

# FURTHER DEVELOPMENTS OF SINAI'S IDEAS: THE BOLTZMANN-SINAI HYPOTHESIS

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0.1. **Preface.** In 1963 Ya. G. Sinai [Sin(1963)] formulated a modern version of Boltzmann's ergodic hypothesis, what we now call the "Boltzmann-Sinai Ergodic Hypothesis": The billiard system of  $N$  ( $N \geq 2$ ) hard balls of unit mass moving on the flat torus  $\mathbb{T}^\nu = \mathbb{R}^\nu / \mathbb{Z}^\nu$  ( $\nu \geq 2$ ) is ergodic after we make the standard reductions by fixing the values of trivial invariant quantities. It took fifty years and the efforts of several people, including Sinai himself, until this conjecture was finally proved. In this short survey we provide a quick review of the closing part of this process, by showing how Sinai's original ideas developed further between 2000 and 2013, eventually leading the proof of the conjecture.

## 1. POSING THE PROBLEM THE INVESTIGATED MODELS

Non-uniformly hyperbolic systems (possibly, with singularities) play a pivotal role in the ergodic theory of dynamical systems. Their systematic study started several decades ago, and it is not our goal here to provide the reader with a comprehensive review of the history of these investigations but, instead, we opt for presenting in nutshell a cross section of a few selected results.

In 1939 G. A. Hedlund and E. Hopf [He(1939)], [Ho(1939)], proved the hyperbolic ergodicity (i.e. full hyperbolicity and ergodicity) of geodesic flows on closed, compact surfaces with constant negative curvature by inventing the famous method of "Hopf chains" constituted by local stable and unstable invariant manifolds.

In 1963 Ya. G. Sinai [Sin(1963)] formulated a modern version of Boltzmann's ergodic hypothesis, what we call now the "Boltzmann-Sinai Hypothesis": the billiard system of  $N$  ( $\geq 2$ ) hard balls of unit mass moving on the flat torus  $\mathbb{T}^\nu = \mathbb{R}^\nu / \mathbb{Z}^\nu$  ( $\nu \geq 2$ ) is ergodic after we make the standard reductions by fixing the values of the trivial invariant quantities. It took seven years until he proved this conjecture for the case  $N = 2$ ,  $\nu = 2$  in [Sin(1970)]. Another 17 years later N. I. Chernov and Ya. G.

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Sinai [S-Ch(1987)] proved the hypothesis for the case  $N = 2$ ,  $\nu \geq 2$  by also proving a powerful and very useful theorem on local ergodicity.

In the meantime, in 1977, Ya. Pesin [P(1977)] laid down the foundations of his theory on the ergodic properties of smooth, hyperbolic dynamical systems. Later on this theory (nowadays called Pesin theory) was significantly extended by A. Katok and J-M. Strelcyn [K-S(1986)] to hyperbolic systems with singularities. That theory is already applicable for billiard systems, too.

Until the end of the seventies the phenomenon of hyperbolicity (exponential instability of trajectories) was almost exclusively attributed to some direct geometric scattering effect, like negative curvature of space, or strict convexity of the scatterers. This explains the profound shock that was caused by the discovery of L. A. Bunimovich [B(1979)]: Certain focusing billiard tables (like the celebrated stadium) can also produce complete hyperbolicity and, in that way, ergodicity. It was partly this result that led to Wojtkowski's theory of invariant cone fields, [W(1985)], [W(1986)].

The big difference between the system of two balls in  $\mathbb{T}^\nu$  ( $\nu \geq 2$ , [S-Ch(1987)]) and the system of  $N$  ( $\geq 3$ ) balls in  $\mathbb{T}^\nu$  is that the latter one is merely a so called semi-dispersive billiard system (the scatterers are convex but not strictly convex sets, namely cylinders), while the former one is strictly dispersive (the scatterers are strictly convex sets). This fact makes the proof of ergodicity (mixing properties) much more complicated. In our series of papers jointly written with A. Krámli and D. Szász [K-S-Sz(1990)], [K-S-Sz(1991)], and [K-S-Sz(1992)], we managed to prove the (hyperbolic) ergodicity of three and four billiard balls on the toroidal container  $\mathbb{T}^\nu$ . By inventing new topological methods and the Connecting Path Formula (CPF), in the two-part paper [Sim(1992)-I], [Sim(1992)-II] I proved the (hyperbolic) ergodicity of  $N$  hard balls in  $\mathbb{T}^\nu$ , provided that  $N \leq \nu$ .

The common feature of hard ball systems is – as D. Szász pointed this out first in [Sz(1993)] and [Sz(1994)] – that all of them belong to the family of so called cylindrical billiards, the definition of which can be found later in this survey. However, the first appearance of a special, 3-D cylindrical billiard system took place in [K-S-Sz(1989)], where we proved the ergodicity of a 3-D billiard flow with two orthogonal cylindrical scatterers. Later D. Szász [Sz(1994)] presented a complete picture (as far as ergodicity is concerned) of cylindrical billiards with cylinders whose generator subspaces are spanned by mutually orthogonal coordinate axes. The task of proving ergodicity for the first non-trivial, non-orthogonal cylindrical billiard system was taken up in [S-Sz(1994)].

Finally, in our joint venture with D. Szász [S-Sz(1999)] we managed to prove the complete hyperbolicity of typical hard ball systems on flat tori.

**1.1. Cylindric billiards.** Consider the  $d$ -dimensional ( $d \geq 2$ ) flat torus  $\mathbb{T}^d = \mathbb{R}^d / \mathcal{L}$  supplied with the usual Riemannian inner product  $\langle \cdot, \cdot \rangle$  inherited from the standard inner product of the universal covering space  $\mathbb{R}^d$ . Here  $\mathcal{L} \subset \mathbb{R}^d$  is supposed to be a

lattice, i. e. a discrete subgroup of the additive group  $\mathbb{R}^d$  with  $\text{rank}(\mathcal{L}) = d$ . The reason why we want to allow general lattices other than just the integer lattice  $\mathbb{Z}^d$  is that otherwise the hard ball systems would not be covered. The geometry of the structure lattice  $\mathcal{L}$  in the case of a hard ball system is significantly different from the geometry of the standard orthogonal lattice  $\mathbb{Z}^d$  in the Euclidean space  $\mathbb{R}^d$ .

The configuration space of a cylindric billiard is  $\mathbf{Q} = \mathbb{T}^d \setminus (C_1 \cup \dots \cup C_k)$ , where the cylindric scatterers  $C_i$  ( $i = 1, \dots, k$ ) are defined as follows:

Let  $A_i \subset \mathbb{R}^d$  be a so called lattice subspace of  $\mathbb{R}^d$ , which means that  $\text{rank}(A_i \cap \mathcal{L}) = \dim A_i$ . In this case the factor  $A_i / (A_i \cap \mathcal{L})$  is a subtorus in  $\mathbb{T}^d = \mathbb{R}^d / \mathcal{L}$ , which will be taken as the generator of the cylinder  $C_i \subset \mathbb{T}^d$ ,  $i = 1, \dots, k$ . Denote by  $L_i = A_i^\perp$  the orthocomplement of  $A_i$  in  $\mathbb{R}^d$ . Throughout this survey we will always assume that  $\dim L_i \geq 2$ . Let, furthermore, the numbers  $r_i > 0$  (the radii of the spherical cylinders  $C_i$ ) and some translation vectors  $t_i \in \mathbb{T}^d = \mathbb{R}^d / \mathcal{L}$  be given. The translation vectors  $t_i$  play a crucial role in positioning the cylinders  $C_i$  in the ambient torus  $\mathbb{T}^d$ . Set

$$C_i = \{x \in \mathbb{T}^d : \text{dist}(x - t_i, A_i / (A_i \cap \mathcal{L})) < r_i\}.$$

In order to avoid further unnecessary complications, we always assume that the interior of the configuration space  $\mathbf{Q} = \mathbb{T}^d \setminus (C_1 \cup \dots \cup C_k)$  is connected. The phase space  $\mathbf{M}$  of our cylindric billiard flow will be the unit tangent bundle of  $\mathbf{Q}$  (modulo some natural gluings at its boundary), i. e.  $\mathbf{M} = \mathbf{Q} \times \mathbb{S}^{d-1}$ . (Here  $\mathbb{S}^{d-1}$  denotes the unit sphere of  $\mathbb{R}^d$ .)

The dynamical system  $(\mathbf{M}, \{S^t\}, \mu)$ , where  $S^t$  ( $t \in \mathbb{R}$ ) is the dynamics defined by uniform motion inside the domain  $\mathbf{Q}$  and specular reflections at its boundary (at the scatterers), and  $\mu$  is the Liouville measure, is called a cylindric billiard flow we investigate.

**1.2. Transitive cylindric billiards.** The main conjecture concerning the (hyperbolic) ergodicity of cylindric billiards is the "Erdős-tarcsa conjecture" (named after the picturesque village in rural Hungary where it was initially formulated) that appeared as Conjecture 1 in Section 3 of [S-Sz(2000)]:

**Conjecture 1.1** (The Erdős-tarcsa Conjecture). A cylindric billiard flow is ergodic if and only if it is transitive, i.e. the Lie group generated by all rotations across the constituent spaces of the cylinders acts transitively on the sphere of compound velocities, see Section 3 of [S-Sz(2000)]. In the case of transitivity the cylindric billiard system is actually a completely hyperbolic Bernoulli flow, see [C-H(1996)] and [O-W(1998)].

The theorem of [Sim(2002)] proves a slightly relaxed version of this conjecture (only full hyperbolicity without ergodicity) for a wide class of cylindric billiard systems, namely the so called "transverse systems", which include every hard ball system.

**1.3. Transitivity.** Let  $L_1, \dots, L_k \subset \mathbb{R}^d$  be subspaces,  $A_i = L_i^\perp$ ,  $\dim L_i \geq 2$ ,  $i = 1, \dots, k$ . Set

$$\mathcal{G}_i = \{U \in \text{SO}(d) : U|_{A_i} = \text{Id}_{A_i}\},$$

and let  $\mathcal{G} = \langle \mathcal{G}_1, \dots, \mathcal{G}_k \rangle \subset \text{SO}(d)$  be the algebraic generate of the compact, connected Lie subgroups  $\mathcal{G}_i$  in  $\text{SO}(d)$ . The following notions appeared in Section 3 of [S-Sz(2000)].

**Definition 1.2.** We say that the system of base spaces  $\{L_1, \dots, L_k\}$  (or, equivalently, the cylindric billiard system defined by them) is *transitive* if and only if the group  $\mathcal{G}$  acts transitively on the unit sphere  $\mathbb{S}^{d-1}$  of  $\mathbb{R}^d$ .

**Definition 1.3.** We say that the system of subspaces  $\{L_1, \dots, L_k\}$  has the *Orthogonal Non-splitting Property (ONSP)* if there is no non-trivial orthogonal splitting  $\mathbb{R}^d = B_1 \oplus B_2$  of  $\mathbb{R}^d$  with the property that for every index  $i$  ( $1 \leq i \leq k$ )  $L_i \subset B_1$  or  $L_i \subset B_2$ .

The next result can be found in Section 3 of [S-Sz(2000)] (see 3.1–3.6 thereof):

**Proposition 1.4.** *For the system of subspaces  $\{L_1, \dots, L_k\}$  the following three properties are equivalent:*

- (1)  $\{L_1, \dots, L_k\}$  is transitive;
- (2)  $\{L_1, \dots, L_k\}$  has the ONSP;
- (3) the natural representation of  $\mathcal{G}$  in  $\mathbb{R}^d$  is irreducible.

#### 1.4. Transverseness.

**Definition 1.5.** We say that the system of subspaces  $\{L_1, \dots, L_k\}$  of  $\mathbb{R}^d$  is *transverse* if the following property holds: For every non-transitive subsystem  $\{L_i : i \in I\}$  ( $I \subset \{1, \dots, k\}$ ) there exists an index  $j_0 \in \{1, \dots, k\}$  such that  $P_{E^+}(A_{j_0}) = E^+$ , where  $A_{j_0} = L_{j_0}^\perp$ , and  $E^+ = \text{span}\{L_i : i \in I\}$ . We note that in this case, necessarily,  $j_0 \notin I$ , otherwise  $P_{E^+}(A_{j_0})$  would be orthogonal to the subspace  $L_{j_0} \subset E^+$ . Therefore, every transverse system is automatically transitive.

We note that every hard ball system is transverse, see [Sim(2002)]. The main result of that paper is the following theorem.

**Theorem 1.6.** *Assume that the cylindric billiard system is transverse. Then this billiard flow is completely hyperbolic, i.e. all relevant Lyapunov exponents are nonzero almost everywhere. Consequently, in such a dynamical system every ergodic component has positive measure, and the restriction of the flow to each ergodic component has the Bernoulli property, see [C-H(1996)] and [O-W(1998)].*

An immediate consequence of this result is

**Corollary 1.7.** *Every hard ball system is completely hyperbolic.*

Thus Theorem 1.6 above generalizes the main result of [S-Sz(1999)], where the complete hyperbolicity of almost every hard ball system was proven.

## 2. TOWARD ERGODICITY

In the series of articles [K-S-Sz(1989)], [K-S-Sz(1991)], [K-S-Sz(1992)], [Sim(1992)-I], and [Sim(1992)-II] the authors developed a powerful, three-step strategy for proving the (hyperbolic) ergodicity of hard ball systems. First of all, all these proofs are inductions on the number  $N$  of balls involved in the problem. Secondly, the induction step itself consists of the following three major steps:

**2.1. Step I.** To prove that every non-singular (i. e. smooth) trajectory segment  $S^{[a,b]}x_0$  with a “combinatorially rich” symbolic collision sequence is automatically sufficient (or, in other words, “geometrically hyperbolic”), provided that the phase point  $x_0$  does not belong to a countable union  $J$  of smooth sub-manifolds with codimension at least two. (Containing the exceptional phase points.)

Here combinatorial richness means that the symbolic collision sequence of the orbit segment contains a large enough number of consecutive, connected collision graphs, see also the introductory section of [S-Sz(1999)].

The exceptional set  $J$  featuring this result is negligible in our dynamical considerations – it is a so called slim set, i.e. a subset of the phase space  $\mathbf{M}$  that can be covered by a countable union  $\bigcup_{n=1}^{\infty} F_n$  of closed, zero-measured subsets  $F_n$  of  $\mathbf{M}$  that have topological co-dimension at least 2.

**2.2. Step II.** Assume the induction hypothesis, i. e. that all hard ball systems with  $N'$  balls ( $2 \leq N' < N$ ) are (hyperbolic and) ergodic. Prove that then there exists a slim set  $S \subset \mathbf{M}$  with the following property: For every phase point  $x_0 \in \mathbf{M} \setminus S$  the whole trajectory  $S^{(-\infty,\infty)}x_0$  contains at most one singularity and its symbolic collision sequence is combinatorially rich, just as required by the result of Step I.

**2.3. Step III.** By using again the induction hypothesis, prove that almost every singular trajectory is sufficient in the time interval  $(t_0, \infty)$ , where  $t_0$  is the moment of the singular reflection, i.e. a tangential reflection or multiple reflections occurring at the same time. (Here the phrase “almost every” refers to the volume defined by the induced Riemannian metric on the singularity manifolds.)

We note here that the almost sure sufficiency of the singular trajectories (featuring Step III) is an essential condition for the proof of the celebrated Theorem on Local Ergodicity for algebraic semi-dispersive billiards proved by Chernov and Sinai in [S-Ch(1987)]. Under this assumption the theorem of [S-Ch(1987)] states that in any algebraic semi-dispersive billiard system (i.e. in a system such that the smooth

components of the boundary  $\partial\mathbf{Q}$  are algebraic hypersurfaces) a suitable, open neighborhood  $U_0$  of any hyperbolic phase point  $x_0 \in \mathbf{M}$  (with at most one singularity on its trajectory) belongs to a single ergodic component of the billiard flow.

In an inductive proof of ergodicity, steps I and II together ensure that there exists an arc-wise connected set  $C \subset \mathbf{M}$  with full measure, such that every phase point  $x_0 \in C$  is hyperbolic with at most one singularity on its trajectory. Then the cited Theorem on Local Ergodicity (now taking advantage of the result of Step III) states that for every phase point  $x_0 \in C$  an open neighborhood  $U_0$  of  $x_0$  belongs to one ergodic component of the flow. Finally, the connectedness of the set  $C$  and  $\mu(C) = 1$  easily imply that the billiard flow with  $N$  balls is indeed ergodic, and actually fully hyperbolic, as well.

In the papers [S-Sz(1999)], [Sim(2003)], and [Sim(2004)] we investigated systems of hard balls with masses  $m_1, m_2, \dots, m_N$  ( $m_i > 0$ ) moving on the flat torus  $\mathbb{T}_L^\nu = \mathbb{R}^\nu / L \cdot \mathbb{Z}^\nu$ ,  $L > 0$ .

The main results of the papers [Sim(2003)] and [Sim(2004)] are summarized as follows:

**Theorem 2.1.** *For almost every selection  $(m_1, \dots, m_N; L)$  of the external geometric parameters from the region  $m_i > 0$ ,  $L > L_0(r, \nu)$ , where the interior of the phase space is connected, it is true that the billiard flow  $(\mathbf{M}_{\vec{m}, L}, \{S^t\}, \mu_{\vec{m}, L})$  of the  $N$ -ball system is ergodic and completely hyperbolic. Then, following from the results of [C-H(1996)] and [O-W(1998)], such a semi-dispersive billiard system actually enjoys the Bernoulli mixing property, as well.*

*Remark 2.2.* We note that the results of the papers [Sim(2003)] and [Sim(2004)] nicely complement each other. They precisely assert the same, almost sure ergodicity of hard ball systems in the cases  $\nu = 2$  and  $\nu \geq 3$ , respectively. It should be noted, however, that the proof of [Sim(2003)] is primarily dynamical-geometric (except the verification of the Chernov-Sinai Ansatz), whereas the novel parts of [Sim(2004)] are fundamentally algebraic. We note that the Chernov-Sinai Ansatz claims that almost every singular trajectory is eventually hyperbolic.

*Remark 2.3.* The above inequality  $L > L_0(r, \nu)$  corresponds to physically relevant situations. Indeed, in the case  $L < L_0(r, \nu)$  the particles would not have enough room to even freely exchange positions.

### 3. THE CONDITIONAL PROOF

In the paper [Sim(2009)] we again considered the system of  $N$  ( $\geq 2$ ) elastically colliding hard spheres with masses  $m_1, \dots, m_N$  and radius  $r$  on the flat unit torus  $\mathbb{T}^\nu$ ,  $\nu \geq 2$ . We proved the Boltzmann-Sinai Ergodic Hypothesis, i. e. the full hyperbolicity and ergodicity of such systems for every selection  $(m_1, \dots, m_N; r)$  of

the external parameters, provided that almost every singular orbit is geometrically hyperbolic (sufficient), i. e. the so called Chernov-Sinai Ansatz is true. The proof does not use the formerly developed, rather involved algebraic techniques, instead it extensively employs dynamical methods and tools from geometric analysis.

To upgrade the full hyperbolicity to ergodicity, one needs to refine the analysis of the degeneracies, i.e. the set of non-hyperbolic phase points. For hyperbolicity, it was enough that the degeneracies made a subset of codimension  $\geq 1$  in the phase space. For ergodicity, one has to show that its codimension is  $\geq 2$ , or to find some other ways to prove that the (possibly) arising one-codimensional, smooth submanifolds of non-sufficiency are incapable of separating distinct, open ergodic components from each other. The latter approach was successfully pursued in [Sim(2009)]. In the paper [Sim(2003)] I took the first step in the direction of proving that the codimension of exceptional manifolds is at least two: It was proved there that the systems of  $N \geq 2$  disks on a 2D torus (i.e.,  $\nu = 2$ ) are ergodic for typical (generic)  $(N + 1)$ -tuples of external parameters  $(m_1, \dots, m_N, r)$ . The proof involved some algebro-geometric techniques, thus the result is restricted to generic parameters  $(m_1, \dots, m_N; r)$ . But there was a good reason to believe that systems in  $\nu \geq 3$  dimensions would be somewhat easier to handle, at least that was indeed the case in early studies.

In the paper [Sim(2004)] I was able to further improve the algebro-geometric methods of [S-Sz(1999)], and proved that for any  $N \geq 2$ ,  $\nu \geq 2$ , and for almost every selection  $(m_1, \dots, m_N; r)$  of the external geometric parameters the corresponding system of  $N$  hard balls on  $\mathbb{T}^\nu$  is (fully hyperbolic and) ergodic.

In the paper [Sim(2009)] the following result was obtained.

**Theorem 3.1.** *For any integer values  $N \geq 2$ ,  $\nu \geq 2$ , and for every  $(N + 1)$ -tuple  $(m_1, \dots, m_N, r)$  of the external geometric parameters the standard hard ball system  $(\mathbf{M}_{\vec{m}, r}, \{S_{\vec{m}, r}^t\}, \mu_{\vec{m}, r})$  is (fully hyperbolic and) ergodic, provided that the Chernov-Sinai Ansatz holds true for all such systems.*

*Remark 3.2.* The novelty of the theorem (as compared to the result in [Sim(2004)]) is that it applies to every  $(N+1)$ -tuple of external parameters (provided that the interior of the phase space is connected), without an exceptional zero-measure set. Somehow, the most annoying shortcoming of several earlier results was exactly the fact that those results were only valid for hard sphere systems apart from an undescribed, countable collection of smooth, proper submanifolds of the parameter space  $\mathbb{R}^{N+1} \ni (m_1, m_2, \dots, m_N; r)$ . Furthermore, those proofs do not provide any effective means to check if a given  $(m_1, \dots, m_N; r)$ -system is ergodic or not, most notably for the case of equal masses in Sinai's classical formulation of the problem.

*Remark 3.3.* The present result speaks about exactly the same models as the result of [Sim(2002)], but the statement of this new theorem is obviously stronger than that of the theorem in [Sim(2002)]: It has been known for a long time that, for the family

of semi-dispersive billiards, ergodicity cannot be obtained without also proving full hyperbolicity.

*Remark 3.4.* As it follows from the results of [C-H(1996)] and [O-W(1998)], all standard hard ball systems (the models covered by the theorems of this survey), once they are proved to be mixing, they also enjoy the much stronger Bernoulli mixing property. However, even the K-mixing property of semi-dispersive billiard systems follows from their ergodicity, as the classical results of Sinai in [Sin(1970)], and [Sin(1979)] show.

In the subsequent part of this survey we review the necessary technical prerequisites of the proof, along with some of the needed references to the literature. The fundamental objects of the paper [Sim(2009)] are the so called "exceptional manifolds" or "separating manifolds"  $J$ : they are codimension-one submanifolds of the phase space that are separating distinct, open ergodic components of the billiard flow.

In §3 of [Sim(2009)] we proved Main Lemma 3.5, which states, roughly speaking, the following: Every separating manifold  $J \subset \mathbf{M}$  contains at least one sufficient (or geometrically hyperbolic) phase point. The existence of such a sufficient phase point  $x \in J$ , however, contradicts the Theorem on Local Ergodicity of Chernov and Sinai (Theorem 5 in [S-Ch(1987)]), since an open neighborhood  $U$  of  $x$  would then belong to a single ergodic component, thus violating the assumption that  $J$  is a separating manifold. In §4 this result was exploited to carry out an inductive proof of the (hyperbolic) ergodicity of every hard ball system, provided that the Chernov-Sinai Ansatz holds true for all hard ball systems.

In what follows, we make an attempt to briefly outline the key ideas of the proof of Main Lemma 3.5 of [Sim(2009)]. Of course, this outline will lack the majority of the nitty-gritty details, technicalities, that constitute an integral part of the proof. The proof is a proof by contradiction.

We consider the one-sided, tubular neighborhoods  $U_\delta$  of  $J$  with radius  $\delta > 0$ . Throughout the whole proof of the main lemma the asymptotics of the measures  $\mu(X_\delta)$  of certain (dynamically defined) sets  $X_\delta \subset U_\delta$  are studied, as  $\delta \rightarrow 0$ . We fix a large constant  $c_3 \gg 1$ , and for typical points  $y \in U_\delta \setminus U_{\delta/2}$  (having non-singular forward orbits and returning to the layer  $U_\delta \setminus U_{\delta/2}$  infinitely many times in the future) we define the arc-length parametrized curves  $\rho_{y,t}(s)$  ( $0 \leq s \leq h(y,t)$ ) in the following way:  $\rho_{y,t}$  emanates from  $y$  and it is the curve inside the manifold  $\Sigma_0^t(y)$  with the steepest descent towards the separating manifold  $J$ . Here  $\Sigma_0^t(y)$  is the inverse image  $S^{-t}(\Sigma_t^t(y))$  of the flat, local orthogonal manifold passing through  $y_t = S^t(y)$ . The terminal point  $\Pi(y) = \rho_{y,t}(h(y,t))$  of the smooth curve  $\rho_{y,t}$  is either

- (a) on the separating manifold  $J$ , or
- (b) on a singularity of order  $k_1 = k_1(y)$ .

The case (b) is further split in two sub-cases, as follows:



(b/1)  $k_1(y) < c_3$ ;

(b/2)  $c_3 \leq k_1(y) < \infty$ .

About the set  $\overline{U}_\delta(\infty)$  of (typical) points  $y \in U_\delta \setminus U_{\delta/2}$  with property (a) it is shown that, actually,  $\overline{U}_\delta(\infty) = \emptyset$ . Roughly speaking, the reason for this is the following: For a point  $y \in \overline{U}_\delta(\infty)$  the iterates  $S^t$  of the flow exhibit arbitrarily large contractions on the curves  $\rho_{y,t}$ , thus the infinitely many returns of  $S^t(y)$  to the layer  $U_\delta \setminus U_{\delta/2}$  would "pull up" the other endpoints  $S^t(\Pi(y))$  to the region  $U_\delta \setminus J$ , consisting entirely of sufficient points, and showing that the point  $\Pi(y) \in J$  itself is sufficient, thus violating the assumed hypothesis of the proof by contradiction.

The set  $\overline{U}_\delta \setminus \overline{U}_\delta(c_3)$  of all phase points  $y \in U_\delta \setminus U_{\delta/2}$  with the property  $k_1(y) < c_3$  are dealt with by a lemma, where it is shown that

$$\mu(\overline{U}_\delta \setminus \overline{U}_\delta(c_3)) = o(\delta),$$

as  $\delta \rightarrow 0$ . The reason, in rough terms, is that such phase points must lie at the distance  $\leq \delta$  from the compact singularity set

$$\bigcup_{0 \leq t \leq 2c_3} S^{-t}(\mathcal{SR}^-),$$

and this compact singularity set is transversal to  $J$ , thus ensuring the measure estimate  $\mu(\overline{U}_\delta \setminus \overline{U}_\delta(c_3)) = o(\delta)$ .

Finally, the set  $F_\delta(c_3)$  of (typical) phase points  $y \in U_\delta \setminus U_{\delta/2}$  with  $c_3 \leq k_1(y) < \infty$  is dealt with by lemmas 3.36, 3.37, and Corollary 3.38 of [Sim(2009)], where it is shown that  $\mu(F_\delta(c_3)) \leq C \cdot \delta$ , with constants  $C$  that can be chosen arbitrarily small by selecting the constant  $c_3 \gg 1$  big enough. The ultimate reason of this measure estimate is the following fact: For every point  $y \in F_\delta(c_3)$  the projection

$$\tilde{\Pi}(y) = S^{t_{\bar{k}_1(y)}} \in \partial \mathbf{M}$$

(where  $t_{\bar{k}_1(y)}$  is the time of the  $\bar{k}_1(y)$ -th collision on the forward orbit of  $y$ ) will have a tubular distance  $z_{tub}(\tilde{\Pi}(y)) \leq C_1 \delta$  from the singularity set  $\mathcal{SR}^- \cup \mathcal{SR}^+$ , where the constant  $C_1$  can be made arbitrarily small by choosing the contraction coefficients of the iterates  $S^{t_{\bar{k}_1(y)}}$  on the curves  $\rho_{y, t_{\bar{k}_1(y)}}$  arbitrarily small with the help of the result in Appendix II. The upper measure estimate (inside the set  $\partial \mathbf{M}$ ) of the set of such points  $\tilde{\Pi}(y) \in \partial \mathbf{M}$  (Lemma 2 in [S-Ch(1987)]) finally yields the required upper bound  $\mu(F_\delta(c_3)) \leq C \cdot \delta$  with arbitrarily small positive constants  $C$  (if  $c_3 \gg 1$  is big enough).

The listed measure estimates and the obvious fact

$$\mu(U_\delta \setminus U_{\delta/2}) \approx C_2 \cdot \delta$$

(with some constant  $C_2 > 0$ , depending only on  $J$ ) show that there must exist a point  $y \in U_\delta \setminus U_{\delta/2}$  with the property (a) above, thus ensuring the sufficiency of the point  $\Pi(y) \in J$ .

In the closing section of [Sim(2009)] we completed the inductive proof of ergodicity (with respect to the number of balls  $N$ ) by utilizing Main Lemma 3.5 and earlier results from the literature. Actually, a consequence of the Main Lemma will be that exceptional  $J$ -manifolds do not exist, and this will imply the fact that no distinct, open ergodic components can coexist.

#### 4. PROOF OF ANSATZ

Finally, in the paper [Sim(2013)] we proved the Boltzmann–Sinai Hypothesis for hard ball systems on the  $\nu$ -torus  $\mathbb{R}^\nu/\mathbb{Z}^\nu$  ( $\nu \geq 2$ ) without any assumed hypothesis or exceptional model.

As said before, in [Sim(2009)] the Boltzmann-Sinai Hypothesis was proved in full generality (i.e. without exceptional models), by assuming the Chernov-Sinai Ansatz.

The only missing piece of the whole puzzle is to prove that no open piece of a singularity manifold can precisely coincide with a codimension-one manifold describing the trajectories with a non-sufficient forward orbit segment corresponding to a fixed symbolic collision sequence. This is exactly what we claim in our Theorem below.

**4.1. Formulation of Theorem.** Let  $U_0 \subset \mathbf{M} \setminus \partial\mathbf{M}$  be an open ball,  $T > 0$ , and assume that

- (a)  $S^T(U_0) \cap \partial\mathbf{M} = \emptyset$ ,
- (b)  $S^T$  is smooth on  $U_0$ .

Next we assume that there is a *codimension-one*, smooth submanifold  $J \subset U_0$  with the property that for every  $x \in U_0$  the trajectory segment  $S^{[0,T]}x$  is geometrically hyperbolic (sufficient) if and only if  $x \notin J$ . ( $J$  is a so called non-hyperbolicity or degeneracy manifold.) Denote the common symbolic collision sequence of the orbits  $S^{[0,T]}x$  ( $x \in U_0$ ) by  $\Sigma = (e_1, e_2, \dots, e_n)$ , listed in the increasing time order. Let  $t_i = t(e_i)$  be the time of the  $i$ -th collision,  $0 < t_1 < t_2 < \dots < t_n < T$ .

Finally we assume that for every phase point  $x \in U_0$  the first reflection  $S^{\tau(x)}x$  in the past on the orbit of  $x$  is a singular reflection (i. e.  $S^{\tau(x)}x \in \mathcal{SR}_0^+$ ) if and only if  $x$  belongs to a codimension-one, smooth submanifold  $K$  of  $U_0$ . For the definition of the manifold of singular reflections  $\mathcal{SR}_0^+$  see, for instance, the end of §1 in [Sim(2009)].

**Theorem 4.1.** *Using all the assumptions and notations above, the submanifolds  $J$  and  $K$  of  $U_0$  do not coincide.*

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