let $\mathcal{L}^-$ be the set of limit points of $\{\pi(x), x < 0\}$ and $\mathcal{L}^+$ be the set of limit points of $\{\pi(x), x > 0\}$.

Theorem 3.1.2. *Suppose that* $\inf_{|x-y|=1} p(x,y) > 0$ *for a nearest-neighbor exclusion process on* $\mathbb{Z}$*. Then nonreversible invariant measures can exist only when either (a)* $\mathcal{L}^- = \{0\}$ *and* $\mathcal{L}^+ = \{\infty\}$*, or (b)* $\mathcal{L}^- = \{\infty\}$ *and* $\mathcal{L}^+ = \{0\}$*.*

The above theorem in no way guarantees the existence of nonreversible invariant measures as seen by the following example. Let

$$p(-1,-2) = p(-1,0) = p(0,-1) = p(0,1) = 1/2, \tag{3.5}$$
$$p(x, x+1) = 1 - p(x, x-1) = \frac{|x|+1}{2|x|} \text{ otherwise.}$$

This transition kernel gives us situation (a) in the theorem above. The reversible invariant measures $\{\nu^c : c \in [0,\infty]\}$ certainly exist, but it is easy to see that condition (b) of Theorem 3.1.1 is satisfied by (3.5), therefore there are no nonreversible invariant measures.

A curious aside is as follows. If in this example we start the process off with initial measure ν_ρ and take the limit of some converging sequence of measures

$$\frac{1}{T_n}\int_0^{T_n} \nu_\rho S(t)\,dt \tag{3.6}$$

then this limit is an invariant measure for the process (by the argument given in (2.3)). In view of the previous discussion, this limit must converge to some mixture of the extremal invariant measures $\{\nu^{(n)}, -\infty \leq n \leq \infty\}$. It would be interesting indeed to find out which mixture (3.6) converges to. Note here that we started off with an initial state that concentrates on an uncountable number of states, but the limiting distribution concentrates on a countable number of states (which may very well be just the point masses of all 0's or all 1's).

Assume now that $p(x,y)$ is an asymmetric, nearest-neighbor random walk kernel with nonzero mean. We then have one of the situations described in Theorem 3.1.2, and one might correctly guess that there exists some nonreversible invariant measure. In fact, as it turns out the measures

$$\{\nu_\rho : \rho \in [0,1]\} \tag{3.7}$$

are nonreversible invariant measures.

We note here that the set of measures in (3.7) is the same as the set of measures in (3.1) but are of an entirely different nature. In the setting of (3.1) the measures $\{\nu_\rho : \rho \in [0,1]\}$ are reversible for the nearest-neighbor random walk kernel with mean zero, and they constitute the entire set of extremal invariant measures. On the other hand, for a nearest-neighbor random walk kernel with nonzero mean, the measures $\{\nu_\rho : \rho \in [0,1]\}$ are not reversible and

$$\mathcal{I}_e = \{\nu_\rho : \rho \in [0,1]\} \cup \{\nu^c : c \in [0,\infty]\}.$$

This was proven by Liggett(1976).

The discussion in the previous paragraphs might make us wonder for which transition kernels a nonreversible invariant measure exists. To gain more insight into the situation we introduce a concept known as the *flux* of an invariant measure μ. We will continue to assume that the transition probabilities are nearest-neighbor, but we will no longer assume they are translation invariant. Define

$$\text{flux}(\mu) = p(x,x{+}1)\mu\{\eta(x) = 1, \eta(x{+}1) = 0\} - p(x{+}1,x)\mu\{\eta(x) = 0, \eta(x{+}1) = 1\}. \tag{3.8}$$

Let $1_x(\eta) = \eta(x)$ be the indicator function of $\{\eta(x) = 1\}$. By computing the positive and negative terms of the left-hand side of $\int \Omega 1_x d\mu = 0$ it can be seen that flux(μ) is independent of x.

When an invariant measure μ is reversible it can easily be seen from (3.8) that flux$(\mu) = 0$. So if an invariant measure exists whose flux is nonzero it must be nonreversible. If $p(x,y)$ is a random walk kernel with nonzero mean, the invariant measures $\{\nu_\rho : \rho \in [0,1]\}$ all have a positive flux with the flux being maximized when $\rho = 1/2$ (a full discussion of this can be found in either Janowski and Lebowitz(1994) or Part III of Liggett(1999)). This positive flux is the reason why (3.7) is fundamentally different from (3.1). It would be quite nice if one could prove that some nonreversible invariant measure exists whenever $p(x,x{+}1) > 1/2{+}\epsilon$ for all x. The ϵ here serves the role of providing some positive flux in the limit.

Finally, Section 3.5 will apply Theorem 3.2.1 to give information concerning the domains of attraction (in a Cesaro sense) of reversible measures in the case where $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$. The results of Section 3.5 only give sufficient conditions for Cesaro convergence to an invariant measure, but are nonetheless interesting since so little is known concerning domains of attraction for the asymmetric exclusion process. The key known results concerning domains of attraction of asymmetric exclusion processes are stated in Andjel, Bramson, Liggett(1988). They concern the limiting distribution of exclusion processes with asymmetric, nearest-neighbor random walk kernels when the initial measures are certain product measures. To get an idea of how difficult it is to prove anything of this sort, we refer the reader to Andjel, Bramson, Liggett(1988).

3.2 Extremal reversible measures

In this section we state and prove Theorem 3.2.1. The common technique used in the proof of this theorem and in the proofs of most of the other results in this chapter is the coupling technique. In particular, we will be using the basic coupling of η_t and ξ_t defined in (2.10). This is just the coupling η_t and ξ_t which lets the two exclusion processes move together as much as possible.

Theorem 3.2.1. *Suppose $\mathcal{S}$ is irreducible with respect to $p(x, y)$. Then ν^c is extremal invariant if and only if $\sum_x \pi(x)/[1 + \pi(x)]^2 = \infty$.*

Proof. Suppose $\sum_x \pi(x)/[1 + \pi(x)]^2 < \infty$. By the definition of $\nu^{(n)} \in \mathcal{I}$ given in (3.3) we have that

$$\nu^c = \sum_{n=-\infty}^{\infty} \nu^c(A_n)\nu^{(n)}$$

where $A_n = \emptyset$ if they have not been defined previously. Therefore ν^c is not extremal invariant giving us one direction of the theorem. We will prove the other direction.

Assume throughout that $0 < c < \infty$. Since ν^c is an invariant measure and since all bounded continuous functions can be approximated uniformly by functions that depend on finitely many coordinates then by Theorem B52 in Liggett(1999), we need only show that for any two functions $f, g \in \mathcal{D}(\mathbb{X})$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T E^{\nu^c} f(\eta_0) g(\eta_t) dt = \int f \, d\nu^c \int g \, d\nu^c.$$

We claim that to show the above equation holds, it is enough to show that for any $A \in \mathbb{Y}$ and for $\mu^c_{1,A}(\cdot) = \nu^c(\cdot | \{\eta(x) = 1 \forall x \in A\})$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mu^c_{1,A} S(t) dt = \nu^c. \tag{3.9}$$

To see this define the measures $\mu^c_{\zeta,A}(\cdot) = \nu^c(\cdot | \{\eta(x) = \zeta(x) \forall x \in A\})$ where ζ is a configuration on $\{0,1\}^A$. We can write the measure ν^c as a linear combination

$$\nu^c = \sum_{\zeta \in \{0,1\}^A} a_\zeta \mu^c_{\zeta,A}$$

where we use the convention that $\zeta = i$ is the configuration in $\{0,1\}^A$ such that $\zeta(x) = i$ for all $x \in A$. For

$$f_A = \begin{cases} 1 & \text{when } \eta(x) = 1 \text{ for all } x \in A \\ 0 & \text{otherwise} \end{cases}$$

we have that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T E^{\nu^c} f_A(\eta_0) g(\eta_t) dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T a_1 \int S(t) g(\eta) d\mu^c_{1,A} dt = \int f_A d\nu^c \int g \, d\nu^c$$

which proves the claim.

Define $\mu^c_{0,A}$ similarly to the way we defined $\mu^c_{1,A}$. If we assume a fixed $A \in \mathbb{Y}$ then we can drop the subscript A so as to write $\mu^c_i = \mu^c_{i,A}$. The rest of the proof will now argue that (3.9) holds.

Choose $\delta > 0$ and couple the processes η_t and ξ_t using the basic coupling starting with measures μ^c_0 and μ^c_1 so that η_0 and ξ_0 disagree only for $x \in A$. In particular, since the basic coupling is the coupling which allows η_t and ξ_t to move together as much as possible, then η_t and ξ_t can differ at most at n sites where $|A| = n$.

If there exists $\bar{T}$ such that for all $T > \bar{T}$

$$\frac{1}{T}\int_0^T [\mu_1^c S(t)\{\xi(0) = 1\} - \mu_0^c S(t)\{\eta(0) = 1\}]dt \leq \delta$$

then we must have that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu_1^c S(t)\{\xi(0) = 1\}dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mu_0^c S(t)\{\eta(0) = 1\}dt.$$

Keeping in mind the way that η_t and ξ_t are coupled, irreducibility then tells us that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu_1^c S(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mu_0^c S(t)dt.$$

But the measure ν^c lies stochastically between the left-hand side and the right-hand side of the equation above, so in fact we must have that (3.9) holds.

We can therefore assume to the contrary that there exists a $\delta > 0$ and a sequence $\{T_n\}$ such that

$$\frac{1}{T_n}\int_0^{T_n} [\mu_1^c S(t)\{\xi(0) = 1\} - \mu_0^c S(t)\{\eta(0) = 1\}]dt > \delta \tag{3.10}$$

for all n.

Pick $\epsilon > 0$ so that

$$\nu^{c+\epsilon}\{\xi(0) = 1\} - \nu^{c-\epsilon}\{\eta(0) = 1\} < \delta/3.$$

Using the basic coupling once more, couple the processes η_t and ξ_t starting off in the measures μ_1^c and $\nu^{c+\epsilon}$ so that $\lambda_1\{(\eta, \xi) : \eta(x) \leq \xi(x) \text{ for all } x \in \mathcal{S}\backslash A\} = 1$ where λ_1 is the coupling measure. If $\hat{\mu}^c = \nu^c(\cdot|\{\eta(x) = 0 \text{ for some } x \in A\})$ then

$$\nu^c = \gamma\mu_1^c + (1-\gamma)\hat{\mu}^c$$

for $\gamma = \nu^c\{\eta(x) = 1 \text{ for all } x \in A\}$ (note that γ is equal to a_1 used above). Couple the measures $\hat{\mu}^c$ and $\nu^{c+\epsilon}$ in a way similar to λ_1 so that we get another coupling measure λ_2.

Choose a subsequence $\{T_{n_k}\}$ so that we can define some limiting invariant measure

$$\omega_1 = \lim_{k\to\infty} \frac{1}{T_{n_k}} \int_0^{T_{n_k}} \lambda_1 S(t)dt.$$

Let ν_1^c be the η-marginal measure of ω_1 so that in particular

$$\nu_1^c = \lim_{k\to\infty} \frac{1}{T_{n_k}} \int_0^{T_{n_k}} \mu_1^c S(t)dt.$$

To complete the proof of the theorem we will need the following lemma:

Lemma 3.2.2. $\nu^{c+\epsilon} \geq \nu_1^c$.

Proof of lemma. Let $f_x(\eta,\xi) = [1-\eta(x)]\xi(x)$,

$$D_m = \{(\eta,\xi) : \eta(x) > \xi(x) \text{ at exactly } m \text{ sites}\},$$

and $D = \bigcup_{m\geq 1} D_m$. If $\nu^{c+\epsilon} \not\geq \nu_1^c$ then it must be that $\omega_1(D) > 0$. We claim that this implies

$$\int_D \sum_x f_x d\omega_1 = 0.$$

To prove the claim, assume to the contrary that $\int_D \sum_x f_x d\omega_1 > 0$ so that there exist sites for which $\eta(x) < \xi(x)$. Let M be the largest m for which $\omega_1(D_m) > 0$. Then by the irreducibility condition and by the fact that there exist sites for which $\eta(x) < \xi(x)$, we have $\omega_1 S(t)(D_M) < \omega_1(D_M)$ for $t > 0$. But this is a contradiction to the invariance of ω_1 proving the claim.

Now if the two processes η_t and ξ_t have the measures ν^c and $\nu^{c+\epsilon}$ respectively then let ω be the coupling measure for $\{(\eta_t,\xi_t)\}$ which concentrates on $\nu^c \leq \nu^{c+\epsilon}$.

For this coupling, the ω-probability that $f_x(\eta, \xi) = 1$ for a given x is equal to the left-hand side below:

$$\frac{(c+\epsilon)\pi(x)}{1+(c+\epsilon)\pi(x)} - \frac{c\pi(x)}{1+c\pi(x)} > \frac{\epsilon\pi(x)}{[1+(c+\epsilon)\pi(x)]^2}.$$

Since $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$, by the Borel-Cantelli Lemma the ω probability that $\sum_x f_x = \infty$ is equal to 1. The measure ω_1 is absolutely continuous with respect to ω since

$$\omega = \gamma\lambda_1 + (1-\gamma)\lambda_2 = \gamma\omega_1 + (1-\gamma)\lim_{k\to\infty}\frac{1}{T_{n_k}}\int_0^{T_{n_k}} \lambda_2 S(t)dt$$

where λ_2 is as defined above. Therefore $\int_E \sum_x f_x d\omega_1 = \infty$ for any set E with positive ω_1 measure. But this contradicts $\int_D \sum_x f_x d\omega_1 = 0$, so it must be that $\omega_1(D) = 0$ proving the lemma. □

We now turn back to the proof of the theorem. Since by the lemma we have $\nu^{c+\epsilon} \geq \nu_1^c$, then there exists a K such that for all $k > K$

$$\frac{1}{T_{n_k}}\int_0^{T_{n_k}} \mu_1^c S(t)\{\eta(0)=1\}dt - \nu^{c+\epsilon}\{\xi(0)=1\} < \delta/3.$$

If ν_0^c is some limiting measure of

$$\frac{1}{T_{n_{k_l}}}\int_0^{T_{n_{k_l}}} \mu_0^c S(t)dt$$

then an argument similar to that used in Lemma 3.2.2 shows that $\nu^{c-\epsilon} \leq \nu_0^c$. There then exists an L such that for $l > L$

$$\nu^{c-\epsilon}\{\eta(0)=1\} - \frac{1}{T_{n_{k_l}}}\int_0^{T_{n_{k_l}}} \mu_0^c S(t)\{\xi(0)=1\}dt < \delta/3.$$

Altogether we have for $l > L$,

$$\frac{1}{T_{n_{k_l}}}\int_0^{T_{n_{k_l}}} [\mu_1^c S(t)\{\xi(0)=1\} - \mu_0^c S(t)\{\eta(0)=1\}]dt < \delta$$

which contradicts inequality (3.10) so it must be that (3.9) holds completing the proof of the theorem. □

3.3 The asymptotically mean zero process on $\mathbb{Z}$

In this section we prove Theorem 3.1.1. To do so we will need to define $\tilde{\mathcal{I}}$ as the set of invariant measures for the basic coupling and $\tilde{\mathcal{I}}_e$ as its extreme points.

Recall that $f_x(\eta,\xi) = [1-\eta(x)]\xi(x)$. In order to simplify the notation we further define the functions

$$h_{yx}(\eta,\xi) = [1-\eta(y)][1-\xi(y)]f_x(\eta,\xi), \quad g_{yx}(\eta,\xi) = \eta(y)\xi(y)f_x(\eta,\xi),$$
$$\text{and } f_{yx}(\eta,\xi) = \eta(y)[1-\xi(y)]f_x(\eta,\xi).$$

In particular, using definition (2.10), we have for T a finite subset of $\mathcal{S}$

$$\tilde{\Omega}\left(\sum_{x\in T} f_x(\eta,\xi)\right) = -\sum_{x\in T, y\in\mathcal{S}} (p(x,y)+p(y,x))f_{yx}(\eta,\xi) \tag{3.11}$$
$$+ \sum_{x\in T, y\notin T} [p(x,y)g_{xy} - p(y,x)g_{yx}] + \sum_{x\in T, y\notin T} [p(y,x)h_{xy} - p(x,y)h_{yx}].$$

Proof of Theorem 3.1.1. Let $\nu \in \tilde{\mathcal{I}}$. Then $\int \tilde{\Omega}(\sum_{x\in T} f_x)d\nu = 0$ for each finite $T \subset \mathbb{Z}$ so that for $T_{[m,n]} = \{x \in \mathbb{Z} : m \le x \le n\}$ we get

$$\sum_{x\in T_{[m,n]}, y\in\mathbb{Z}} (p(x,y)+p(y,x)) \int f_{yx} d\nu \tag{3.12}$$
$$= \sum_{x\in T_{[m,n]}, y\notin T_{[m,n]}} p(x,y)\int (g_{xy} - h_{yx})d\nu + \sum_{x\in T_{[m,n]}, y\notin T_{[m,n]}} p(y,x)\int (h_{xy} - g_{yx})d\nu.$$

Notice that the left-hand side of this equation is increasing in n and $-m$, so that when we take the limit as $n\to\infty$ or as $-m\to\infty$, a limit exists. Also, since the construction of the exclusion process assumes that $\sup_y \sum_x p(x,y)$ is finite and since $\int |g_{xy} - h_{yx}|d\nu \le 1$ and $\int |h_{xy} - g_{yx}|d\nu \le 1$, the right-hand side sums in (3.12) above are absolutely convergent for any fixed n and m.

Choosing $\epsilon > 0$ we can find N so that for $n > N$:

$$\sum_{x>n+N} \sum_{z<n-x} p(x, x+z)$$
$$\leq \sum_{x>n+N} \sum_{z<n-x} |p(x, x+z) - q_1(z)| + \sum_{|z|>N} |z| q_1(z) < \epsilon,$$

$$\sum_{0<x<n} \sum_{z>n+N-x} p(x, x+z)$$
$$\leq \sum_{0<x<n} \sum_{z>n+N-x} |p(x, x+z) - q_1(z)| + \sum_{|z|>N} |z| q_1(z) < \epsilon,$$

$$\sum_{x\leq 0} \sum_{z>n+N-x} p(x, x+z)$$
$$\leq \sum_{x\leq 0} \sum_{z>n+N-x} |p(x, x+z) - q_2(z)| + \sum_{|z|>N} |z| q_2(z) < \epsilon,$$

and

$$\sum_{x<-n-N} \sum_{z>-x-n} p(x, x+z)$$
$$\leq \sum_{x<-n-N} \sum_{z>-x-n} |p(x, x+z) - q_2(z)| + \sum_{|z|>N} |z| q_2(z) < \frac{\epsilon}{3},$$

$$\sum_{-n<x<0} \sum_{z<-x-n-N} p(x, x+z)$$
$$\leq \sum_{-n<x<0} \sum_{z<-x-n-N} |p(x, x+z) - q_2(z)| + \sum_{|z|>N} |z| q_2(z) < \frac{\epsilon}{3},$$

$$\sum_{x\geq 0} \sum_{z<-n-N-x} p(x, x+z)$$
$$\leq \sum_{x\geq 0} \sum_{z<-n-N-x} |p(x, x+z) - q_1(z)| + \sum_{|z|>N} |z| q_1(z) < \frac{\epsilon}{3}.$$

Now by the inequalities above and by (3.4) we can pass to the limit in (3.12) so as to write

$$\begin{aligned}
&\lim_{m\to-\infty}\lim_{n\to\infty}\sum_{x\in T_{[m,n]},y\in\mathbb{Z}}(p(x,y)+p(y,x))\int f_{yx}d\nu\\
=&\lim_{n\to\infty}\sum_{x\in T_{[0,n]},y>n}\left[q_1(y-x)\int(g_{xy}-h_{yx})d\nu+q_1(x-y)\int(h_{xy}-g_{yx})d\nu\right]\\
+&\lim_{m\to-\infty}\sum_{x\in T_{[m,0]},y<m}\left[q_2(y-x)\int(g_{xy}-h_{yx})d\nu+q_2(x-y)\int(h_{xy}-g_{yx})d\nu\right].
\end{aligned}$$

The right-hand side above is equal to

$$\begin{aligned}
&\lim_{l\to\infty}\frac{1}{l}\sum_{n=1}^{l}\sum_{x\in T_{[0,n]},y>n}\left[\int q_1(y-x)(g_{xy}-h_{yx})+q_1(x-y)(h_{xy}-g_{yx})d\nu\right]\\
+&\lim_{k\to\infty}\frac{1}{k}\sum_{m=-1}^{-k}\sum_{x\in T_{[m,0]},y<m}\left[\int q_2(y-x)(g_{xy}-h_{yx})+q_2(x-y)(h_{xy}-g_{yx})d\nu\right].
\end{aligned}\tag{3.13}$$

We will devote the next few paragraphs to showing that these limits are in fact equal to zero.

Define the measures ν^+ and ν^- by choosing a subsequence n_j so that the following limits exist:

$$\nu^+=\lim_{j\to\infty}\frac{1}{n_j}\sum_{1\le x\le n_j}\nu_x$$

$$\nu^-=\lim_{j\to\infty}\frac{1}{|n_{-j}|}\sum_{-1\ge x\ge n_{-j}}\nu_x$$

where ν_x is the x translate of ν. In the partial sums of (3.13) above, for j large enough each term

$$q_i(y-x)\int(g_{xy}-h_{yx})d\nu$$

appears $|y-x|$ times when $q_i(y-x)>0$, so we can write (3.13) as

$$\sum_{z\in\mathbb{Z}^+}[zq_1(z)\int(g_{oz}-h_{zo})d\nu^+ - zq_1(-z)\int(g_{zo}-h_{oz})d\nu^+] \tag{3.14}$$
$$+\sum_{z\in\mathbb{Z}^-}[-zq_2(z)\int(g_{oz}-h_{zo})d\nu^- + zq_2(-z)\int(g_{zo}-h_{oz})d\nu^-]$$

Now consider two coupled processes with transition rates equal to $q_1(z)$ and $q_2(z)$ respectively. The measures ν^+ and ν^- are translation invariant in x and are also invariant measures for the coupled processes with respect to $q_1(z)$ and $q_2(z)$ respectively. To see that ν^+ is invariant with respect to $q_1(z)$, let $\tilde{\Omega}_1$ be the generator for the coupled process of $q_1(z)$ and for $A, B \in \mathbb{Y}$ let

$$f_{(A,B)} = \begin{cases} 1 & \text{when } \eta(x)=\xi(y)=1 \text{ for all } x\in A, y\in B \\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\int \tilde{\Omega}_1 f_{(A,B)}d\nu_+ &= \lim_{k\to\infty}\frac{1}{n_k}\sum_{1\le x\le n_k}\int\tilde{\Omega}_1 f_{(A+x,B+x)}d\nu \\ &= \lim_{k\to\infty}\frac{1}{n_k}\sum_{1\le x\le n_k}\int\tilde{\Omega} f_{(A+x,B+x)}d\nu = 0\end{aligned}$$

where $A+x$ is the x translate A.

Since ν^+ is invariant and translation invariant in x then Lemma VIII.3.2 in IPS tells us $\int f_{xy}d\nu^+ = 0$ for all x,y. We can therefore write ν^+ as $\nu^+ = \lambda\nu_1 + (1-\lambda)\nu_2$ where ν_1 concentrates on $\{(\eta,\xi):\eta<\xi\}$ and ν_2 on $\{(\eta,\xi):\eta\ge\xi\}$. Therefore

$$\int(g_{oz}-h_{zo})d\nu^+ = \lambda\int(g_{oz}-h_{zo})d\nu_1 \tag{3.15}$$
$$\begin{aligned} &= \lambda[\nu_1\{(\eta,\xi):\eta(0)=1,\eta(z)=0\} - \nu_1\{(\eta,\xi):\eta(0)=\xi(0)=1,\eta(z)=\xi(z)=0\} \\ &+ \nu_1\{(\eta,\xi):\eta(0)=\xi(0)=1,\eta(z)=\xi(z)=0\} - \nu_1\{(\eta,\xi):\xi(0)=1,\xi(z)=0\}] \\ &= \lambda[\nu_1\{(\eta,\xi):\eta(0)=1,\eta(z)=0\} - \nu_1\{(\eta,\xi):\xi(0)=1,\xi(z)=0\}].\end{aligned}$$

Because ν_1 and ν_2 are mutually singular and $\nu^+ = \lambda\nu_1 + (1-\lambda)\nu_2$, then it must be that the measure ν_1 is also invariant and translation invariant in x with respect to $q_1(z)$. Hence by Theorem VIII.3.9 in IPS, the marginals of ν_1 are exchangeable causing the right-hand side of (3.15) to be equal to a constant c^+. Similarly, the expression $\int (g_{zo} - h_{oz})d\nu^+ = c^+$. Using the same arguments we have that $\int (g_{oz} - h_{zo})d\nu^-$ and $\int (g_{zo} - h_{oz})d\nu^-$ are equal to a constant c^-. Now by the mean zero assumption, we get that expression (3.14) is equal to 0, but since (3.13) and (3.14) are equal, we have in fact that

$$\sum_{x\in T, y\in\mathbb{Z}} (p(x,y) + p(y,x)) \int f_{xy} d\nu = 0 \tag{3.16}$$

for every $T \subset \mathbb{Z}$.

By the nonnegativity of $\int f_{xy}d\nu$ it must be that $\int f_{xy}d\nu = 0$ for all x, y. Therefore $\nu \in \tilde{\mathcal{I}}$ implies that

$$\nu\{(\eta,\xi) : \eta < \xi \text{ or } \eta \geq \xi\} = 1.$$

If $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ we can use Theorem 3.2.1 to pick $\mu \in \mathcal{I}_e$ and $\nu^c \in \mathcal{I}_e$. On the other hand if $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$ we can use the analysis in the introduction of this chapter to pick $\mu \in \mathcal{I}_e$ and $\nu^{(n)} \in \mathcal{I}_e$. In either case Proposition VIII.2.14 in IPS says that the coupling measure ν can be taken in $\mathcal{I}_e$. Since $\nu\{(\eta,\xi) : \eta < \xi \text{ or } \eta \geq \xi\} = 1$, Proposition VIII.2.13 in IPS then tells us that ν has marginals $\mu \leq \nu^c$ or $\mu \geq \nu^c$ in the first case, and $\mu \leq \nu^{(n)}$ or $\mu \geq \nu^{(n)}$ in the second case.

Take first the case where $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$. Supposing that $\mu \neq \nu^0 \neq \nu^\infty$, we have that there exists a c_0 for which $\nu^{c_1} \leq \mu$ for all $c_1 < c_0$ and $\mu \leq \nu^{c_2}$ for all $c_2 > c_0$. By the continuity of the one parameter family of measures $\{\nu^c\}$ it must be that $\mu = \nu^{c_0}$.

If $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$ then we have three cases (i), (ii), and (iii) as given in the introduction. Theorem 3.0.6 gives the first two cases so we will consider only (iii). If $\mu \neq \nu^{(-\infty)} \neq \nu^{(\infty)}$ then there exists an $n \in \mathbb{Z}$ such that either $\mu = \nu^{(n)}$ or $\nu^{(n)} < \mu < \nu^{(n+1)}$. If the latter is true then μ concentrates on $A = \{\eta : \sum_{x \in T} \eta(x) < \infty, \sum_{x \notin T}[1-\eta(x)] < \infty\}$ for some $T \subset \mathbb{Z}$ which means that it must be a mixture of stationary distributions for the Markov chains on A_n as described in the introduction to this chapter. But $\mu \in \mathcal{I}_e$ so it must in fact be equal to some $\nu^{(n)}$ completing the proof. □

We include in this section two more results which have proofs similar to that of Theorem 3.1.1. We first need the following definition: given transition probabilities $p(x,y)$ define the boundary of a set $\mathcal{T}$ to be

$$\partial\mathcal{T} = \{x \notin \mathcal{T} : p(x,y) > 0 \text{ for some } y \in \mathcal{T}\}.$$

Proposition 3.3.1. *Let $\mathcal{S}$ be irreducible with respect to $p(x,y)$ and suppose that $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$. If there exists a sequence of increasing sets $\mathcal{T}_n$ such that $\cup\mathcal{T}_n = \mathcal{S}$ and either $\lim_{n\to\infty} \sum_{x \in \partial\mathcal{T}_n} \pi(x) = 0$ or $\lim_{n\to\infty} \sum_{x \in \partial\mathcal{T}_n} 1/\pi(x) = 0$, then $\mathcal{I}_e = \{\nu^c : c \in [0,\infty]\}$.*

Proof. Choose $\mu \in \mathcal{I}_e$. If $\lim_{n\to\infty} \sum_{x \in \partial\mathcal{T}_n} \pi(x) = 0$ then couple η_t with ξ_t so that they have the measures μ and ν^c respectively. If $\lim_{n\to\infty} \sum_{x \in \partial\mathcal{T}_n} 1/\pi(x) = 0$ then couple them vice versa. We will prove the case in which $\lim_{n\to\infty} \sum_{x \in \partial\mathcal{T}_n} \pi(x) = 0$. The other case follows similarly.

By (3.11),

$$\begin{aligned}
&\sum_{x \in \mathcal{T}_n, y \in \mathcal{S}} [p(x,y)+p(y,x)] \int f_{yx} d\nu \\
= &\sum_{x \in \mathcal{T}_n, y \notin \mathcal{T}_n} p(x,y) \int (g_{xy} - h_{yx}) d\nu + \sum_{x \in \mathcal{T}_n, y \notin \mathcal{T}_n} p(y,x) \int (h_{xy} - g_{yx}) d\nu.
\end{aligned}$$

Just as in the above proof, the left-hand side of this equation is increasing in n so that a limit exists as $n \to \infty$. The right-hand side above goes to 0 as $n \to \infty$ since

$$\begin{aligned}
&\sum_{x \in \mathcal{T}_n, y \notin \mathcal{T}_n} p(x,y) \int (g_{xy} - h_{yx}) d\nu + \sum_{x \in \mathcal{T}_n, y \notin \mathcal{T}_n} p(y,x) \int (h_{xy} - g_{yx}) d\nu \\
\leq \quad & \sum_{x \in \mathcal{T}_n, y \notin \mathcal{T}_n} p(x,y) \int f_y d\nu + \sum_{x \in \mathcal{T}_n, y \notin \mathcal{T}_n} p(y,x) \int f_y d\nu \\
\leq \quad & C \sum_{y \in \partial \mathcal{T}_n} \int f_y d\nu + \sum_{y \in \partial T_n} \int f_y d\nu \leq C \sum_{y \in \partial \mathcal{T}_n} \pi(y) + \sum_{y \in \partial T_n} \pi(y).
\end{aligned}$$

Here $C = \sup_y \sum_x p(x,y)$ which is finite by the assumptions in the introduction.

Irreducibility now gives us $\int f_{xy} d\nu = 0$ for all x, y. The rest of the proof just follows that of Theorem 3.1.1. □

Note that if we change the hypothesis $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ to

$$\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$$

then Theorem 3.0.6 says that $\mathcal{I}_e = \{\nu^{(n)}, 0 \leq n \leq \infty\}$.

Corollary 3.3.2. *If in Theorem 3.1.1 we replaced condition* (3.4) *with the condition that* $p(x,y)$ *has finite range,* $\lim_{x \to +\infty} p(x, x+z) = q_1(z)$, *and* $\lim_{x \to -\infty} \pi(x)$ *equals* 0 *or* ∞ *(or alternatively* $\lim_{x \to -\infty} p(x, x+z) = q_2(z)$, *and* $\lim_{x \to +\infty} \pi(x)$ *equals* 0 *or* ∞*) then the result still holds.*

Proof. Replace expression (3.13) in the proof of Theorem 3.1.1 with

$$\begin{aligned}
&\lim_{k \to \infty} \frac{1}{k} \sum_{n=1}^{k} \sum_{x \in T_{[0,n]}, y > n} \left[q_1(y-x) \int (g_{xy} - h_{yx}) d\nu + q_1(x-y) \int (h_{xy} - g_{yx}) d\nu \right] \\
+ \quad & \lim_{m \to -\infty} \sum_{x \in T_{[m,0]}, y < m} \left[p(x,y) \int (g_{xy} - h_{yx}) d\nu + p(y,x) \int (h_{xy} - g_{yx}) d\nu \right].
\end{aligned}$$

The proofs of Theorem 3.1.1 and Proposition 3.3.1 imply that this expression is 0. The rest is proven above. □

Before moving on to the next section let us discuss what the above results tell us in the case where $p(x, y)$ has finite range on $\mathbb{Z}$. Proposition 3.3.1 together with Theorem 3.0.6 says that if $\lim_{|x|\to\infty} \pi(x)$ equals 0 or ∞ then the reversible measures are the only invariant measures. If the limits $\lim_{x\to\infty} \pi(x)$ and $\lim_{x\to-\infty} \pi(x)$ exist and one of them is nonzero and finite, then the combination of Theorem 3.1.1 and Corollary 3.3.2 imply that the only invariant measures are the reversible ones. All together we have the following: if $\pi(x)$ exists and has limits in both directions for the finite-range exclusion process on $\mathbb{Z}$, then unless the limit is 0 in one direction and ∞ in the other direction, the only invariant measures are the reversible ones. Of course, as seen in an example in the introduction, it is also possible to have $\lim_{x\to\infty} p(x, x+z) = q_1(z)$ and $\lim_{x\to-\infty} p(x, x+z) = q_2(z)$ as given in Theorem 3.1.1 and at the same time have the limit of $\pi(x)$ to be 0 in one direction, ∞ in the other. In those cases Theorem 3.1.1 rules out nonreversible invariant measures. A similar comment can be made for Corollary 3.3.2. We remind the reader, however, that if the transition probabilities are translation invariant with a drift so that the limit of $\pi(x)$ is 0 in one direction and ∞ in the other direction, then Liggett(1976) tells us that $\{\nu_\rho : 0 \leq \rho \leq 1\}$ is a class of nonreversible invariant measures.

3.4 The nearest-neighbor process on $\mathbb{Z}$

Assume throughout this section that we are dealing with the irreducible, nearest-neighbor exclusion process on $\mathbb{Z}$. In this case, a reversible $\pi(x)$ always exists so we need not make this assumption. Similar to the discussion at the end of the last section, we will show that if $\inf_{|x-y|=1} p(x,y) > 0$ then the only possible nonreversible measures are in the case where the limit of $\pi(x)$ is 0 in one direction and ∞ in the other direction.

In order to prove the next two propositions we need the following lemma the proof of which is given in Corollary 5.2 of Liggett(1976):

Lemma 3.4.1 (Liggett). *If* $\inf_{|x-y|=1} p(x,y) > 0$ *and* $\nu \in \tilde{\mathcal{I}}_e$*, then exactly one of the following holds:*

(a) $\nu\{(\eta,\xi) : \eta = \xi\} = 1,$

(b) $\nu\{(\eta,\xi) : \eta \leq \xi, \eta \neq \xi\} = 1,$

(c) $\nu\{(\eta,\xi) : \eta \geq \xi, \eta \neq \xi\} = 1,$

(d) $\nu(B) = 1,$

(e) $\nu\{(\eta,\xi) : (\xi,\eta) \in B\} = 1,$

where $B = \{(\eta,\xi) : \exists x \in \mathbb{Z}$ *such that* $\eta(y) \leq \xi(y)$ *for all* $y < x, \eta(y) < \xi(y)$ *for some* $y < x, \eta(z) \geq \xi(z)$ *for all* $z \geq x, \eta(z) > \xi(z)$ *for some* $z \geq x\}$.

Proposition 3.4.2. *If* $\inf_{|x-y|=1} p(x,y) > 0$ *and* $\pi(x)$ *has some finite, nonzero limit point as* x *goes to* ∞ *and some finite, nonzero limit point as* x *goes to* $-\infty$*, then* $\mathcal{I}_e = \{\nu^c : c \in [0,\infty]\}$.

Proof. The assumptions imply that $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ so Theorem 3.2.1 tells us $\mathcal{I}_e \supset \{\nu^c : c \in [0,\infty]\}$. We will show the reverse containment.

Choose a sequence $\{n_k\}$ extending in both directions so that finite, nonzero limits of $\pi(n_k)$ exist. For any probability measure μ on $\mathbb{X}$, the set of limit points L_+ of $\{\mu\{\xi(n_k) = 1\}, k > 0\}$ satisfies one of the following properties:

(i) $L_+ = \{1\}$ or $L_+ = \{0\}$.

(ii) $L_+ = \{1, 0\}$.

(iii) L_+ contains some limit point between 0 and 1.

The same is true for the set of limit points L_- of $\{\mu\{\xi(n_k) = 1\}, k < 0\}$.

Now suppose we couple ν^c with another extremal invariant measure μ_e, the two measures corresponding to the processes η_t and ξ_t respectively. Since Theorem 3.2.1 tells us that ν^c is extremal, Section VIII.2 in IPS implies there exists a coupling measure such that $\nu \in \tilde{\mathcal{I}}_e$.

If μ_e satisfies condition (i) for both L_+ and L_- then there are two possibilities: either $L_+ = L_-$ or $L_+ \neq L_-$. Suppose first that $L_+ = L_- = \{1\}$ for μ_e. If in this case we have that $\mu_e\{\xi(z) = 1\} < 1$ for some z then we can choose $c < \infty$ large enough so that $\nu^c\{\eta(z) = 1\} > \mu_e\{\xi(z) = 1\}$. But this contradicts the assumption that $\nu^c\{\eta(n_k) = 1\} = c\pi(n_k)/[1 + c\pi(n_k)]$ has limits less than 1 for k going to ∞ and $-\infty$. To see this suppose the coupling measure satisfies $\nu(B) = 1$ as defined in Lemma 3.4.1. Given

$$0 < \epsilon < 1 - \lim_{k\to\infty} c\pi(n_k)/[1 + \ c\pi(n_k)] \qquad (3.17)$$

we can choose K large enough so that

$$1 - \epsilon < \nu\{(\eta, \xi) : \exists x < K \text{ such that } \eta(y) \leq \xi(y) \forall y < x, \eta(y) < \xi(y) \text{ for some } y < x,$$
$$\eta(z) \geq \xi(z) \forall z \geq x, \eta(z) > \xi(z) \text{ for some } z \geq x\}.$$

This, however, contradicts $L_+ = 1$. Similarly we cannot have that $\nu\{(\eta, \xi) : (\xi, \eta) \in B\} = 1$. So Lemma 3.4.1 tells us that $\eta \leq \xi$ which contradicts $\nu^c\{\eta(z) = 1\} > \mu_e\{\xi(z) = 1\}$. It must be that $\mu_e = \nu^\infty$. A similar argument shows that if $L_+ = L_- = \{0\}$ for μ_e then $\mu_e = \nu^0$.

Consider the second case where $L_- \neq L_+$; without loss of generality we will assume that $L_- = \{0\}$.

We claim that given $\epsilon > 0$, we can find n such that $\mu_e\{\xi(n) = 0\} < \epsilon$ and $\mu_e\{\xi(n+1) = 0\} < \epsilon$. To see this suppose that for some $\epsilon > 0$ there exists no n for which this is true. Then since $L_+ = \{1\}$, there are infinitely many $x > 0$ for which $\mu_e\{\xi(x) = 0\} < \epsilon/4$ and infinitely many $y > 0$ for which $\mu_e\{\xi(y) = 0\} \geq \epsilon$. Choosing ν^c so that $\lim_{k\to\infty} c\pi(n_k)/[1+c\pi(n_k)] = 1-\epsilon/2$ gives us a contradiction to Lemma 3.4.1 and thus proves the claim.

Given the same $\epsilon > 0$ we can choose $m < n$ so that $\mu_e\{\xi(m-1) = 1\} < \epsilon$. Since we have that $\nu \in \tilde{\mathcal{I}}_e$ then $\int \tilde{\Omega}(\sum_{x\in T} f_x)d\nu = 0$ for each finite $T \subset \mathbb{Z}$. By (3.11),

$$\sum_{m\leq x\leq n, y\in\mathbb{Z}} (p(x,y)+p(y,x)) \int f_{yx} d\nu \tag{3.18}$$
$$= \sum_{x=m \text{ or } n, y=m-1 \text{ or } n+1} \left[p(x,y)\int (g_{xy}-h_{yx})d\nu + p(y,x)\int (h_{xy}-g_{yx})d\nu\right]$$

which is increasing in n and $-m$.

Due to our choice of m and n above, $\int h_{n,n+1}d\nu < \epsilon$ and $\mu_e\{\xi(m-1) = 1\} < \epsilon$; moreover
$P(A) - P(A \bigcap B \bigcap C) \leq P(B^c) + P(C^c)$ implies that $\nu^c\{\eta(n+1) = 1, \eta(n) = 0\} - \int g_{n+1,n}d\nu < 2\epsilon$ so that

$$\begin{aligned}
&\sum_{m\leq x\leq n, y\in\mathbb{Z}} (p(x,y)+p(y,x)) \int f_{yx} d\nu \\
< \;& p(n,n+1)\int g_{n,n+1}d\nu - p(n+1,n)\int g_{n+1,n}d\nu + 3\epsilon \\
< \;& p(n,n+1)\nu^c\{\eta(n) = 1, \eta(n+1) = 0\} \\
- \;& p(n+1,n)\nu^c\{\eta(n) = 0, \eta(n+1) = 1\} + 5\epsilon.
\end{aligned}$$

By the reversibility of ν^c

$$p(n,n+1)\nu^c\{\eta(n)=1,\eta(n+1)=0\} = p(n+1,n)\nu^c\{\eta(n)=0,\eta(n+1)=1\}$$

so equation (3.18) is in fact equal to 0. Since we have assumed here that $L_- = \{0\}$ and $L_+ = \{1\}$ for μ_e then choosing $0 < c < \infty$ for then we have a contradiction s.

Suppose μ_e satisfies condition (ii) for either L_+ or L_- so that either $L_+ = \{0,1\}$ or $L_- = \{0,1\}$. Choose ν^c with $0 < c < \infty$. Again we contradict Lemma 3.4.1.

Combining all the above arguments we have that either $\mu_e = \nu^0$, $\mu_e = \nu^\infty$, or μ_e satisfies (iii) in some direction. Assuming the latter we can, without loss of generality, choose $0 < c_0 < \infty$ so that

$$\lim_{k\to\infty} c_0\pi(n_k)/[1 + c_0\pi(n_k)] = \lim_{l\to\infty} \mu_e\{\xi(n_{k_l}) = 1\}.$$

For all $c > c_0$,

$$\lim_{k\to\infty} c\pi(n_k)/[1 + c\pi(n_k)] > \lim_{l\to\infty} \mu_e\{\xi(n_{k_l}) = 1\}.$$

By Lemma 3.4.1 either $\mu_e \le \nu^c$ or $\nu(B) = 1$ where B is defined in the lemma. Similarly, for all $c < c_0$, either $\mu_e \ge \nu^c$ or $\nu\{(\eta,\xi) : (\xi,\eta) \in B\} = 1$. Combining these two arguments gives $\nu^{c_1} \le \mu_e \le \nu^{c_2}$ for all $c_1 < c_0 < c_2$. By the continuity of the one parameter family of measures ν^c, $\mu_e = \nu^{c_0}$. □

Proposition 3.4.3. *If* $\inf_{|x-y|=1} p(x,y) > 0$, $\lim_{x\to\infty} \pi(x) = \infty$, *and* $\pi(x)$ *has a finite, nonzero limit point as* x *goes to* $-\infty$, *then* $\mathcal{I}_e = \{\nu^c : c \in [0,\infty]\}$.

Proof. Again, by Theorem 3.2.1 we need only show that $\mathcal{I}_e \subset \{\nu^c : c \in [0,\infty]\}$.

We argue first that without loss of generality we can assume the limit points of $\{\pi(x), x < 0\}$ are all finite. Assume to the contrary that ∞ is a limit point.

For any $R > 0$ we can find $x < -R$ such that $\min(\pi(x), \pi(x+1)) > R$ since $\inf_{|x-y|=1} p(x,y) > 0$. The conditions of Proposition 3.3.1 are then satisfied so that $\mathcal{I}_e = \{\nu^c : c \in [0,\infty]\}$ holds.

Couple ν^c with another extremal invariant measure μ_e, the two measures corresponding to the processes η_t and ξ_t respectively. As argued above there exists a coupling measure such that $\nu \in \tilde{\mathcal{I}}_e$.

Let L^- be the the set of limit points of $\{\mu_e\{\xi(x) = 1\}, x < 0\}$. Note that L^- is slightly different from L_- described in Proposition 3.4.2 in that L_- is the set of limit points for a subset of $\{\mu\{\xi(x) = 1\}, x < 0\}$. L^- satisfies one of the following properties:

(i) L^- contains some limit point between 0 and 1.

(ii) $L^- = \{1, 0\}$.

(iii) $L^- = \{1\}$.

(iv) $L^- = \{0\}$.

The same is true for the set L^+ of limit points $\{\mu_e\{\xi(x) = 1\}, x > 0\}$.

Suppose L^- satisfies (i). Choose a sequence $x_n \to -\infty$ so that $0 < \lim_{n\to\infty} \mu_e\{\xi(x_n) = 1\} < 1$ exists. Since we can assume that the limit points of $\{\pi(x), x < 0\}$ are all finite, there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} \pi(x_{n_k}) < \infty$ exists.

Consider the two cases where $\lim_{k\to\infty} \pi(x_{n_k}) = 0$ and where $\lim_{k\to\infty} \pi(x_{n_k}) > 0$. Assume the latter case first. Choose $0 < c_0 < \infty$ so that

$$\lim_{k\to\infty} c_0\pi(x_{n_k})/[1 + c_0\pi(x_{n_k})] = \lim_{n\to\infty} \mu_e\{\xi(x_n) = 1\}.$$

For all $c > c_0$,

$$\lim_{k\to\infty} c\pi(x_{n_k})/[1 + c\pi(x_{n_k})] > \lim_{n\to\infty} \mu_e\{\xi(x_n) = 1\}.$$

Using the argument at the end of Proposition 3.4.2, we have that for all $c_1 < c_0 < c_2$, $\nu_{c_1} \le \mu_e \le \nu_{c_2}$. Consequently, it must be that $\mu_e = \nu_{c_0}$.

Now assume that $\lim_{k\to\infty} \pi(x_{n_k}) = 0$ so that for all $0 < c < \infty$ the coupling satisfies either $\nu^c \le \mu_e$ or $\nu\{B\} = 1$ where B is given in Lemma 3.4.1. If $\nu^c \le \mu_e$ for all $0 < c < \infty$ then $\mu_e = \nu^\infty$, a contradiction to L^- satisfying (i). So it must be that $\nu\{B\} = 1$.

We claim that for any $r < 1$ there exists $m < 0$ such that $\mu_e\{\xi(m) = 1\} > r$ and $\mu_e\{\xi(m-1) = 1\} > r$. By the hypothesis of the theorem we can choose a sequence $\{x_l\}$ going to $-\infty$ so that $0 < \lim_{l\to\infty} \pi(x_l) < \infty$ exists. If $\inf_{|x-y|=1} p(x, y) > p$ then choose c so that

$$\lim_{l\to\infty} \frac{cp\pi(x_l)}{1 + cp\pi(x_l)} > r.$$

Since $\pi(x_l - 1) > p\pi(x_l)$, it follows that $\lim_{l\to\infty} \frac{c\pi(x_l-1)}{1+c\pi(x_l-1)} > r$. Now since $\nu\{B\} = 1$ there exists a K such that $l > K$ implies $\mu_e\{\xi(x_l) = 1\} > r$ and $\mu_e\{\xi(x_l - 1) = 1\} > r$ which proves the claim.

Since we have that $\nu \in \tilde{\mathcal{I}}_e$ then $\int \tilde{\Omega}(\sum_{x\in T} f_x) d\nu = 0$ for each finite $T \subset \mathbb{Z}$. By (3.11),

$$\begin{aligned} &\sum_{m\le x\le n, y\in\mathbb{Z}} (p(x,y) + p(y,x)) \int f_{yx} d\nu \\ = &\sum_{x=m \text{ or } n, y=m-1 \text{ or } n+1} \left[p(x,y) \int (g_{xy} - h_{yx}) d\nu + p(y,x) \int (h_{xy} - g_{yx}) d\nu \right] \end{aligned}$$

which is increasing in n and $-m$.

Using the claim above along with the fact that $\lim_{x\to\infty} \pi(x) = \infty$, we can argue just as we argued in the case where $L_- \neq L_+$ of (i) in Proposition 3.4.2, to get

$$
\begin{aligned}
& \sum_{m \le x \le n, y \in \mathbb{Z}} (p(x,y) + p(y,x)) \int f_{yx} d\nu \\
< \; & p(m, m-1) \int g_{m,m-1} d\nu - p(m-1, m) \int g_{m-1,m} d\nu + 3\epsilon \\
< \; & p(m, m-1) \nu^c \{\eta(m) = 1, \eta(m-1) = 0\} \\
- \; & p(m-1, m) \nu^c \{\eta(m) = 0, \eta(m-1) = 1\} + 5\epsilon.
\end{aligned}
$$

By the reversibility of ν^c the left-hand side must be 0, but this contradicts $\nu\{B\} = 1$.

Suppose L^- satisfies condition (ii). Choosing ν^c with $0 < c < \infty$ gives us a contradiction to Lemma 3.4.1.

If L^- satisfies condition (iii) then we will handle the two cases (a) $L^+ = \{1\}$ and (b) $L^+ \neq \{1\}$. Considering case (a) if we switch the coupling so that μ_e corresponds to η_t then we have that the left-hand side of the following inequality goes to 0:

$$
\begin{aligned}
& \sum_{|x|=n, |y|=n+1} (p(x,y) + p(y,x)) \int f_y d\nu \geq \\
& \sum_{|x|=n, |y|=n+1} p(x,y) \int (g_{xy} - h_{yx}) d\nu + \sum_{|x|=n, |y|=n+1} p(y,x) \int (h_{xy} - g_{yx}) d\nu
\end{aligned}
\tag{3.19}
$$

By (3.11) and by irreducibility we get $\int f_{xy} d\nu = 0$ for all x, y. The measure μ_e must lie stochastically above all ν^c for all finite c and must therefore be equal to ν^∞.

If (b) holds then we refer the reader to the argument given above in the case where L^- satisfies (i) and $\lim_{k \to \infty} \pi(x_{n_k}) = 0$.

Finally suppose that (iv) holds so that $L^- = \{0\}$. If L^+ satisfies (i) or (ii) then by Lemma 3.4.1, $\mu_e \leq \nu^c$ for all $c > 0$ so that $\mu_e = \nu^0$, a contradiction. If L^+

satisfies (iv) then similarly $\mu_e = \nu^0$. Let L^+ satisfy (iii) so that $L^+ = \{1\}$. For a given z choose c small enough so that $\nu^c\{\eta(z) = 1\} < \mu_e\{\xi(z) = 1\}$. We thus have that $\nu\{(\eta, \xi) : (\xi, \eta) \in B\} = 1$ as given in Lemma 3.4.1. But by (3.11) and (3.19), for a given $\epsilon > 0$ we can find $-m$ and n large enough so that

$$\sum_{m \le x \le n, y \in \mathbb{Z}} (p(x, y) + p(y, x)) \int f_{yx} d\nu < \epsilon$$

which of course contradicts $\nu\{(\eta, \xi) : (\xi, \eta) \in B\} = 1$. □

Proof of Theorem 3.1.2. Note first that since $\inf_{|x-y|=1} p(x, y) > 0$ then it cannot be that $\mathcal{L}^-$ or $\mathcal{L}^+$ is equal to $\{0, \infty\}$. In light of this fact, if either $\mathcal{L}^-$ or $\mathcal{L}^+$ contains a finite, nonzero point then Proposition 3.4.2 and analogs of Proposition 3.4.3 imply there are no nonreversible measures. If $\mathcal{L}^+ = \mathcal{L}^- = \{0\}$ or $\mathcal{L}^+ = \mathcal{L}^- = \{\infty\}$ then Proposition 3.3.1 implies there are no nonreversible measures. □

3.5 A result concerning domains of attraction

In this section we will prove the following theorem used to prove Corollary 3.5.2 which in turn gives us information about the Cesaro domain of attraction of ν^c when $\sum_x \pi(x)/[1 + \pi(x)]^2 = \infty$.

Recall that $\mathcal{P}$ is the set of all measures on $\mathbb{X}$.

Theorem 3.5.1. *Let $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ and let θ be a probability measure on $[0, \infty]$. Also, assume that ν_c is a family of extremal invariant measures indexed by $c \in [0, \infty]$. Suppose $\{\mu_c\}, \mu_c \in \mathcal{P}$ is such that for each $c \in [0, \infty]$, μ_c is absolutely continuous with respect to ν_c. If*

$$\mu = \int_0^\infty \mu_c \, \theta(dc) \text{ and } \nu = \int_0^\infty \nu_c \, \theta(dc) \tag{3.20}$$

then $\lim_{T\to\infty} \frac{1}{T} \int_0^T \mu S(t) dt$ exists and is equal to ν.

The fact that Theorem 3.5.1 concerns Cesaro convergence rather than the usual weak convergence, while undesirable, is not so bad since many results in particle systems concern Cesaro convergence (see Section I.1 in IPS). One notable example of this is the main result of Andjel(1986) which concerns the Cesaro convergence of certain initial product measures when the transition kernel of the exclusion process is an asymmetric, nearest-neighbor random walk. In fact, these results were later shown to be true for weak convergence (this was the goal of Andjel, Bramson, Liggett(1988)).

Proof of Theorem 3.5.1. For a fixed c we first prove that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu_c S(t)dt = \nu_c. \tag{3.21}$$

By the compactness of $\mathcal{P}$ we can choose a sequence of times such that

$$\lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} \mu_c S(t)dt \tag{3.22}$$

converges in distribution to some measure λ. Pick a continuous (and therefore bounded) function f on $\mathbb{X}$ with $\|f\| \leq 1$ and let g be the Radon-Nikodym derivative of μ_c with respect to ν_c. Given $\epsilon > 0$ we have that for n large enough

$$|\frac{1}{t_n}\int_0^{t_n}\int (S(t)f)g\, d\nu_c dt - \int f\, d\lambda| < \epsilon/3.$$

We can choose a simple function

$$\hat{g} = \sum_{k=1}^{N} c_k 1_{E_k}$$

approximating g such that $\cup_k E_k = \mathbb{X}$, $\hat{g} \geq 0$, $\int \hat{g}\, d\nu_c = 1$, and $\int |g-\hat{g}|\, d\nu_c < \epsilon/3$. Since $\|S(t)f\| \leq \|f\| \leq 1$ this gives us

$$|\frac{1}{t_n}\int_0^{t_n}\int (S(t)f)g\, d\nu_c dt - \frac{1}{t_n}\int_0^{t_n}\int (S(t)f)\hat{g}\, d\nu_c dt| \leq \int |g-\hat{g}|\, d\nu_c < \epsilon/3.$$

Without loss of generality we can henceforth assume that $\nu_c(E_k) > 0$ for each k. Define the measure μ_k concentrating on E_k by letting

$$\mu_k(A) = \frac{\nu_c(A)}{\nu_c(E_k)}$$

for all $A \subset E_k$ and $\mu_k = 0$ otherwise. If we think of $\hat{g}$ as the Radon-Nikodym derivative of some measure λ_ϵ with respect to ν_c then we can write

$$\sum_{k=1}^{N} \nu_c(E_k)\mu_k = \nu_c \text{ and } \sum_{k=1}^{N} c_k \nu_c(E_k)\mu_k = \lambda_\epsilon.$$

We can now find a subsequence $\{t_{n_l}\}$ such that for each k the following limit exists:

$$\lim_{l\to\infty} \frac{1}{t_{n_l}} \int_0^{t_{n_l}} \mu_k S(t)dt = \nu_k.$$

Moreover, the argument used in (2.3) tells us $\nu_k \in \mathcal{I}$. Since ν_c is extremal invariant and since $\sum_{k\geq 1} \nu_c(E_k)\nu_k = \nu_c$, it must be that $\nu_k = \nu_c$ for each k. This then yields

$$\sum_{k=1}^{N} c_k \nu_c(E_k)\nu_k = \lim_{l\to\infty} \frac{1}{t_{n_l}} \int_0^{t_{n_l}} \lambda_\epsilon S(t)dt = \nu_c$$

which gives us

$$|\frac{1}{t_{n_l}} \int_0^{t_{n_l}} \int (S(t)f)\hat{g}\, d\nu_c dt - \int f\, d\nu_c| < \epsilon/3$$

for l large enough.

Combining the three inequalities we have

$$|\int f\, d\lambda - \int f\, d\nu_c| < \epsilon.$$

But $\epsilon > 0$ is arbitrary so it must be that $\int f\, d\lambda = \int f\, d\nu_c$ for each continuous f with $\|f\| \leq 1$ which implies that (3.22) is equal to ν_c. Now let M_n be the closure of the set of measures

$$\{\frac{1}{T} \int_0^T \mu_c S(t)dt : T \geq n\}.$$

Using the compactness of $\mathcal{P}$ along with the fact that $\{t_n\}$ is an arbitrary sequence of times causing convergence in (3.22), we have that $\bigcap_{n\in\mathbb{N}} M_n = \nu_c$ proving (3.21).

To finish the proof note that since $\|S(t)f\| \le \|f\|$, we can use the Dominated Convergence Theorem together with Fubini's Theorem to show that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^\infty \int S(t)f \, d\mu_c \, \theta(dc) \, dt = \int f \, d\nu.$$

$\square$

For the following corollary let ν_α be the product measure with marginals $0 < \nu_\alpha\{\eta(x) = 1\} = \alpha(x) < 1$ for $\alpha(x)$ a function on $\mathcal{S}$.

Corollary 3.5.2. *Suppose* $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$. *If* $\sum_x |\alpha(x) - \frac{c\pi(x)}{1+c\pi(x)}| < \infty$ *then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \nu_\alpha S(t) dt = \nu^c. \tag{3.23}$$

Proof. Let $\beta(x) = \frac{c\pi(x)}{1+c\pi(x)}$, $m_x = \min[\alpha(x), \beta(x)]$, and $M_x = \max[\alpha(x), \beta(x)]$. We then have

$$\begin{aligned}
1 - |\alpha(x) - \beta(x)| &= 1 - M_x + m_x \\
&= [(1 - M_x)(1 - M_x)]^{1/2} + (m_x m_x)^{1/2} \\
&\le [(1 - M_x)(1 - m_x)]^{1/2} + (m_x M_x)^{1/2} \\
&= [(1 - \alpha(x))(1 - \beta(x))]^{1/2} + (\alpha(x)\beta(x))^{1/2}.
\end{aligned}$$

Since $\sum_x |\alpha(x) - \beta(x)| < \infty$ then

$$\prod_x \{(\alpha(x)\beta(x))^{1/2} + [(1 - \alpha(x))(1 - \beta(x))]^{1/2}\} \ge \prod_x \{1 - |\alpha(x) - \beta(x)|\} > 0.$$

An application of Kakutani's Dichotomy tells us that ν_α is absolutely continuous with respect to ν^c which completes the proof. $\square$

We remark here that if $\alpha(x)$ and $\beta(x)$ are both bounded away from 0 and 1 then Kakutani's Dichotomy tells us that $\sum_x [\alpha(x) - \beta(x)]^2 < \infty$ is a necessary and sufficient condition for ν_α to be absolutely continuous with respect to ν^c (e.g. page 245 of Durrett(1996)).

CHAPTER 4

The Noisy Voter-Exclusion Process

Since Proposition 1.3.1 is the main tool used in analyzing the symmetric exclusion process, a natural question to ask is: what is the most general setting for which an analog of Proposition 1.3.1 can be proved for some sort of dual process? The answer to this question is the process that will be studied in this chapter. In order to describe this process, we must first introduce the voter model which happens to have a dual process which is very similar to the that of the symmetric exclusion process.

The voter model is an interacting particle system introduced independently by Clifford and Sudbury(1973) and Holley and Liggett(1975). In particular it is a spin system with rates given by

$$c(x,\eta) = \begin{cases} \sum_y q_v(x,y)\eta(y) & \text{if } \eta(x) = 0, \\ \sum_y q_v(x,y)[1-\eta(y)] & \text{if } \eta(x) = 1, \end{cases}$$

where $q_v(x,y) \geq 0$ and $\sup_x \sum_y q_v(x,y) < \infty$ for $x, y \in \mathcal{S}$.

To describe the voter model in a more intuitive manner let $\mathcal{S}$ be a countable set for which a voter resides at each site in the set. The voter at site x waits an exponential time with mean $[\sum_y q_v(x,y)]^{-1}$ at which point it chooses one of its neighbors with probability $q_v(x,y)/\sum_z q_v(x,z)$ and subsequently takes the opinion (either 1 or 0) of y.

The generalization of symmetric exclusion that we will consider not only combines the voter model and symmetric exclusion, but it also adds births and deaths at various sites. In Schwartz(1976) the β-δ process is introduced, a particle system which is exactly the symmetric exclusion process combined with births and deaths. The transition rates of the exclusion process are given by $q_e(x, y)$ while births ($\eta(x)$ goes from 0 to 1) occur at site x with exponential rate $\beta(x)$ and deaths ($\eta(x)$ goes from 1 to 0) occur with rate $\delta(x)$.

If we define the transition rates $q(x, y) = q_e(x, y) + q_v(x, y)$ and let $q_x = \sum_y q(x, y)$, then we can combine the voter model and the β-δ process to obtain a new process which must satisfy the following: (a) $\mathcal{S}$ is irreducible with respect to $q(x, y)$, (b) $q_e(x, y) = q_e(y, x)$, (c) $\max\{\sup_x q_x, \sup_y \sum_x q_e(x, y)\} < \infty$, and (d) $\inf_x q_x > 0$. Also, the transition rates $\beta(x)$ and $\delta(x)$ must satisfy (e) $\sup_x(\beta(x) + \delta(x)) < \infty$. Condition (d) is not necessary, but it is convenient for the purposes of our discussion. We will call such a process a noisy voter-exclusion process (NVE process).

In the setting of the NVE process, the voter at x waits an exponential time with mean q_x at which point it again chooses a neighbor with probability $q(x, y)/q_x$, but now the voter decides to either switch places with y with probability $q_e(x, y)/[q_e(x, y)+ q_v(x, y)]$ or, as before, take the opinion of y with probability $q_v(x, y)/[q_e(x, y) + q_v(x, y)]$. In addition to this, a voter at x with opinion 0 decides to spontaneously change its opinion to 1 with exponential rate $\beta(x)$, and a voter at x with opinion 1 spontaneously changes its opinion to 0 with rate $\delta(x)$.

Recall from the discussion on spin systems in the introduction that

$$\eta_x(u) = \begin{cases} \eta(u) & \text{if } u \neq x \\ 1 - \eta(u) & \text{if } u = x. \end{cases}$$

Using the results of Chapter I in IPS (Liggett(1985)), the generator for an NVE process is given by the closure of the following operator on $\mathcal{D}(\mathbb{X})$:

$$\Omega f(\eta) = \sum_{\eta(x)=1,\eta(y)=0} q_e(x,y)[f(\eta_{xy}) - f(\eta)] + \sum_x c(x,\eta)[f(\eta_x) - f(\eta)]$$

where

$$c(x,\eta) = \begin{cases} \beta(x) + \sum_y q_v(x,y)\eta(y) & \text{if } \eta(x) = 0, \\ \delta(x) + \sum_y q_v(x,y)[1-\eta(y)] & \text{if } \eta(x) = 1. \end{cases}$$

As usual, we will call the corresponding semigroup $S(t)$.

If $\beta(x) = \delta(x) \equiv 0$ then we will say that we have a voter-exclusion process. A previous study (Belitsky, Ferrari, Menshikov, and Popov(2001)) has been done concerning the ergodic theory of the voter-exclusion process in the case where $\mathcal{S} = \mathbb{Z}$ and $q_e(x,y)$ is not necessarily symmetric, but there is no overlap with the results of this chapter.

If $q_e(x,y) \equiv 0$ then we just get the noisy voter model. Granovsky and Madras(1995) study some important equilibrium functionals and critical values of the noisy voter model, but only for the case where β and δ are constant. We, on the other hand, will study the invariant measures of the NVE process where $\beta(x)$ and $\delta(x)$ are in general not constant.

In Chapter V of IPS, one can find a complete characterization of the extremal invariant measures and their domains of attraction for the voter model (Holley and Liggett(1975)). Schwartz(1976) does the same for the β-δ process. Just as in the symmetric exclusion process, these results are all based upon an analog of Proposition 1.3.1. Using such an analog, one can prove that

$$\nu_\alpha S(t)\{\eta(x) = 1 \text{ for all } x \in A\} \tag{4.1}$$

is increasing in t for the voter model (similarly, it can be shown that (4.1) is nonincreasing in t for symmetric exclusion, however, we did not prove this in Chapter 1). For the NVE process, a dual exists, but there is no monotonicity concerning the dual so we will have to use other techniques in order to classify the invariant measures under various conditions. Assume throughout this chapter that $q_v(x,y) > 0$ for some $x, y \in \mathcal{S}$. For the case where $q_v(x,y) \equiv 0$ we refer the reader to Schwartz(1976).

4.1 The results

Recall that $\mathcal{P}$ is the set of probability measures on $\mathbb{X}$,

$$\mathcal{H} = \left\{ \alpha : \mathcal{S} \to [0,1] \text{ such that } \sum_y q(x,y)\alpha(y) = q_x \alpha(x) \text{ for all } x \right\},$$

and that $\mathcal{S}_n = \mathcal{S}^n \backslash \{\vec{x} : x_i = x_j \text{ for some } i < j\}$. Also, let $\mu_\alpha = \lim_{t\to\infty} \nu_\alpha S(t)$. Theorem 4.1.4 below will show that these limits exist.

If $E_t = (x_t, y_t) \in \mathcal{S}_2$ is the finite, two particle exclusion process with transition rates $q(x,y)$ then define the functions q_v and q_e on $\mathcal{S}_2$ by $q_v(E_t) = q_v(x_t, y_t) + q_v(y_t, x_t)$ and $q_e(E_t) = q_e(x_t, y_t) + q_e(y_t, x_t)$.

Suppose $X(t)$ and $Y(t)$ are independent continuous time Markov chains on $\mathcal{S}$ with transition rates $q(x,y)$ and denote $p_t(x,y) = P^x(X(t) = y)$. Let $\Lambda = \{\omega | \int_0^\infty \beta(X(t)) + \delta(X(t))\, dt < \infty\}$. For $\alpha \in \mathcal{H}$, $\alpha(X(t))$ is a bounded martingale so $\lim_{t\to\infty}$ exists with probability one. We can define an equivalence relation R on $\mathcal{H}$ by

$$\alpha_1 R \alpha_2 \text{ if } \lim_{t\to\infty} [\alpha_1(X(t)) - \alpha_2(X(t))] = 0 \text{ almost surely on } \Lambda.$$

$\mathcal{H}_R$ is any set of representatives of the equivalence classes determined by R.

Let $\mathcal{E}$ be the following event:

$$\{\text{there exists } t_n \to \infty \text{ such that } X(t_n) = Y(t_n)\}.$$

Then we will say that $\mathcal{H}^*$ is the set of all $\alpha \in \mathcal{H}$ such that

$$P^{\{x,y\}}(\lim_{t\to\infty} \alpha(X(t)) = 0 \text{ or } 1 \text{ on } \mathcal{E}) = 1 \text{ for all } x, y \in \mathcal{S},$$

and $\mathcal{H}_R^*$ is again the set of equivalence classes on $\mathcal{H}^*$.

Define the following function on $\mathcal{S}^2$,

$$g(x, y) = P^{(x,y)}[X(t) = Y(t) \text{ for some } t > 0].$$

Note that if $g(x, y) = 1$ for some $(x, y) \in \mathcal{S}_2$ then by irreducibility $g(x, y) \equiv 1$ (For more detail concerning this see Lemma VIII.1.18 in IPS).

We are now in a position to state the theorems:

Theorem 4.1.1. *An NVE process is ergodic if and only if*

$$P^x[\int_0^\infty \beta(X(t)) + \delta(X(t))\,dt = \infty] = 1 \textit{ for all } x \in \mathcal{S}. \tag{4.2}$$

Theorem 4.1.2. *Suppose $\mu \in \mathcal{P}$ and δ_0, δ_1 are the point masses on all 0's and all 1's. Assume that (4.2) does not hold and that*

$$P^E[\int_0^\infty q_v(E_t)\,dt = \infty] = 1 \textit{ for all } E \in \mathcal{S}_2. \tag{4.3}$$

Then

(a) $\lim_{t\to\infty} \delta_0 S(t) = \mu^0$ *and* $\lim_{t\to\infty} \delta_1 S(t) = \mu^1$ *exist,*

(b) $\mathcal{I}_e = \{\mu^0, \mu^1\}$*, and*

(c) $\lim_{t\to\infty} \mu S(t) = \lambda\mu^1 + (1-\lambda)\mu^0$ *if and only if*

$$\lim_{t\to\infty} \sum_y p_t(x, y)\mu\{\eta(y) = 1\} = \lambda \textit{ for all } x \in \mathcal{S}. \tag{4.4}$$

We will say that the transition rates $q(x, y)$ on $\mathbb{Z}^d$ have finite range N if $q(x, y) = 0$ when $|x - y| > N$. In order to show that (4.3) is not an unreasonable condition the following corollary gives circumstances under which (4.3) holds.

Corollary 4.1.3. *Let $\mathcal{S} = \mathbb{Z}^d$, $q_e(x,y) = q_e(0, y-x)$, and $q_v(x,y) = q_v(0, y-x)$. Suppose $X(t) - Y(t)$ is recurrent and $q_e(x,y)$ has finite range N. Then $\mathcal{I}_e = \{\mu^0, \mu^1\}$ and for $\mu \in \mathcal{P}$, $\lim_{t\to\infty} \mu S(t) = \lambda\mu^1 + (1-\lambda)\mu^0$ if and only if (4.4) holds.*

Theorem 4.1.4. *(a) μ_α exists for all $\alpha \in \mathcal{H}$, and $\mu_{\alpha_1} = \mu_{\alpha_2}$ if and only if $\alpha_1 R \alpha_2$.*

(b) If $g(x,y) < 1$ for some $x, y \in \mathcal{S}$ and

$$P^E[\int_0^\infty q_e(E_t)\,dt = \infty] = 0 \text{ for some } E \in \mathcal{S}_2 \tag{4.5}$$

then $\mathcal{I}_e = \{\mu_\alpha : \alpha \in \mathcal{H}_R^*\}$.

(c) If $q(x,y) = q(y,x)$ for all $x, y \in \mathcal{S}$ and

$$P^E[\int_0^\infty q_v(E_t)\,dt = \infty] = 0 \text{ for some } E \in \mathcal{S}_2 \tag{4.6}$$

then $\mathcal{I}_e = \{\mu_\alpha : \alpha \in \mathcal{H}_R\}$.

The condition that $g(x,y) < 1$ for some $x, y \in \mathcal{S}$ is not needed in part (a), but we put it there because if $g \equiv 1$ then we are left with the situation in Theorem 4.1.2. It should also be remarked that if $q(x,y) = q(y,x)$ and $g(x,y) < 1$ for some $(x,y) \in \mathcal{S}_2$ then Lemma VIII.1.23 in IPS implies that (4.5) and (4.6) are satisfied. On the other hand when $q(x,y) = q(y,x)$, we claim that $g \equiv 1$ implies that $X(t)$ is recurrent so that $\beta(x) + \delta(x) > 0$ for some x gives us (4.2). To prove the claim use the Chapman-Kolmogorov equation to get

$$\begin{aligned} p_{2t}(x,x) &= \sum_y p_t(x,y)p_t(y,x) \\ &= \sum_y [p_t(x,y)]^2 = P^{(x,x)}[X(t) = Y(t)]. \end{aligned}$$

So if $X(t)$ is transient then $g(x,y) < 1$ for some $x, y \in \mathcal{S}$ since

$$\int_0^\infty P^{(x,x)}[X(t) = Y(t)]\,dt < \infty$$

(This argument will be made more explicit by Lemma 4.3.1).

Theorem 4.1.5. *Suppose* $\mu \in \mathcal{P}$ *and that* $E^{(x,y)}g(X(t),Y(t)) \to 0$ *for some* $x, y \in \mathcal{S}$*. If*

$$\lim_{t\to\infty} \sum_y p_t(x,y)\mu\{\eta(y)=1\} = \alpha(x) \text{ and} \tag{4.7}$$

$$\lim_{t\to\infty} \sum_{u,v} p_t(x,u)p_t(x,v)\mu\{\eta(u)=\eta(v)=1\} = \alpha^2(x) \text{ for all } x \in \mathcal{S} \tag{4.8}$$

then $\lim_{t\to\infty} \mu S(t) = \mu_\alpha$*. A necessary and sufficient condition for* $\lim_{t\to\infty} \mu S(t) = \mu_\alpha$ *is that*

$$\lim_{s\to\infty} \limsup_{t\to\infty} \int_X \{\sum_x p_s(w,x)P^x[\Lambda] \sum_y p_t(x,y)[\eta(y)-\alpha(y)]\}^2 \, d\mu(\eta) = 0 \tag{4.9}$$

We should mention two instances for which $g(x,y) < 1$ for some $x, y \in \mathcal{S}$ implies

$$E^{(x,y)}g(X(t),Y(t)) \to 0 \text{ for some } x, y \in \mathcal{S}. \tag{4.10}$$

Firstly, if $q(x,y)$ is symmetric then as stated in the comments following Theorem 4.1.4, Lemma VIII.1.18 in IPS gives (4.10). Secondly, if the only bounded harmonic functions are constants then Corollary II.7.3 in IPS together with Proposition 5.19 in Kemeny, Snell, and Knapp(1976) give (4.10). We also note here that condition (4.9) is equivalent to (4.7) and (4.8) when $P^x[\Lambda] = 1$ for all $x \in \mathcal{S}$.

Corollary 4.1.6. *If* $g(x,y) < 1$ *for some* $x, y \in \mathcal{S}$*,* $\mathcal{H} = \{\alpha : \alpha \in [0,1]\}$*, and* $\beta(x) = \delta(x) \equiv 0$ *then* $\mathcal{I}_e = \{\mu_\alpha : \alpha \in \mathcal{H}\}$*.*

The proofs of the above theorems appear in Section 4.4. The above theorems give partial results concerning the invariant measures and their respective domains of attraction for certain NVE processes. Clearly there are NVE processes which are not covered by these theorems. Examples of these situations include the process on $\mathbb{Z}^2$ where $q_e(x,y)$ is translation invariant, $\beta(x) = \delta(x) \equiv 0$, $q_v(x,y) = 0$ outside of a finite set, and $q_v(x,y)$ is not symmetric. A more interesting example is provided in V.1.6 of IPS; in fact using Liggett's example we can create similar examples to show that there exist NVE processes which do not satisfy (4.5) yet have $g(x, y) < 1$ for some $x, y \in \mathcal{S}$. Section 4.5 discusses how one might go about proving a general result that would include the exceptions we have just mentioned.

We now turn to a discussion of a slightly more general process. In particular, modify the NVE process by allowing for exclusion rates where $q_e(x, y) \neq q_e(y, x)$. Call such a process a generalized NVE process. It should be noted that not requiring the symmetry of $q_e(x, y)$ really does change the nature of the process. We will state two main reasons for this. Firstly, the properties of the dual finite particle system that allow us to prove the above theorems no longer exist. Secondly, the results for the asymmetric case are completely different; in fact it is known that Theorems 4.1.4 and 4.1.5 and Corollary 4.1.6 do not hold in general when $q_e(x, y)$ is not symmetric. We can however prove certain things about the generalized NVE process in specific cases using methods other than duality.

In Section 4.6 we prove an ergodic theorem for the case where $q_v(x, y) \equiv 0$ using the coupling method. When $q_v(x, y) \equiv 0$ we will call the process a noisy exclusion process. We will also show in this final section that Theorem 4.1.1 does not hold in general when $q_e(x, y)$ is not symmetric.

The main result of Section 4.6 is an extension, in the case where $\mathcal{S} = \mathbb{Z}^d$ and the transition rates have finite range, of Schwartz's(1976) ergodic theorem which is exactly Theorem 4.1.1 when $q_v(x, y) \equiv 0$. Before we state the theorem we need the following definitions:

$$T_n = \{x \in \mathbb{Z}^d : |x_i| \leq n \text{ for all } i\}.$$

$$T_n^N = T_{n+N} \backslash T_n.$$

Theorem 4.1.7. *Suppose η_t is a noisy exclusion process with transition rates $q_e(x, y)$ irreducible with respect to $\mathbb{Z}^d$ and having finite range N. Let $\{b_l\}$ be a nonnegative sequence satisfying (a) $\sum b_l = \infty$ if $d = 1$ and (b) $\lim_{l\to\infty} lb_l = \infty$ if $d \geq 2$. If $p(l)$ is a nonnegative function on $\mathbb{N}$ satisfying $p(l+1) \geq p(l) + N$ and is bounded by kl^k for some $k > 0$, and if β, δ satisfy $\beta(x) + \delta(x) \geq b_l$ for all $x \in T^N_{p(l)}$ and $\beta(x) = \delta(x) = 0$ otherwise, then η_t is ergodic.*

For some simple examples to see the applicability of Theorem 4.1.7 set $N = 1$ and let $p(l)$ be an arithmetic sequence e.g. $k, 2k, 3k, \ldots$ Suppose $\beta(x) = \delta(x) = 1$ for all $\|x\| = nk$, $n \in \mathbb{N}$ with $\|\cdot\|$ being the l^∞ norm and $\beta(x) = \delta(x) = 0$ otherwise. Then the theorem tells us that the noisy exclusion process is ergodic. Note that if $k = 1$ then the $M < \epsilon$ Theorem in Section I.4 of IPS also gives us ergodicity, but if $k > 1$ then the $M < \epsilon$ Theorem in general gives us no information. Also, Theorem 4.1.7 allows us to let $\beta(x) + \delta(x) \to 0$ whereas the $M < \epsilon$ Theorem again gives no information in such a circumstance. We should however mention here that in the nearest neighbor case, a version of the $M < \epsilon$ Theorem proven in Ferarri(1990) allows for $\beta(x) + \delta(x) \to 0$, but once again, Ferarri's theorem gives no information in the case where $k > 1$.

4.2 The dual of the NVE process

As stated before the proofs of the theorems are possible only because there exists an analog of Proposition 1.3.1. Its proof follows that of Theorem VIII.1.1 in IPS. Before stating and proving Proposition 4.2.1 we will need to describe the dual and semi-dual of the NVE process.

The semi-dual process A_t is a continuous time Markov chain on $\mathbb{Y}$ such that the particles in A_t move independently on $\mathcal{S}$ according to the motions of the independent $X_i(t)$ processes except that transitions to sites that are already occupied are handled in the following way: If a particle at x attempts to move to y which is already occupied then the transition is either suppressed with probability $q_e(x,y)/[q_e(x,y)+q_v(x,y)]$ or the two particles coalesce and move together thereafter with probability $q_v(x,y)/[q_e(x,y)+q_v(x,y)]$. In particular $|A_t| \leq |A_{t+s}|$ for all $s \geq 0$.

Now let $\mathbb{Y}^*$ be defined by adding to $\mathbb{Y}$ a cemetery state, Δ, and the empty set, $\emptyset$. We define the process A_t^* starting in a state $A \in \mathbb{Y}$ to move just as A_t does except that in addition A_t^* goes to $A_t^* \backslash \{x\}$ at rate $\beta(x)$ if $x \in A_t^*$ and A_t^* goes to the cemetery state Δ at rate $\sum_{x \in A_t^*} \delta(x)$. We will call A_t^* the dual process. Define D to be the event that A_t^* is never in the state Δ.

If $\mu \in \mathcal{P}$ and $A \in \mathbb{Y}$, then define

$$\hat{\mu}(A) = \mu\{\eta(x) = 1 \text{ for all } x \in A\}.$$

Extend this function to $\mathbb{Y}^*$ by letting $\hat{\mu}(\Delta) = 0$ and $\hat{\mu}(\emptyset) = 1$.

Proposition 4.2.1. *Extend the domain of $\eta \in X$ by letting $\eta(\Delta) = 0$. If $A \in \mathbb{Y}$ and η_t is an NVE process then for all $t \geq 0$*

$$P^\eta[\{\eta_t = 1 \text{ on } A\}] = P^A[\{\eta = 1 \text{ on } A_t^*\} \cup \{A_t^* = \emptyset\}].$$

Proof. Let

$$u_\eta(t, A^*) = P^\eta[\{\eta_t = 1 \text{ on } A^*\} \cup \{A^* = \emptyset\}] = S(t)H(\cdot, A^*)(\eta),$$

where for $A^* \neq \emptyset$

$$H(\eta, A^*) = \prod_{x \in A^*} \eta(x) = \begin{cases} 1 & \text{if } \eta(x) = 1 \text{ for all } x \in A^* \\ 0 & \text{otherwise} \end{cases}$$

and $H(\eta, \emptyset) = 1$.

For each $A \in \mathbb{Y}$, $H(\cdot, A) \in \mathcal{D}$ so we have

$$\begin{aligned}
\Omega H(\cdot, A)(\eta) &= \sum_{\eta(x)=1,\eta(y)=0} q_e(x,y)[H(\eta_{xy}, A) - H(\eta, A)] \\
&+ \sum_{x,y:\eta(x)\neq\eta(y)} q_v(x,y)[H(\eta_x, A) - H(\eta, A)] \\
&+ \sum_x [\beta(x)(1-\eta(x)) + \delta(x)\eta(x)][H(\eta_x, A) - H(\eta, A)] \\
&= \frac{1}{2}\sum_{x,y} q_e(x,y)[H(\eta_{xy}, A) - H(\eta, A)] \\
&+ \sum_{x\in A, y\in\mathcal{S}} q_v(x,y)H(\eta, A\backslash\{x\})[1-2\eta(x)]\{\eta(x)[1-\eta(y)] + \eta(y)[1-\eta(x)]\} \\
&+ \sum_{x\in A} \beta(x)[H(\eta, A\backslash\{x\}) - H(\eta, A)] + \sum_{x\in A}\delta(x)[H(\eta, \Delta) - H(\eta, A)]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{x,y} q_e(x,y)[H(\eta, A_{xy}) - H(\eta, A)] + \sum_{x\in A, y\in\mathcal{S}} q_v(x,y)H(\eta, A\backslash\{x\})[\eta(y) - \eta(x)] \\
&+ \sum_{x\in A} \beta(x)[H(\eta, A\backslash\{x\}) - H(\eta, A)] + \sum_{x\in A}\delta(x)[H(\eta, \Delta) - H(\eta, A)] \\
&= \sum_{x\in A, y\notin A} q_e(x,y)[H(\eta, A_{xy}) - H(\eta, A)] \\
&+ \sum_{x\in A, y\in\mathcal{S}} q_v(x,y)[H(\eta, (A\backslash\{x\})\cup\{y\}) - H(\eta, A)] \\
&+ \sum_{x\in A} \beta(x)[H(\eta, A\backslash\{x\}) - H(\eta, A)] + \sum_{x\in A}\delta(x)[H(\eta, \Delta) - H(\eta, A)].
\end{aligned}$$

Here A_{xy} is obtained from A in the same way that η_{xy} is obtained from η. The symmetry of $q_e(x,y)$ is used in second and fourth steps above.

By Theorem I.2.9 in IPS

$$
\begin{aligned}
&\frac{d}{dt}u_\eta(t,A) = \Omega S(t)H(\cdot,A)(\eta) \\
= &\sum_{x\in A, y\notin A} q_e(x,y)[S(t)H(\cdot,A_{xy})(\eta) - S(t)H(\cdot,A)(\eta)] \\
+ &\sum_{x\in A, y\in \mathcal{S}} q_v(x,y)[S(t)H(\cdot,(A\backslash\{x\})\cup\{y\})(\eta) - S(t)H(\cdot,A)(\eta)] \\
+ &\sum_{x\in A} \beta(x)[S(t)H(\cdot,A\backslash\{x\})(\eta) - S(t)H(\cdot,A)(\eta)] \\
+ &\sum_{x\in A} \delta(x)[S(t)H(\cdot,\Delta)(\eta) - S(t)H(\cdot,A)(\eta)] \\
= &\sum_{x\in A, y\notin A} q_e(x,y)[u_\eta(t,A_{xy}) - u_\eta(t,A)] \\
+ &\sum_{x\in A, y\in \mathcal{S}} q_v(x,y)[u_\eta(t,(A\backslash\{x\})\cup\{y\}) - u_\eta(t,A)] \\
+ &\sum_{x\in A} \beta(x)[u_\eta(t,A\backslash\{x\}) - u_\eta(t,A)] + \sum_{x\in A} \delta(x)[u_\eta(t,\Delta) - u_\eta(t,A)].
\end{aligned}
$$

For each $A \in \mathbb{Y}$, the unique solution to this system of differential equations with initial condition $H(\eta, A)$ is

$$E^A H(\eta, A_t^*) = P^A[\{\eta = 1 \text{ on } A_t^*\} \cup \{A_t^* = \emptyset\}]$$

(See Theorem 1.3 of Dynkin(1965)). □

4.3 Preliminary lemmas

The first five lemmas are adaptations of lemmas proved by Schwartz(1976). We omit the proofs of Lemmas 4.3.1, 4.3.2, and 4.3.5 since they are the same as found in Schwartz(1976) except for perhaps a change in notation.

Suppose $\mathcal{E}_t$ is a continuous time nonexplosive jump process on a countable set $\mathcal{N}$ and let $\mathcal{E}_k$ be the imbedded Markov chain. The transition rates of $\mathcal{E}_t$ are given by Q_{xy}. For $\mathcal{L} \subset \mathcal{N}$ define

$$Q_{\mathcal{L}}(x) = \sum_{y \in \mathcal{L}, y \neq x} Q_{xy}.$$

Lemma 4.3.1. *Assume there exist constants $0 < \alpha_1 < \alpha_2 < \infty$ such that for each $x \in \mathcal{N}$, $\alpha_1 \leq Q_{\mathcal{N}}(x) \leq \alpha_2$. Then almost surely*

$$\{\omega | \int_0^\infty Q_{\mathcal{L}}(\mathcal{E}_t)\, dt = \infty\} = \{\omega | \mathcal{E}_k \in \mathcal{L} \textit{ infinitely often}\} \subset \{\omega | \mathcal{E}_t \in \mathcal{L} \textit{ for some } t\}.$$

Lemma 4.3.2. *Assume $0 < \sup_x(\beta(x) + \delta(x)) < \infty$. Then (4.2) holds if and only if*

$$P^A[A_t^* = \emptyset \textit{ or } A_t^* = \Delta \textit{ eventually}] = 1$$

for all $A \in \mathbb{Y}$.

For the next lemma define the function

$$h(A) = P^A(|A_t| < |A| \text{ for some } t > 0) \text{ for } A \in \mathbb{Y}$$

which is in some sense a voter model analog of the function $g(x, y)$.

Lemma 4.3.3. *If (4.3) holds then $P^A(|A_t| = 1$ eventually$) = 1$ for all $A \in \mathbb{Y}$.*

Proof. We first prove the case for which A_t starts in a two particle state $|A| = 2$.

Take $\mathcal{E}_t$ in Lemma 4.3.1 to be A_t, and let $\mathcal{L}$ be the set of states such that $|A_t| = 1$. We then interpret $Q_{\mathcal{L}}(A_t)$ as the rate at which A_t jumps to a one particle state. If $A_t = \{x\}$ then $Q_{\mathcal{L}}(A_t)$ is just q_x. Now suppose that $|A_t| = 2$ for all t. Then A_t is exactly E_t defined above to be the two particle exclusion process with respect to $q(x, y)$. Therefore

$$\int_0^\infty q(E_t)\, dt = \int_0^\infty Q_{\mathcal{L}}(A_t)\, dt = \infty$$

and by Lemma 4.3.1, $|A_t| = 1$ eventually, a contradiction. We have thus proved the case where $|A| = 2$.

For the general case suppose $|A| \geq 2$. Couple B_t, a semi-dual process starting from a two particle state $|B| = 2$, with A_t so that $B_t \subset A_t$. In order to do this let A_t and B_t move as usual except when a particle tries to move with rate $q_e(x, y)$ to an occupied site, instead of the motion being "excluded", let the two particles switch places. Of course this is the same motion as before, just a different way of thinking of it.

Using the coupling we have now that $h(A) = 1$ for all $|A| \geq 2$. Thus with probability one, $|A|$ decreases for all $|A| \geq 2$ which proves the lemma. $\square$

Recall that D is the event where A_t^* is never in the state Δ.

Lemma 4.3.4. *If* $\beta(x) \equiv 0$ *then*

$$\lim_{t\to\infty} E^{\{x\}} P^{A_t}[D^c, \Lambda] = 0 \textit{ for all } x \in \mathcal{S}.$$

Proof. Let $\mathcal{E}_t = (X(t), \zeta(t))$ be a Markov jump process on $\mathcal{N} = \mathcal{S} \times \{0, 1, 2, \ldots\}$ with jump rates $Q_{(x,n),(y,0)} = q(x, y)$ and $Q_{(x,n),(x,n+1)} = \delta(x)$. Let $\mathcal{L} = \mathcal{S} \times \{1, 2, \ldots\}$ so that $Q_{\mathcal{L}}((x, n)) = \delta(x)$. We then have that

$$\begin{aligned} \lim_{t\to\infty} E^{\{x\}} P^{A_t}[D^c, \Lambda] &= \lim_{t\to\infty} P^x[\mathcal{E}_s \text{ jumps to } \mathcal{L} \text{ after time } t, \Lambda] \\ &= P^x[\mathcal{E}_k \in \mathcal{L} \text{ infinitely often}, \Lambda]. \end{aligned}$$

But the right-hand side is equal to 0 by Lemma 4.3.1 completing the proof. $\square$

We will need three definitions in stating the next lemma and in proving Theorem 4.3.9. Before stating the definitions we ask the reader to think of $\mu\{\eta : \eta(X(t)) = 0\}$ as a family of random variables on the space of paths. We then have

$$\mathcal{P}' = \{\mu \in \mathcal{P} : \lim_{t\to\infty} \mu\{\eta(X(t)) = 0\} = 1 \text{ almost surely on } \Lambda^c\}.$$

$$\mathcal{H}' = \{\alpha \in \mathcal{H} : \lim_{t\to\infty} \alpha(X(t)) = 0 \text{ almost surely on } \Lambda^c\}.$$

If $S(t)$ is the semigroup for an NVE process then let $S'(t)$ be the semigroup for the same process except that $\beta(x) = \delta(x) \equiv 0$.

For part (b) of the following lemma we couple A_t and A_t^* so that they move together until the first time that $A_t^* = \delta$ or $|A_t^*| < |A_t|$.

Lemma 4.3.5. *(a) $\mathcal{H}'$ is a set of class representatives for the equivalence relation R on $\mathcal{H}$.*

(b) If we extend the state space of A_t to include Δ and $\emptyset$ as absorbing states then $\lim_{t\to\infty} P^{A_t^*}[A_s^* \neq A_s$ for some $s \geq 0] = 0$ almost surely.

(c) Suppose that $\beta(x) \equiv 0$. If $\mu \in \mathcal{I}$ or if $\mu = \lim_{t\to\infty} \nu S'(t)$ exists for $\nu \in \mathcal{I}$, then $\mu \in \mathcal{P}'$.

Define E_t^n to be the finite exclusion process on n particles starting in the state A where $|A| = n$. To be consistent with our previous definition of E_t we will leave the superscript off if $n = 2$ so that $E_t = E_t^2$ and $|E| = 2$.

Lemma 4.3.6. *If (4.6) holds and $q(x,y) = q(y,x)$ then*

$$P^A[\int_0^\infty \sum_{E\subset E_t^n} q_v(E)\,dt = \infty] = 0 \text{ for all } A \in \mathbb{Y}.$$

Proof. Suppose $A = \{x_1, \ldots, x_n\}$. Let $E_t^{\{i,j\}}$ be the two particle exclusion process starting from $\{x_i, x_j\}$. We will show there exists a multiple coupling of the processes E_t^n and $E_t^{\{i,j\}}$ for $0 \leq i < j \leq n$ such that

$$\{E_t^n\} \subset \bigcup_{0\leq i<j\leq n} \{E_t^{\{i,j\}}\}. \tag{4.11}$$

Let $X_i(t)$ be a process equal in distribution to $X(t)$. The key to seeing why (4.11) is true is noticing that there exists a way to couple $X_i(t)$ and $X_j(t)$ so that whenever one tries to coalesce with the other, they simply switch places. This can be done since $q(x, y) = q(y, x)$. With that said, it is clear that we can couple the $X_i(t)$'s with E_t^n so that

$$\{E_t^n\} = \{X_1(t), \ldots, X_n(t)\}.$$

Here the processes $X_i(t)$ start at x_i and are clearly not independent of each other.

For each $E_t^{\{i,j\}}$ we can label one particle first class and the other particle second class. We can now think of the evolution of $E_t^{\{i,j\}}$ in the following way. If a second class particle tries to go to a site occupied by a first class particle, it is not allowed to do so. However, if a first class particle attempts to move to a site occupied by a second class particle, the two particles switch places. With this evolution a first class particle is equal in distribution to $X(t)$. By choosing the first class particles to have the paths of the $X_i(t)$ processes above it is clear that (4.11) holds.

Suppose now that

$$P^A[\int_0^\infty \sum_{E \subset E_t^n} q_v(E)\, dt = \infty] > 0 \text{ for some } A \in \mathbb{Y}.$$

In light of (4.11), it must be that

$$P^E[\int_0^\infty q_v(E_t)\, dt = \infty] > 0 \text{ for some } E \in \mathcal{S}_2.$$

By irreducibility

$$P^E[\int_0^\infty q_v(E_t)\, dt = \infty] > 0 \text{ for all } E \in \mathcal{S}_2.$$

□

Lemma 4.3.7. *If $q(x, y) = q(y, x)$ and (4.6) holds then $h(E_t^n) \to 0$ almost surely for all initial states $A \in \mathcal{S}_n$.*

Proof. By Lemma 4.3.6,

$$P^A[\int_0^\infty \sum_{E\subset E_t^n} q_v(E)\,dt = \infty] = 0 \text{ for all } A \in \mathbb{Y}. \tag{4.12}$$

Let E_k^n be the imbedded Markov chain for the process E_t^n starting with initial state A. Let Ω be the path space for E_k^n and let $\mathcal{M}$ be the probability measure on Ω for our process. Choose $\epsilon > 0$. If there exists a set $F \subset \Omega$ such that $\mathcal{M}(F) > 0$ and $h(E_k^n) > \epsilon$ infinitely often on F then it must be that

$$\sum_{k=0}^{\infty} \sum_{E\subset E_k^n} q_v(E) = \infty$$

almost surely on F since whenever $\sum_{k=0}^{\infty} \sum_{E\subset E_k^n} q_v(E) < \infty$ it must be that $h(E_k^n) > \epsilon$ finitely many times.

We claim that

$$\{\omega | \int_0^\infty \sum_{E\subset E_t^n} q_v(E)\,dt = \infty\} = \{\omega | \sum_{k=0}^{\infty} \sum_{E\subset E_k^n} q_v(E) = \infty\} \tag{4.13}$$

almost surely. To see this define τ_k to be the kth jump time of E_t^n. Now note that

$$\int_0^\infty \sum_{E\subset E_t^n} q_v(E)\,dt = \sum_{k=0}^{\infty} \sum_{E\subset E_k^n} q_v(E)[\tau_{k+1} - \tau_k].$$

By our assumptions $E[\tau_{k+1} - \tau_k]$ and $\text{Var}[\tau_{k+1} - \tau_k]$ are bounded above and below uniformly in k. Since $[\tau_{k+1} - \tau_k | E_1^n, E_2^n, \ldots]$ are independent, Kolmogorov's Three Series Theorem proves the claim.

Since (4.13) contradicts (4.12) we have shown that $h(E_k^n) \to 0$ almost surely. This however implies that $h(E_t^n) \to 0$ almost surely. □

Suppose V_t is the dual process for the voter model with rates $q(x, y)$ starting from the set A. If we couple A_t and V_t so that they move together as much as possible then we can define the function

$$f(A) = P^A[A_t \neq V_t \text{ for some } t > 0].$$

Again, $f(A)$ plays much the same role as $h(A)$ and $g(x,y)$.

Lemma 4.3.8. *If (4.5) holds then $E^A f(A_t) \to 0$ for all $A \in \mathbb{Y}$.*

Proof. We prove first the case where $|A| \leq 2$. Let $\mathcal{E}_t = (A_t, \zeta(t))$ be a Markov jump process on $\mathcal{N} = (\mathcal{S}_2 \cup \mathcal{S}) \times \{0,1,2,\ldots\}$ with jump rates (i) $Q_{(A,n),(B,0)}$ equal to the jump rate from A to B of the semi-dual process and (ii) $Q_{(A,n),(A,n+1)} = q_e(A)$ if $|A| = 2$. Let $\mathcal{L} = \mathcal{S}_2 \times \{1,2,\ldots\}$ so when $|A| = 2$, $Q_{\mathcal{L}}((A,n)) = q_e(A)$ and when $|A| = 1$, $Q_{\mathcal{L}}((A,n)) = 0$. We then have that

$$\begin{aligned}\lim_{t\to\infty} E^A f(A_t) &= \lim_{t\to\infty} P^A[\mathcal{E}_s \text{ jumps to } \mathcal{L} \text{ after time } t] \\ &= P^A[\mathcal{E}_k \in \mathcal{L} \text{ infinitely often}].\end{aligned}$$

Since (4.5) holds, Lemma 4.3.1 implies that the right-hand side is 0.

Now suppose $|A| > 2$. Change the coupling of the $X_i(t)$ processes that we used in Lemma 4.3.6 by letting $X_i(t)$ and $X_j(t)$ switch places at rate $q_e(X_i(t), X_j(t))$ and coalesce and move together thereafter at rate $q_v(X_i(t), X_j(t))$. Again, we are allowed to do this since $q_e(x,y) = q_e(y,x)$. With this new coupling we can couple the $X_i(t)$'s with A_t so that

$$\{A_t\} = \{X_1(t), \ldots, X_n(t)\}.$$

As in Lemma 4.3.6, we use the idea of first class particles along with the fact that $X_i(t)$ can be coupled with $E_t^{\{i,j\}}$ so that $\{X_i(t)\} \subset \{E_t^{\{i,j\}}\}$, we have that the proof for $|A| \leq 2$ implies the proof for all $A \in \mathbb{Y}$. □

The next theorem is actually a special case of Theorem 4.1.4. We prove this special case right now in order make the proof of the general case easier to read.

Theorem 4.3.9. *Suppose $q_e(x, y) \equiv 0$.*

(a) μ_α exists for all $\alpha \in \mathcal{H}$, and $\mu_{\alpha_1} = \mu_{\alpha_2}$ if and only if $\alpha_1 R \alpha_2$.

(b) $\mathcal{I}_e = \{\mu_\alpha : \alpha \in \mathcal{H}^*_R\}$.

Proof. The proof is virtually the same as that of Theorem 1.3 in Schwartz(1976), but it is included here for completeness. We will however leave out some repetitive details.

Let $\mathcal{J}$ represent the set of invariant measures for the case where $\beta(x) = \delta(x) \equiv 0$, in other words the voter model. In Chapter V of IPS, it is shown that $\mathcal{J}_e = \{\mu_\alpha : \alpha \in \mathcal{H}^*\}$. Consider a certain subset of $\mathcal{J}$, namely

$$\mathcal{J}' = \{\mu \in \mathcal{J} : \lim_{t\to\infty} \mu\{\eta(X(t)) = 0\} = 1 \text{ almost surely on } \Lambda^c\}.$$

The main part of the proof is showing that there exists a bijective affine map between $\mathcal{J}'$ and $\mathcal{I}$. To avoid confusion, we will put a bar over the extremal invariant measures of the pure voter model so that we have $\mathcal{J}_e = \{\bar{\mu}_\alpha : \alpha \in \mathcal{H}^*\}$.

In order to do this we will first consider the case where $\beta(x) \equiv 0$, but $\delta(x) \geq 0$. We start by coupling the the semi-dual process A_t with n independent processes $X_1(t), \ldots, X_n(t)$ which start from $A = \{x_1, \ldots, x_n\}$ and are equal in distribution to $X(t)$. In particular, couple the processes so that $A_t \subset \{X_1(t), \ldots, X_n(t)\}$. Let $X^*_i(t)$ be the dual process starting from $\{x_i\}$ and henceforth define $T(t)$ to be the semi-group for the voter model.

By coupling the processes A^*_t and A_t so that they move together as much as possible, it is clear that for any measure $\mu \in \mathcal{P}$ and any $A \in \mathbb{Y}$, $S(t)\hat{\mu}(A) \leq T(t)\hat{\mu}(A)$. Thus if $\mu \in \mathcal{I}$ and $\nu \in \mathcal{J}'$ then $\hat{\mu}(A) \leq T(t)\hat{\mu}(A)$ and $S(t)\hat{\nu}(A) \leq \hat{\nu}(A)$. Applying the respective semigroups once more to both these inequalities gives $T(s)\hat{\mu}(A) \leq T(t+s)\hat{\mu}(A)$ and $S(t+s)\hat{\nu}(A) \leq S(s)\hat{\nu}(A)$ so that $\lim_{t\to\infty} \mu T(t)$ and $\lim_{t\to\infty} \nu S(t)$ exist by monotonicity and duality.

Now take $\mu_1 \in \mathcal{J}'$. Let $\lim_{t\to\infty} \mu_1 S(t) = \mu_2$ and define the map $\sigma(\mu_1) = \mu_2$. We will show that σ is an affine bijection from $\mathcal{J}'$ to $\mathcal{I}$.

Since $\mu_1 \in \mathcal{P}'$, it follows that

$$\begin{aligned}\lim_{t\to\infty} |T(t)\hat{\mu}_1(A) - S(t)\hat{\mu}_1(A)| &\leq P^A[\bigcup_{1\leq i\leq n} \{X_i^*(t) = \Delta \text{ eventually}, \Lambda\}] \\ &\leq \sum_{i=1}^{n} P^{\{x_i\}}[D^c, \Lambda].\end{aligned}$$

By the definition of μ_2 and by the fact that $\mu_1 \in \mathcal{J}$

$$|\hat{\mu}_1(A) - \hat{\mu}_2(A)| \leq \sum_{i=1}^{n} P^{\{x_i\}}[D^c, \Lambda].$$

Applying $T(t)$ to both sides of this last inequality and passing to the limit gives

$$\lim_{t\to\infty} |\hat{\mu}_1(A) - T(t)\hat{\mu}_2(A)| \leq \lim_{t\to\infty} \sum_{i=1}^{n} E^{\{x_i\}} P^{A_t}[D^c, \Lambda].$$

Lemma 4.3.4 says that the right-hand side above is equal to 0 so that $\lim_{t\to\infty} \mu_2 T(t) = \mu_1$. This proves that σ is injective. If we think of $X^*(t) = \Delta$ as an absorbing state where $X^*(t)$ continually jumps to Δ at exponential rate one then a similar argument using Lemma 4.3.5 (c) shows σ to be surjective. To see that σ is affine note simply that if $\mu_1, \nu_1 \in \mathcal{J}'$ then

$$\lim_{t\to\infty} (\lambda\mu_2 + (1-\lambda)\nu_2)S(t) = \lambda\mu_1 + (1-\lambda)\nu_1.$$

We have thus far shown that there exists an affine bijection between $\mathcal{J}'$ and $\mathcal{I}$ for the case $\beta \equiv 0$. For the general case we compare the process η_t with birth rates $\beta(x)$ and death rates $\delta(x)$ to a similar process $\tilde{\eta}_t$ having the same transition rates except that the death rates are now $\tilde{\delta}(x) = \beta(x) + \delta(x)$ and the birth rates are identically 0. Let the associated dual process, semigroup, and set of invariant measures for $\tilde{\eta}_t$ be $\tilde{A}_t^*$, $\tilde{S}(t)$, and $\tilde{\mathcal{I}}$.

Couple the two dual processes so that they make the same transitions except when a particle in A_t^* dies off due to a $\beta(x)$ jump, then $\tilde{A}_t^*$ goes to the state Δ. If we can show that

$$\lim_{t\to\infty} E^A P^{A_t^*}[A_s^* \neq \tilde{A}_s^*] = 0 \text{ for all } A \in \mathbb{Y} \tag{4.14}$$

and similarly that

$$\lim_{t\to\infty} E^A P^{\tilde{A}_t^*}[A_s^* \neq \tilde{A}_s^*] = 0 \text{ for all } A \in \mathbb{Y} \tag{4.15}$$

then we can make arguments similar to the ones above to show that for $\nu_1 \in \mathcal{I}$ and $\nu_2 \in \tilde{\mathcal{I}}$, the limits $\lim_{t\to\infty} \nu_1 \tilde{S}(t) = \nu_2$ and $\lim_{t\to\infty} \nu_2 S(t) = \nu_1$ exist. We can also show that the map $\lim_{t\to\infty} \nu_2 S(t) = \nu_1 = \tilde{\sigma}(\nu_2)$ is an affine bijection between $\tilde{\mathcal{I}}$ and $\mathcal{I}$. If we extend the state space of A_t as in Lemma 4.3.5 (b) then the following inequalities combined with Lemma 4.3.5 (b) prove (4.14) and (4.15):

$$P^{\tilde{A}_t^*}[A_s^* \neq \tilde{A}_s^*] \leq P^{A_t^*}[A_s^* \neq \tilde{A}_s^*] \leq P^{A_t^*}[A_s^* \neq A_s \text{ for some } s \geq 0].$$

Our desired affine bijection from $\mathcal{J}'$ to $\mathcal{I}$ is just $\tilde{\sigma} \circ \sigma$. We are now ready to prove the two parts of the theorem. We start with part (a).

To prove μ_α exists we need only show

$$\lim_{t\to\infty} \nu_\alpha S(t) = \lim_{t\to\infty} \lim_{s\to\infty} \lim_{r\to\infty} \nu_\alpha T(r) \tilde{S}(s) S(t). \tag{4.16}$$

Let $\bar{\mu}_\alpha = \lim_{r\to\infty} \nu_\alpha T(r)$ and let $\tilde{\bar{\mu}} = \lim_{s\to\infty} \bar{\mu} \tilde{S}(s)$. We have already argued that these limits exist. Applying $S(t)$ and passing to the limit in the following inequalities proves (4.16).

$$\begin{aligned}
& \lim_{t\to\infty} |S(t)\hat{\nu}_\alpha(A) - S(t)\hat{\tilde{\bar{\mu}}}_\alpha(A)| \\
\leq\ & \lim_{t\to\infty} |S(t)\hat{\nu}_\alpha(A) - \tilde{S}(t)\hat{\bar{\mu}}_\alpha(A)| + \lim_{t\to\infty} |\hat{\tilde{\bar{\mu}}}_\alpha(A) - S(t)\hat{\tilde{\bar{\mu}}}_\alpha(A)| \\
\leq\ & \lim_{t\to\infty} |S(t)\hat{\nu}_\alpha(A) - T(t)\hat{\nu}_\alpha(A)| + \lim_{t\to\infty} |\hat{\bar{\mu}}_\alpha(A) - \tilde{S}(t)\hat{\bar{\mu}}_\alpha(A)| + \lim_{t\to\infty} |\hat{\tilde{\bar{\mu}}}_\alpha(A) - S(t)\hat{\tilde{\bar{\mu}}}_\alpha(A)| \\
\leq\ & 3P^A[A_s^* \neq A_s \text{ for some } s \geq 0].
\end{aligned}$$

Suppose now that $\lim_{t\to\infty} \nu_{\alpha_1} S(t) = \lim_{t\to\infty} \nu_{\alpha_2} S(t)$. We have

$$\begin{aligned}\hat{\nu}_{\alpha_i}(\{X(s)\}) &= \lim_{t\to\infty} E^{X(s)} \hat{\nu}_{\alpha_i}(\{X^*(t)\}) \\ &= P^{X(s)}[X^*(t) = \emptyset \text{ eventually}] + E^{X(s)}(\lim_{t\to\infty} \hat{\nu}_{\alpha_i}(\{X(t)\}) 1_{\{X^*(t)\neq\emptyset\,\forall\, t, D\}}).\end{aligned}$$

But since $P^{X(s)}(\{X^*(t) \neq \emptyset\,\forall\, t, D\}) \to 1$ on Λ_x by the arguments given for Lemma 4.3.4 and since $E^{X(s)}(\lim_{t\to\infty} \hat{\nu}_{\alpha_i}(\{X(t)\})) = \alpha_i(X(s))$, then it follows that $\alpha_1 R \alpha_2$.

For the opposite direction if we assume that $\alpha_1 R \alpha_2$, then

$$\begin{aligned}\lim_{t\to\infty} (\nu_{\alpha_1} S(t) - \nu_{\alpha_2} S(t))(A) &= \lim_{t\to\infty} E^A(\hat{\nu}_{\alpha_1}(A_t^*)) - \lim_{t\to\infty} E^A(\hat{\nu}_{\alpha_2}(A_t^*)) \\ &= \lim_{t\to\infty} E^A(\prod_{x\in A_t^*} \alpha_1(x)) - \lim_{t\to\infty} E^A(\prod_{x\in A_t^*} \alpha_2(x)) = 0.\end{aligned}$$

For part (b) it is enough to show that the extreme points of $\mathcal{J}'$ are $\{\bar{\mu}_\alpha \in \mathcal{J} : \alpha \in \mathcal{H}^* \cap \mathcal{H}'\}$. Then applying (4.16) along with Lemma 4.3.5 (a) completes the proof. To prove $\mathcal{J}'_e = \{\bar{\mu}_\alpha \in \mathcal{J} : \alpha \in \mathcal{H}^* \cap \mathcal{H}'\}$ note that if $\lambda\pi_1 + (1-\lambda)\pi_2 = \mu \in \mathcal{J}'_e$ for $\pi_1, \pi_2 \in \mathcal{J}$ then $\pi_1, \pi_2 \in \mathcal{J}'$ and hence $\pi_1 = \pi_2 = \mu$. Therefore $\mu \in \mathcal{J}_e \cap \mathcal{J}' = \{\bar{\mu}_\alpha : \alpha \in \mathcal{H}^* \cap \mathcal{H}'\}$. On the other hand if $\alpha \in \mathcal{H}^*$ and $\bar{\mu}_\alpha \in \mathcal{J}'$, then $\bar{\mu}_\alpha$ is an extreme point of $\mathcal{J}'$. □

4.4 Proofs of the theorems

Proof of Theorem 4.1.1. Suppose condition (4.2) holds. By Proposition 4.2.1 we need only show that for any two measures $\mu_1, \mu_2 \in \mathcal{P}$, the limits $\lim_{t\to\infty} S(t)\hat{\mu}_i(A)$ exist and are equal for all $A \in \mathbb{Y}$. But Lemma 4.3.2 implies that

$$\lim_{t\to\infty} S(t)\hat{\mu}_i(A) = P^A[A_t^* = \emptyset \text{ eventually}]$$

which is independent of μ_i proving one direction of the theorem.

For the opposite direction suppose that (4.2) does not hold. Lemma 4.3.2 implies that $P^A[A_t^* = \emptyset \text{ or } A_t^* = \Delta \text{ eventually}] < 1$ for some $A \in \mathbb{Y}$. Therefore

$$\lim_{t\to\infty} S(t)\hat{\delta}_1(A) = P^A[A_t^* = \emptyset \text{ eventually}] + P^A[A_t^* \neq \emptyset \,\forall\, t, D]$$

is not equal to

$$\lim_{t\to\infty} S(t)\hat{\delta}_0(A) = P^A[A_t^* = \emptyset \text{ eventually}]$$

for some $A \in \mathbb{Y}$ showing that the process is not ergodic. □

Proof of Theorem 4.1.2. By Proposition 4.2.1, $\lim_{t\to\infty} \delta_1 S(t)$ exists since

$$\lim_{t\to\infty} S(t)\hat{\delta}_1(A) = 1 - P^A[D^c].$$

Similarly, $\lim_{t\to\infty} \delta_0 S(t)$ exists since

$$\lim_{t\to\infty} S(t)\hat{\delta}_0(A) = P^A[A_t = \emptyset \text{ eventually}]$$

completing the proof of part (a).

Turning to part (b) let $\mu \in \mathcal{I}$. By Lemma 4.3.3 and a coupling argument it can be seen that if $\lim_{t\to\infty} E^x\hat{\mu}(\{X(t)\})$ exists, it is independent of our choice of x. So now using Proposition 4.2.1 and Lemma 4.3.3 together with the Strong Markov Property, we have

$$\begin{aligned} \hat{\mu}(A) &= E^A\hat{\mu}(A_t^*) \qquad (4.17)\\ &= P^A[A_t^* = \emptyset \text{ eventually}] + P^A[\lim_{t\to\infty} |A_t^*| = 1, D] \lim_{t\to\infty} E^x\hat{\mu}(\{X(t)\}). \end{aligned}$$

Since we have assumed that (4.2) does not hold then $P^A[\lim_{t\to\infty} |A_t^*| = 1, D] > 0$ so that the last limit on the right-hand side exists.

Let $\lambda = \lim_{t\to\infty} E^x\hat{\mu}(\{X(t)\})$ and consider the invariant measure $\mu^\lambda = \lambda\mu^1 + (1-\lambda)\mu^0$. We have that for all $A \in \mathbb{Y}$,

$$\hat{\mu}^\lambda(A) = E^A\hat{\mu}^\lambda(A_t^*) = P^A[A_t^* = \emptyset \text{ eventually}] + \lambda P^A[\lim_{t\to\infty} |A_t^*| = 1, D].$$

Since $P^A[A_t^* = \emptyset$ eventually] and $P^A[\lim_{t\to\infty} |A_t^*| = 1, D]$ do not depend on μ or μ^λ, comparing the above equation with (4.17) gives us $\mu = \mu^\lambda$ showing that every invariant measure is a mixture of μ^1 and μ^0. This proves part (b).

We have already proved one direction of part (c) above. For the converse, suppose that (4.4) holds. Let τ be the first time that $|A_t| = 1$. Lemma 4.3.3 implies that τ is finite with probability one. As before, by the Strong Markov Property

$$\lim_{t\to\infty} E^A \hat{\mu}(A_t) = E^A[\lim_{t\to\infty} E^{A_\tau} \hat{\mu}(A_t)].$$

This limit exists and is equal to λ by (4.4).

This completes the proof of part (c) since the proof of part (b) implies that $\lim_{t\to\infty} \mu S(t) = \lambda\mu^1 + (1-\lambda)\mu^0$ is equivalent to

$$\lim_{t\to\infty} E^A \hat{\mu}(A_t) = \lambda.$$

□

Proof of Corollary 4.1.3. We need only show that the recurrence of $Z(t) = X(t) - Y(t)$ implies (4.3).

Let R be the set of all $y \in \mathbb{Z}^d$ such that $|y| \leq N$. By our assumptions we can choose $z \in \mathcal{S}$ so that $q_v(0, z) > 0$. If $E_t = \{x_t, y_t\}$ is the two particle exclusion process then we will say that $E_t = z$ if $x_t - y_t = z$ and $E_t \in R$ if $|x_t - y_t| \leq N$.

Since $Z(t)$ is recurrent, $Z(t)$ jumps to 0 infinitely often and therefore $X(t)$ and $Y(t)$ meet infinitely often. If there are infinitely many jumps of $Z(t)$ to 0 caused by the $q_v(x, y)$ rates then (4.3) automatically holds by arguments similar to those of Lemma 4.3.1. Thus we will henceforth assume that there are infinitely many jumps of $Z(t)$ to 0 caused by the $q_e(x, y)$ rates giving us

$$P^{\{x,y\}}(Z(t) \in R \text{ for some } t > 0) = 1 \text{ for all } x, y \in \mathcal{S}.$$

By coupling $Z(t)$ and E_t together until the first time that $X(t)$ and $Y(t)$ meet, we have in fact that

$$P^{\{x,y\}}(E_t \in R \text{ for some } t > 0) = 1 \text{ for all } \{x, y\} \in \mathcal{S}_2.$$

If E_k is the embedded Markov Chain for E_t then the above equation implies that

$$P^{\{x,y\}}(E_k \in R \text{ infinitely often}) = 1 \text{ for all } \{x, y\} \in \mathcal{S}_2.$$

For a fixed $\bar{t} > 0$ let $m = \min_{x \in R}\{P^x(E_{\bar{t}} = z)\}$. Since our process is irreducible $m > 0$, therefore

$$P^{\{x,y\}}(E_k = z \text{ infinitely often}) = 1 \text{ for all } \{x, y\} \in \mathcal{S}_2. \tag{4.18}$$

Now by the same argument given to show (4.13) in the proof of Lemma 4.3.7,

$$\{\omega | \int_0^\infty q_v(E_t)\, dt = \infty\} = \{\omega | \sum_{k=0}^{\infty} q_v(E_k) = \infty\}$$

almost surely. By (4.18) we get that (4.3) holds as desired. □

Proof of Theorem 4.1.4. Again let $T(t)$ be the semigroup for the voter model with rates $q(x, y)$. Chapter V in IPS tells us $\lim_{t\to\infty} \nu_\alpha T(t) = \bar{\mu}_\alpha$ exists for all $\alpha \in \mathcal{H}$. By coupling the dual of our process together with the dual of the voter model so that they move together as much as possible, it is clear that $S(t)\hat{\bar{\mu}}_\alpha(A) \leq T(t)\hat{\bar{\mu}}_\alpha(A) = \hat{\bar{\mu}}_\alpha(A)$. Applying $S(s)$ to both sides gives $S(t + s)\hat{\bar{\mu}}_\alpha(A) \leq S(s)\hat{\bar{\mu}}_\alpha(A)$. Part (a) follows from this monotonicity along with the arguments laid out in Theorem 4.3.9.

Concerning the rest of the proof we will only prove part (b) since the proof of part (c) is basically the same as that of (b) except for replacing the use of Lemma 4.3.8 with Lemma 4.3.7. Just as in the proof of Theorem 4.3.9 the general idea is to show that there exists a bijective, affine map σ between $\mathcal{I}$ and $\mathcal{J}$ where $\mathcal{J}_e = \{\lim_{t\to\infty} \nu_\alpha T(t) : \alpha \in \mathcal{H}^*\}$.

For part (b) we will prove only the case where $\beta(x) = \delta(x) \equiv 0$ so that $A_t^* = A_t$. The general result follows from the arguments laid out in the proof of Theorem 4.3.9 except for a slight change in the independent processes $X_1(t), \ldots, X_n(t)$ starting from $A = \{x_1, \ldots, x_n\}$. For the proof here we must use the coupling of the $X_i(t)$ processes that we used in the proof of Lemma 4.3.8 instead of letting them be independent. We now prove the case $\beta(x) = \delta(x) \equiv 0$.

Take $\mu \in \mathcal{I}$ and suppose that both A_t and V_t start with initial set A. By coupling the two processes so that A_t contains V_t, we see that

$$|S(t)\hat{\mu}(A) - T(t)\hat{\mu}(A)| \leq f(A) = P^A[A_t \neq V_t \text{ for some } t > 0].$$

By the invariance of μ

$$|\hat{\mu}(A) - T(t)\hat{\mu}(A)| \leq f(A) \tag{4.19}$$

so that

$$|T(s)\hat{\mu}(A) - T(t+s)\hat{\mu}(A)| \leq T(s)f(A).$$

By Lemma 4.3.8 and the fact that $S(s)f(A) \to 0$ implies that $T(s)f(A) \to 0$, the right-hand side goes to 0. This in turn shows that $\lim_{t\to\infty} T(t)\hat{\mu}(A)$ exists. The duality of the voter model which is a special case of Proposition 4.2.1, implies that $\lim_{t\to\infty} \mu T(t) = \nu$ exists and is invariant for the voter model with rates $q(x, y)$.

By passing to the limit in (4.19)

$$|\hat{\mu}(A) - \hat{\nu}(A)| \leq f(A).$$

Hence Lemma 4.3.8 tells us $\lim_{t\to\infty} \nu S(t) = \mu$.

For $\mu \in \mathcal{I}$, if we define $\sigma(\mu) = \lim_{t\to\infty} \mu T(t) = \nu$, then the above arguments have shown that σ is injective. A similar arguments proves σ maps onto $\mathcal{J}$. To see that it is affine note that

$$\lim_{t\to\infty} (\lambda\mu_1 + (1-\lambda)\mu_2)T(t) = \lambda\nu_1 + (1-\lambda)\nu_2.$$

We now conclude the proof of the case $\beta(x) = \delta(x) \equiv 0$ by showing that for $\bar{\mu}_\alpha = \lim_{t\to\infty} \nu_\alpha T(t)$,

$$\lim_{t\to\infty} \nu_\alpha S(t) = \lim_{t\to\infty} \bar{\mu}_\alpha S(t).$$

Applying $S(s)$ to the following inequality and passing to the limit proves the above equation.

$$\begin{aligned} & \lim_{t\to\infty} |S(t)\hat{\nu}_\alpha(A) - S(t)\hat{\bar{\mu}}_\alpha(A)| \\ \leq \;& \lim_{t\to\infty} |S(t)\hat{\nu}_\alpha(A) - T(t)\hat{\nu}_\alpha(A)| + \lim_{t\to\infty} |\hat{\bar{\mu}}_\alpha(A) - S(t)\hat{\bar{\mu}}_\alpha(A)| \leq 2f(A). \end{aligned}$$

□

Proof of Theorem 4.1.5. Putting $A = \{x_1, \dots, x_n\}$ let

$$W_n(t)\hat{\mu}(A) = E^A\hat{\mu}(\{X_1(t), \dots, X_n(t)\})$$

be the semigroup for n independent processes. Then the assumptions of the theorem tell us $W_2(t)g(x,y) \to 0$ so that

$$P^{\{x,y\}}[X(t) = Y(t) \text{ infinitely often}] = 0. \tag{4.20}$$

The proof that (4.9) is necessary and sufficient for $\lim_{t\to} \mu S(t) = \mu_\alpha$ is proven in Theorem 8.7 in Schwartz(1976). The only thing to note is that the assumption that $X(t)$ is transient is needed only to show that when $q(x,y) = q(y,x)$, (4.20) holds.

The rest of the proof is similar to the proof of Theorems V.1.9 in IPS. Assume that μ satisfies (4.7) and (4.8). By Proposition 4.2.1 and the definition of μ_α, it suffices to show that for each $A \in \mathbb{Y}$,

$$\lim_{t\to\infty} E^A \hat{\mu}(A_t^*) = \lim_{t\to\infty} E^A \prod_{x\in A_t^*} \alpha(x). \tag{4.21}$$

where we make the convention that $\alpha(\Delta) = 0$ and $\prod_{x\in\emptyset} \alpha(x) = 1$.

Conditions (4.7) and (4.8) are equivalent to the assertion that for each $x \in \mathcal{S}$

$$\sum_y p_t(x,y)\eta(y)$$

converges in probability to $\alpha(x)$ with respect to μ. This in turn is equivalent to

$$\lim_{t\to\infty} E^{\{x_1,\ldots,x_n\}} \hat{\mu}(\{X_1(t),\ldots,X_n(t)\}) = \prod_{i=1}^n \alpha(x_i) \tag{4.22}$$

where the $X_i(t)$ are all independent.

Let τ_1 be the first time that either $X_i(t) = X_j(t)$ for some $1 \le i < j \le n$, $A_t^* = \Delta$, or $|A_t^*|$ decreases. Still putting $A = \{x_1,\ldots,x_n\}$, let τ_2 be the first time starting from $A_{\tau_1}^*$ that any of the three events described above occur unless $A_\tau^* = \Delta$ in which case we will let $\tau_2 = \infty$. Continuing in this way we can define τ_k for all $k \ge 1$.

By (4.22) and the Strong Markov Property, if the limits below exist then

$$\begin{aligned}
&\lim_{t\to\infty} E^A[\hat{\mu}(A_t^*), \tau_1 = \infty] = \lim_{t\to\infty} E^A[\hat{\mu}(\{X_1(t),\ldots,X_n(t)\}), \tau_1 = \infty] \quad (4.23)\\
= &\prod_{i=1}^n \alpha(x_i) - \lim_{t\to\infty} E^A[\hat{\mu}(\{X_1(t),\ldots,X_n(t)\}), \tau_1 < \infty]\\
= &\prod_{i=1}^n \alpha(x_i) - E^A \lim_{t\to\infty} E^{(X_1(\tau_1),\ldots,X_n(\tau_1))}[\hat{\mu}(\{X_1(t),\ldots,X_n(t)\}), \tau_1 < \infty]\\
= &\lim_{t\to\infty} E^A[\prod_{x\in A_t^*} \alpha(x), \tau_1 = \infty] = \lim_{t\to\infty} E^A[\prod_{x\in A_t} \alpha(x), \tau_1 = \infty].
\end{aligned}$$

By the convergence theorem for bounded submartingales $\lim_{t\to\infty}\prod_{x\in A_t}\alpha(x)$ exists almost surely so by the Dominated Convergence Theorem the above limits exist.

Using the Strong Markov Property once more we can get

$$\begin{aligned}\lim_{t\to\infty} E^A[\hat{\mu}(A_t^*),\tau_1<\infty] &= E^A \lim_{t\to\infty}\left(E^{A^*_{\tau_1}}[\hat{\mu}(A_t^*),\tau_1<\infty,\tau_2=\infty]\right) \\ &+ E^A \lim_{t\to\infty}\left(E^{A^*_{\tau_1}}[\hat{\mu}(A_t^*),\tau_2<\infty]\right).\end{aligned} \tag{4.24}$$

But by the argument given for (4.23) the first term on the right-hand side above equals

$$\begin{aligned}& E^A \lim_{t\to\infty} E^{A^*_{\tau_1}}[\prod_{x\in A_t^*}\alpha(x),\tau_1<\infty,\tau_2=\infty] \\ = & \lim_{t\to\infty} E^A[\prod_{x\in A_t}\alpha(x),\tau_1<\infty,\tau_2=\infty]P[A^*_{\tau_1}\neq\emptyset\neq\Delta]+P[A^*_{\tau_1}=\emptyset].\end{aligned}$$

The second term on the right-hand side of (4.24) equals

$$E^A \lim_{t\to\infty}\left(E^{A^*_{\tau_2}}[\hat{\mu}(A_t^*),\tau_2<\infty,\tau_3=\infty]\right)+E^A \lim_{t\to\infty}\left(E^{A^*_{\tau_2}}[\hat{\mu}(A_t^*),\tau_3<\infty]\right).$$

Since (4.20) holds we have that $P[\tau_k=\infty \text{ for some } k]=1$. By repeated use of the arguments above it follows that (4.21) holds. □

Proof of Corollary 4.1.6. Take $\mu\in\mathcal{I}$ and again let $W_n(t)$ be the semigroup for n independent random walks $\vec{X}(t)=(X_1(t),\ldots,X_n(t))$. Couple A_t and $\vec{X}(t)$ so that they move together until the first time that two coordinates of $\vec{X}(t)$ meet. We then have that

$$|S(t)\hat{\mu}(A)-W_n(t)\hat{\mu}(A)|\leq g(A). \tag{4.25}$$

Since $S(t)\hat{\mu}(A)=\hat{\mu}(A)$ then

$$|W_n(s)\hat{\mu}(A)-W_n(t+s)\hat{\mu}(A)|\leq W_n(s)g(A).$$

Corollary II.7.3 of IPS tells us that $\vec{X}(t)$ has no nonconstant bounded harmonic functions. By Proposition 5.19 of Kemeny, Snell, and Knapp(1976) $W_n(t)g(A) \to 0$ so that $\lim_{t\to\infty} W_n(t)\hat{\mu}(A)$ exists and is harmonic for the random walk $\vec{X}(t)$ on $\mathcal{S}^n$. Such harmonic functions are constant so we can write

$$\lim_{t\to\infty} W_n(t)\hat{\mu}(A) = \alpha_n \text{ for } |A| = n.$$

The proof of Theorem 2.6 in Liggett(2002) shows that there exists a random variable G taking values in $[0,1]$ with moment sequence α_n. Since $\alpha_n \leq 1$ we can use Carleman's Condition to show that the random variables $\sum_y p_t(x,y)\eta(y)$ with respect to the measure μ converge in distribution to G.

If γ is the probability measure on $[0,1]$ for G, let $\mu_\gamma = \int_0^1 \mu_\alpha \, \gamma(d\alpha)$. Using the arguments presented in Theorem 4.1.5 we can show that for each $A \in \mathbb{Y}$,

$$\lim_{t\to\infty} E^A \hat{\mu}(A_t) = \lim_{t\to\infty} EG^{|A_t|} = \lim_{t\to\infty} E^A \hat{\mu}_\gamma(A_t).$$

Thus $\mu = \mu_\gamma$ and is hence a mixture of the measures $\{\mu_\alpha : \alpha \in [0,1]\}$. By Theorem 4.1.5, each measure μ_α has a different domain of attraction proving that $\mathcal{I}_e = \{\mu_\alpha : \alpha \in [0,1]\}$. □

4.5 Further results

The brief discussion below shows how one might adapt Schwartz(1976) and Chapter V in IPS in order to obtain a general result. Let

$$\hat{\mathcal{E}} = \{\omega : \int_0^\infty q_v(E_t)\, dt = \infty\}.$$

In the introduction we argued that $\lim_{t\to\infty} \alpha(X(t))$ exists almost surely so we can define $\hat{\mathcal{H}}$ to be the set of all $\alpha \in \mathcal{H}$ such that

$$\lim_{t\to\infty} \alpha(X(t)) = 0 \text{ or } 1 \text{ a.s. on } \hat{\mathcal{E}}$$

where $X(t)$ starts from x if $E_0 = \{x, y\}$. For those that are keeping track, $\hat{\mathcal{E}}$ and $\hat{\mathcal{H}}$ are analogous to $\mathcal{E}$ and $\mathcal{H}^*$.

Following Schwartz(1976) and Chapter V in IPS, we conjecture that $\mathcal{I}_e = \{\mu_\alpha : \alpha \in \hat{\mathcal{H}}_R\}$. In order to prove this one would have to generalize Theorem 4.1.5 and show that for $\mu \in \mathcal{I}_e$

$$\hat{\mu}(\{X(t), Y(t)\}) \to \hat{\mu}(\{X(t)\})\hat{\mu}(\{Y(t)\}). \tag{4.26}$$

As mentioned in the introduction, it is the monotonicity of $S(t)\hat{\nu}_\alpha(A)$ that allows us to do this for the pure voter model or the pure symmetric exclusion process. If one were to prove (4.26) and Theorem 4.1.5 in general, new techniques would be needed.

4.6 An ergodic theorem for a related process

The proof of Theorem 4.1.7 requires the following lemma:

Lemma 4.6.1. *Suppose $\{a_n\}$ is bounded above by $k_1 n^{k_2-1}$ for some $k_1, k_2 > 0$ and that $a_n > 0$ for all n. Then there exists a sequence $\{w_n\}$ such that*

$$\text{(i)} \liminf_{n\to\infty} \frac{a_n}{w_n} = 1 \text{ and (ii)} \limsup_{n\to\infty} \frac{n w_n}{\sum_{l=0}^{n-1} w_l} < \infty.$$

Proof. If for some sequence $\{w_n\}$ we have that $w_l/l^{k_2} \geq w_n/n^{k_2}$ for $l \leq n$ then

$$\sum_{l=0}^{n-1} w_l \geq \sum_{l=0}^{n-1} \frac{w_n}{n^{k_2}} l^{k_2} \geq \frac{w_n}{n^{k_2}} \frac{(n-1)^{k_2+1}}{k_2+1}$$

so that condition (ii) holds. So it remains to find a sequence $\{w_n\}$ satisfying condition (i) and the inequality $w_l/l^{k_2} \geq w_n/n^{k_2}$ for $l \leq n$. Let $w_0 = a_0$ and let $w_n = w_{n-1}$ unless $a_n/n^{k_2} = \min_{l \leq n} a_l/l^{k_2}$ in which case we let $w_n = a_n$. Then $w_l/l^{k_2} \geq w_n/n^{k_2}$ for $l \leq n$. Now since $\{a_n\}$ is bounded above by $k_1 n^{k_2-1}$ it follows that $a_n/n^{k_2} \to 0$ and hence $a_n/n^{k_2} = \min_{l \le n} a_l/l^{k_2}$ infinitely often so that $w_n = a_n$ infinitely often. Therefore (i) is also satisfied by this choice of $\{w_n\}$. $\square$

Using the basic coupling for the exclusion process combined with the basic coupling for spin systems, we have that the basic coupling for a noisy exclusion process has generator

$$\begin{aligned}
\bar{\Omega} f(\eta,\xi) &= \sum_{\substack{\eta(x)=\xi(x)=1\\ \eta(y)=\xi(y)=0}} q_e(x,y)[f(\eta_{xy},\xi_{xy}) - f(\eta,\xi)] \\
&+ \sum_{\substack{\eta(x)=1,\eta(y)=0 \text{ and}\\ \xi(y)=1 \text{ or } \xi(x)=0}} q_e(x,y)[f(\eta_{xy},\xi) - f(\eta,\xi)] \\
&+ \sum_{\substack{\xi(x)=1,\xi(y)=0 \text{ and}\\ \eta(y)=1 \text{ or } \eta(x)=0}} q_e(x,y)[f(\eta,\xi_{xy}) - f(\eta,\xi)] \\
&+ \sum_{x:\eta(x)\neq\xi(x)} c_1(x,\eta)[f(\eta_x,\xi) - f(\eta,\xi)] + \sum_{x:\eta(x)\neq\xi(x)} c_2(x,\xi)[f(\eta,\xi_x) - f(\eta,\xi)] \\
&+ \sum_{x:\eta(x)=\xi(x)} c(x,\eta,\xi)[f(\eta_x,\xi_x) - f(\eta,\xi)]
\end{aligned}$$

where

$$c_1(x,\eta) = \begin{cases} \beta(x) & \text{when } \eta(x)=0 \\ \delta(x) & \text{when } \eta(x)=1 \end{cases} \qquad c_2(x,\xi) = \begin{cases} \beta(x) & \text{when } \xi(x)=0 \\ \delta(x) & \text{when } \xi(x)=1 \end{cases}$$

$$\text{and } c(x,\eta,\xi) = \begin{cases} \beta(x) & \text{when } \eta(x)=\xi(x)=0 \\ \delta(x) & \text{when } \eta(x)=\xi(x)=1. \end{cases}$$

Let $\bar{\mathcal{I}}$ be the set of invariant measures for this coupling.

In order to simplify the notation we define the functions

$$f_x(\eta,\xi) = [1-\eta(x)]\xi(x), \qquad h_{yx}(\eta,\xi) = [1-\eta(y)][1-\xi(y)]f_x(\eta,\xi),$$
$$g_{yx}(\eta,\xi) = \eta(y)\xi(y)f_x(\eta,\xi), \quad \text{and} \quad f_{yx}(\eta,\xi) = \eta(y)[1-\xi(y)]f_x(\eta,\xi).$$

In particular, for T a finite subset of $\mathcal{S}$ we have

$$\tilde{\Omega}\left(\sum_{x\in T} f_x(\eta,\xi)\right) = -\sum_{x\in T, y\in\mathcal{S}} \left(q_e(x,y)+q_e(y,x)\right) f_{yx}(\eta,\xi) \tag{4.27}$$
$$- \sum_{x\in T} \left(\beta(x)+\delta(x)\right) f_x(\eta,\xi)$$
$$+ \sum_{x\in T, y\notin T} [q_e(x,y)g_{xy} - q_e(y,x)g_{yx}] + \sum_{x\in T, y\notin T} [q_e(y,x)h_{xy} - q_e(x,y)h_{yx}].$$

Proof of Theorem 4.1.7. Recall that $T_n = \{x \in \mathbb{Z}^d : |x_i| \le n\}$. Couple two noisy exclusion processes, η_t and ξ_t, with $\nu \in \bar{\mathcal{I}}$ so that

$$\int \bar{\Omega}\left(\sum_{x\in T_n} f_x(\eta_t,\xi_t)\right) d\nu = 0.$$

If we let $\int f_x(\eta_t,\xi_t)d\nu = a(x)$ then since $f_{yx}(\eta_t,\xi_t) \ge 0$, equation (4.27) gives us

$$\sum_{x\in T_n} \left(\beta(x)+\delta(x)\right) a(x) \tag{4.28}$$
$$\le \sum_{x\in T_n, y\notin T_n} q_e(x,y) \int (g_{xy} - h_{yx}) d\nu + \sum_{x\in T_n, y\notin T_n} q_e(y,x) \int (h_{xy} - g_{yx}) d\nu$$
$$\le \sum_{x\in T_n, y\notin T_n} q_e(x,y)a(y) + \sum_{x\in T_n, y\notin T_n} q_e(y,x)a(y)$$
$$\le \frac{(2N+1)^d}{2} \sum_{y\in T_n^N} a(y) + \sum_{y\in T_n^N} a(y) \le C_1 \sum_{y\in T_n^N} a(y)$$

for some constant C_1. If we define

$$a_l = \sum_{y\in T_{p(l)}^N} a(y)$$

then by the inequality $\beta(x)+\delta(x) \ge b_l$ for $x \in T_{p(l)}^N$ we can rewrite (4.28) as

$$\sum_{l=0}^{n-1} b_l a_l \le C_1 a_n. \tag{4.29}$$

Now suppose $d = 1$ and condition (a) in the theorem holds. Then since $a(x) \le 1$, we have $a_n \le 2N$ for all n. In light of equation (4.29) we then have that

$\sum_{l\geq 0} b_l a_l < \infty$. On the other hand, if we multiply both sides of (4.29) by b_n and then sum over n we get

$$\sum_{n\geq 0} b_n \sum_{l=0}^{n} b_l a_l \leq C_1 \sum_{n\geq 0} b_n a_n < \infty.$$

Rewriting the left hand side we get

$$\sum_{n\geq 0} b_n \sum_{l=0}^{n} b_l a_l = \sum_{l\geq 0} b_l a_l \sum_{n\geq l} b_n < \infty.$$

This implies that $b_l a_l = 0$ for all l since condition (a) gives us $\sum b_l = \infty$. So we have $a(x) = \int f_x d\nu = 0$ for all x so that the marginals of ν are exactly the same.

Suppose now that $d \geq 2$ and that condition (b) of the theorem holds. Since $p(l) \leq kl^k$ we have that a_n is bounded above by $k_1 n^{k_2-1}$ for some k_1, k_2. If we assume that for all n, $a_n > 0$ then by Lemma 4.6.1, there exists a sequence w_n such that $\liminf a_n/w_n = 1$ and $\limsup n w_n / \sum_{l=0}^{n-1} w_l < \infty$. By condition (b), we have then that

$$\liminf n b_n a_n / w_n = \infty \tag{4.30}$$

However, we also have that there exists a subsequence $\{n_j\}$ for which

$$\sum_{l=0}^{n_j-1} b_l a_l \leq C_1 a_{n_j} \leq C_2 w_{n_j} \leq C_3 \frac{\sum_{l=1}^{n_j-1} w_l}{n_j} \leq C_3 \sum_{l=1}^{n_j-1} \frac{w_l}{l}. \tag{4.31}$$

Notice now that if the limit of the right hand side is infinite, (4.30) and (4.31) contradict each other so that we must have $a_n = 0$ for some n and consequently $a(x) = \int f_x d\nu = 0$ for all x by irreducibility. If the right hand side is bounded then we can use the argument given above for the case $d = 1$ to show that $a(x) = \int f_x d\nu = 0$ for all x. In either case we have that the marginals of ν are the same, and we thus have ergodicity of the process. $\square$

We now restrict ourselves to the case where $d = 1$ and the transition rates are $q_e(x, x+1) = p > 1/2$ and $q_e(x, x-1) = 1 - p = q < 1/2$ for all x. In order to show the importance of the condition that there exist a sequence b_l satisfying $b_l \leq \beta(x)+\delta(x)$ for all $x \in T^N_{p(l)}$, we will find examples of processes on $\mathbb{Z}$ that are not ergodic but satisfy $\sum_x \beta(x) = \infty$.

To start off, consider the case where we have $\beta > 0$ and $\delta > 0$ for a single fixed z and no births and deaths at any other site. Choose c so that $c\pi(z)/(1+c\pi(z)) = \beta/(\beta+\delta)$ for a reversible measure $\pi(x)$ on $\mathbb{Z}$. The product measure ν^c with marginals $\nu^c\{\eta(x) = 1\} = c\pi(z)/(1+c\pi(z))$ is reversible with respect to the exclusion process, and its marginal measure at the site z is reversible with respect to the birth and death process so that ν^c is reversible with respect to the noisy exclusion process. The product measure ν_ρ where $\rho = \frac{\beta}{\beta+\delta}$ is also invariant with respect to the exclusion process, and again, its marginal measure at the site z is reversible with respect to the birth and death process. So ν_ρ is also invariant with respect to the noisy exclusion process.

We have two more invariant measures by starting the process off with initial states δ_0 and δ_1. This is because some subsequence of $\lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \delta_1 S(t)dt$ for $T_n \to \infty$ must lie above both of the invariant measures we have constructed above. Similarly some subsequence of $\lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \delta_0 S(t)dt$ lies below the two invariant measures. We note here that using extensions of these arguments we can construct examples of nonergodic processes for which $q_e(x,y)$ is not translation invariant. In particular, such an example is the process described above modified to let $q_e(z-1,z) = q_e(z-1,z-2) = 1/2$ so that $q_e(x,y)$ is translation invariant except at $x = z-1$.

In order to show that the noisy exclusion process with $\beta(z_i) > 0$ if and only if $\delta(z_i) > 0$ for a finite number of sites $\{z_1, ..., z_k\}$ is not ergodic (this is a special case of Proposition 4.6.2 below) we will need the following coupling for two noisy exclusion processes with the same transition and death rates, but different birth rates. If $\beta_1(x)$ for the process η_t is greater than $\beta_2(x)$ for the process ξ_t for all x then we can couple the two processes in such a way that $\eta_t \geq \xi_t$. Formally, we have the coupling given by

$$\begin{aligned}
\bar{\Omega} f(\eta,\xi) = & \sum_{\substack{\eta(x)=\xi(x)=1 \\ \eta(y)=\xi(y)=0}} q_e(x,y)[f(\eta_{xy},\xi_{xy}) - f(\eta,\xi)] \\
+ & \sum_{\substack{\eta(x)=1,\eta(y)=0 \text{ and} \\ \xi(y)=1 \text{ or } \xi(x)=0}} q_e(x,y)[f(\eta_{xy},\xi) - f(\eta,\xi)] \\
+ & \sum_{\substack{\xi(x)=1,\xi(y)=0 \text{ and} \\ \eta(y)=1 \text{ or } \eta(x)=0}} q_e(x,y)[f(\eta,\xi_{xy}) - f(\eta,\xi)] \\
+ & \sum_{x:\eta(x)\neq\xi(x)} c_1(x,\eta)[f(\eta_x,\xi) - f(\eta,\xi)] + \sum_{x:\eta(x)\neq\xi(x)} c_2(x,\xi)[f(\eta,\xi_x) - f(\eta,\xi)] \\
+ & \sum_{x:\eta(x)=\xi(x)} c(x,\eta,\xi)[f(\eta_x,\xi_x) - f(\eta,\xi)] \\
+ & \sum_{x:\eta(x)=\xi(x)=0} (\beta_1(x) - \beta_2(x))[f(\eta_x,\xi) - f(\eta,\xi)]
\end{aligned}$$

where

$$c_1(x,\eta) = \begin{cases} \beta_1(x) & \text{when } \eta(x)=0 \\ \delta(x) & \text{when } \eta(x)=1 \end{cases} \qquad c_2(x,\xi) = \begin{cases} \beta_2(x) & \text{when } \xi(x)=0 \\ \delta(x) & \text{when } \xi(x)=1 \end{cases}$$

$$\text{and } c(x,\eta,\xi) = \begin{cases} \beta_2(x) & \text{when } \eta(x)=\xi(x)=0 \\ \delta(x) & \text{when } \eta(x)=\xi(x)=1. \end{cases}$$

Similarly, we can couple two processes together so that $\eta_t \leq \xi_t$ when η_t and ξ_t have the same transition and birth rates, but death rates such that $\delta_1(x) \geq \delta_2(x)$ for all x.

Proposition 4.6.2. *Suppose that* $q_e(x,x+1) = p > \frac{1}{2}$ *and* $q_e(x,x-1) = 1-p = q$ *for all* x *and that* $\beta(x) > 0$ *if and only if* $\delta(x) > 0$*. If there exists a* z *such that*

$\beta(x) = 0$ for either all $x \le z$ or for all $x \ge z$ and if there exist a_1 and a_2 such that $\frac{a_1\pi(x)}{1+a_1\pi(x)} \le \frac{\beta(x)}{\beta(x)+\delta(x)} \le \frac{a_2\pi(x)}{1+a_2\pi(x)}$ for all x where $\beta(x) > 0$, then the process is not ergodic.

Proof. Without loss of generality suppose that $\beta(x) = 0$ for all positive x and let $\{z_i\}$ denote the set of points where $\beta(x) > 0$. If η_t is the process described in the hypothesis of the proposition, let the process ξ_t be the same as η_t except that we change the death rates of ξ_t so that $\frac{\beta(z_i)}{\beta(z_i)+\delta(z_i)} = \frac{a_1\pi(z_i)}{1+a_1\pi(z_i)}$ for all $\{z_i\}$. Let the process ζ_t be the same as η_t except that we change the birth rates of ζ_t so that $\frac{\beta(z_i)}{\beta(z_i)+\delta(z_i)} = \frac{a_2\pi(z_i)}{1+a_2\pi(z_i)}$ for all $\{z_i\}$. We can triple couple ξ_t, η_t, and ζ_t so that $\xi_t \le \eta_t \le \zeta_t$. Since the measure ν^{a_1} is invariant for ξ_t and ν^{a_2} is invariant for ζ_t, then η_t has an invariant measure μ_1 with $\nu^{a_1} \le \mu_1 \le \nu^{a_2}$.

Let $M = \max_i \left(\frac{\beta(z_i)}{\beta(z_i)+\delta(z_i)}\right)$. Note that this maximum is achieved since we assumed earlier that $\beta(x) = 0$ for all positive x and consequently if there exist an infinite number of z_i's then $\lim_{i\to\infty} \frac{\beta(z_i)}{\beta(z_i)+\delta(z_i)} = 0$. Now let the process ζ_t be the same as η_t except that we change the birth rates of ζ_t so that $\frac{\beta(z_i)}{\beta(z_i)+\delta(z_i)} = M$ for all $\{z_i\}$. Again, we can couple η_t and ζ_t so that $\eta_t \le \zeta_t$. The measure ν_M is invariant for ζ_t. So η_t has an invariant measure μ_2 such that $\mu_2 \le \nu_M$. Since μ_2 is different from μ_1, the process is not ergodic. □

Note that using the above proposition, we can construct examples of nonergodic processes that satisfy all of the hypotheses for Schwartz's ergodic theorem except for $q_e(x, y) = q_e(y, x)$.

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