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HOMOTOPICAL COMPLEXITY OF A BILLIARD FLOW ON THE $3D$ FLAT TORUS WITH TWO CYLINDRICAL OBSTACLES

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Abstract. We study the homotopical rotation vectors and the homotopical rotation sets for the billiard flow on the unit flat torus with two disjoint and orthogonal toroidal (cylindric) scatterers removed from it.

The natural habitat for these objects is the infinite cone erected upon the Cantor set $\text{Ends}(G)$ of all "ends" of the hyperbolic group $G = \pi_1(\mathbf{Q})$. An element of $\text{Ends}(G)$ describes the direction in (the Cayley graph of) the group G in which the considered trajectory escapes to infinity, whereas the height function s ($s \geq 0$) of the cone gives us the average speed at which this escape takes place.

The main results of this paper claim that the orbits can only escape to infinity at a speed not exceeding $\sqrt{3}$, and in any direction $e \in \text{Ends}(\pi_1(\mathcal{Q}))$ the escape is feasible with any prescribed speed s, $0 \leq s \leq \dfrac{1}{\sqrt{6+2\sqrt{3}}}$. This means that the radial upper and lower bounds for the rotation set R are actually pretty close to each other. Furthermore, we prove the convexity of the set AR of constructible rotation vectors, and that the set of rotation vectors of periodic orbits is dense in AR. We also provide effective lower and upper bounds for the topological entropy of the studied billiard flow.

1. Introduction

The concept of rotation number finds its origin in the study of the average rotation around the circle S^1 per iteration, as classically defined by H. Poincaré in the 1880's [20], when one iterates an orientation-preserving circle homeomorphism $f : S^1 \to S^1$. This is equivalent to studying the average displacement $(1/n)(F^n(x) - x)$ ($x \in \mathbb{R}$) for a lifting $F : \mathbb{R} \to \mathbb{R}$ of f on the universal covering space $\mathbb{R}$ of S^1. The study of fine homotopical properties of geodesic lines on negatively curved, closed surfaces goes back at least to Morse [17]. As far as we know, the first appearance of the concept of homological rotation vectors (associated with flows on manifolds) was the paper of Schwartzman [21], see also Boyland [1] for further references and a good survey of homotopical invariants associated with geodesic flows. In the paper [14] M. Misiurewicz describes the homotopical rotation intervals of "almost continuous" (in an appropriate sense) self maps of the circle. The high-dimensional generalization of the classic concept of rotation numbers and sets, from the circle to tori, is accomplished by Misiurewicz and Ziemian in [16]. The concept of "persistency" of the rotation intervals of surjective circle maps is explored in [15]. Rotation sets of homeomorphisms of the 2-torus are investigated, and the $1D$ concepts and results are generalized in [6]. Further generalization to toral flows was done by Franks and Misiurewicz in [7]. In [11] Kwapisz proves that every convex polygon with rational vertices can be obtained as the rotation set of a homemomorphism of the 2-torus. A description of the rotation sets of subshifts of finite type is given by K. Ziemian in [22]. A systematic study of the relationship between the topological entropy and the rotation set of

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interval maps is presented by Blokh and Misiurewicz in [3]. This relationship is furher generalized (to high-dimensional, fairly general dynamical systems), studied and successfully explored by W. Geller and M. Misiurewicz in [8]. In the series of papers [9] and [10] O. Jenkinson discovers fundamental properties of rotation sets $R \subset \mathbb{R}^d$ associated with a continuous map $f : X \to \mathbb{R}^d$ and a homeomorphism $T : X \to X$ of the compact metric space X. Here the rotation vectors are the averages of f with respect to the possible T-invariant probability measures on X. In the paper [19] Passeggi proves, in some sense, the result reverse to the content of [11]: It is shown there that the rotation set of a topologically generic homeomorphism of $\mathbb{T}^2$ is a rational polygon.

Following an analogous pattern, in [4] we defined the (still commutative) rotation numbers of a $2D$ billiard flow on the billiard table $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with one convex obstacle (scatterer) $\mathbf{O}$ removed. Thus, the billiard table (configuration space) of the model in [4] was $\mathbf{Q} = \mathbb{T}^2 \setminus \mathbf{O}$. Technically speaking, we considered trajectory segments $\{x(t) | 0 \le t \le T\} \subset \mathbf{Q}$ of the billiard flow, lifted them to the universal covering space $\mathbb{R}^2$ of $\mathbb{T}^2$ (not of the configuration space $\mathbf{Q}$), and then systematically studied the rotation vectors as limiting vectors of the average displacement $(1/T)(\tilde{x}(T) - \tilde{x}(0)) \in \mathbb{R}^2$ of the lifted orbit segments $\{\tilde{x}(t) | 0 \le t \le T\}$ as $T \to \infty$. These rotation vectors are still "commutative", for they belong to the vector space $\mathbb{R}^2$.

In this paper we consider the billiard flow on the unit flat torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ with the tubular r_0-neighborhood of two circles

$$S_1 = \{(x_1, x_2, x_3) \in \mathbb{T}^3 : x_2 = x_3 = 0\},$$
$$S_2 = \{(x_1, x_2, x_3) \in \mathbb{T}^3 : x_1 = 0,\ x_3 = 1/2\}$$

serving as scatterers.

Despite all the advantages of the homological (or "commutative") rotation vectors (i. e. that they belong to a real vector space, and this provides us with useful tools to construct actual trajectories with prescribed rotational behaviour), in our current view the "right" lifting of the trajectory segments $\{x(t) | 0 \le t \le T\} \subset \mathbf{Q}$ is to lift these segments to the universal covering space of $\mathbf{Q}$, not of $\mathbb{T}^3$. This, in turn, causes a profound difference in the nature of the arising rotation "numbers", primarily because the fundamental group $\pi_1(\mathbf{Q})$ of the configuration space $\mathbf{Q}$ is the highly complex group

$$\pi_1(\mathbf{Q}) = G = \langle a, b, c, d \,\big|\, ab = ba,\ ac = ca,\ dbd^{-1} = c \rangle,$$

see §2 below. After a bounded modification, trajectory segments $\{x(t) | 0 \le t \le T\} \subset \mathbf{Q}$ give rise to closed loops γ_T in $\mathbf{Q}$, thus defining an element $g_T = [\gamma_T]$ in the fundamental group $\pi_1(\mathbf{Q}) = G$. The limiting behavior of g_T as $T \to \infty$ will be investigated, quite naturally, from two viewpoints:

(1) The direction "e" is to be determined, in which the element g_T escapes to infinity in the hyperbolic group G or, equivalently, in its Cayley graph Γ, see §2 below. All possible directions e form the horizon or the so called ideal boundary $\text{Ends}(G)$ of the group G can be obtained this way, see [5].
(2) The average speed $s = \lim_{T\to\infty}(1/T)\text{dist}(g_T, 1)$ is to be determined, at which the element g_T escapes to infinity, as $T \to \infty$. These limits (or limits $\lim_{T_n\to\infty}(1/T_n)\text{dist}(g_{T_n}, 1)$ for sequences of positive reals $T_n \nearrow \infty$) are nonnegative real numbers.

The natural habitat for the two limit data (s, e) is the infinite cone

$$C = ([0, \infty) \times \text{Ends}(G))/(\{0\} \times \text{Ends}(G))$$

erected upon the set $\text{Ends}(G)$, the latter supplied with the usual Cantor space topology. Since the homotopical "rotation vectors" $(s, e) \in C$ (and the corresponding homotopical rotation sets) are defined in terms of the non-commutative fundamental group $\pi_1(\mathbf{Q}) = G$, these notions will be justifiably called homotopical or noncommutative rotation vectors and sets.

The rotation set arising from trajectories obtained by the arc-length minimizing variational method will be the so called admissible homotopical rotation set $AR \subset C$. The homotopical rotation set R defined without the restriction of admissibility will be denoted by R. Plainly, $AR \subset R$ and these sets are closed subsets of the cone C.

The main results of this paper are Theorems 3.1–3.2, and Theorem 4.1. In theorem 3.1 we find the lower radial estimate $\dfrac{1}{\sqrt{6}+2\sqrt{3}} = 0.169101979\ldots$ for the admissible, compact and convex rotation set AR, weheras Theorem 3.2 yields the upper radial bound $\sqrt{3}$ for the bigger full rotation set R.

Utilizing the above results, Theorem 4.1 provides the inequalities

$$0.185777512\dots = \frac{1}{\sqrt{6}+2\sqrt{3}}\log 3 \le h_{top} \le \sqrt{3}\log 7 = 3.370415245\ldots$$

for the topological entropy of the billiard flow.

2. Prerequisites. Model and Geometry of Orbits

In this paper we are studying the homotopical properties of the trajectories of the following billiard flow $(\mathbf{M}, \{S^t\}, \mu)$: From the standard flat 3-torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ we cut out the open, tubular r_0-neighborhoods ($r_0 > 0$ is small enough) of the following two disjoint one-dimensional subtori

$$T_1 = \{(x_1, x_2, x_3) \in \mathbb{T}^3 : \; x_2 = x_3 = 0\},$$

$$T_2 = \{(x_1, x_2, x_3) \in \mathbb{T}^3 : \; x_1 = 0, \; x_3 = 1/2\}$$

serving as scatterers. In the resulting configuration space $\mathbf{Q} = \mathbf{Q}_{r_0}$ a point is moving uniformly with unit speed, bouncing back at the smooth boundary $\partial\mathbf{Q}$ of $\mathbf{Q}$ according to the law of specular reflections. The natural invariant measure (Liouville measure) μ of the resulting Hamiltonian flow $(\mathbf{M}, \{S^t\}, \mu)$ can be obtained by normalizing the product of the Lebesgue measure of $\mathbf{Q}$ and the hypersurface measure of the unit sphere S^2 of velocities.

A fundamental domain Δ_0 of the configuration space $\mathbf{Q}$ can be obtained by taking

$$\Delta_0 = \left\{x = (x_1, x_2, x_3) \in [0,1]^3 \middle\| \mathrm{dist}(x, T_i) \ge r_0, \quad i = 1,\, 2\right\}$$

by glueing together the opposite faces

$$F_i^0 = \left\{(x_1, x_2, x_3) \in \Delta_0 \middle| x_i = 0\right\}$$

and

$$F_i^1 = \left\{(x_1, x_2, x_3) \in \Delta_0 \middle| x_i = 1\right\}.$$

We prove later (see Theorem 3.1) that the fundamental group $\pi_1(\mathcal{Q})$ is the hyperbolic group finitely presented as follows:

$$\pi_1(\mathcal{Q}) \cong \langle a, b, c, d \mid ab = ba, \; ac = ca, \; dbd^{-1} = c\rangle.$$

We are going to study the asymptotic (in the long time run) homotopical properties of orbit segments $S^{[0,T]}x$ of our billiard flow, where $T \to \infty$. Given any infinite sequence $S^{[0,T_n]}x_n$ of orbit segments with $T_n \to \infty$, by adding a bounded curve to the beginning and ending parts of these orbit segments, we may assume that $q(S^{T_n}x_n) = q(x_n) = q_0 \in \mathbf{Q}$ $(n = 1, 2, \ldots)$ is a fixed base point q_0 for the fundamental group $\pi_1(\mathbf{Q}, q_0)$. Here $q: \; \mathbf{M} \to \mathbf{Q}$ is the natural projection of the unit tangent bundle onto $\mathbf{Q}$. The loops

$$\left\{q(S^t x_n) \middle| 0 \le t \le T_n\right\}$$

naturally give rise to the curves

$$\gamma_n = \left\{\gamma_n(t) \middle| 0 \le t \le T_n\right\} \subset \Gamma$$

in the Cayley graph Γ of $\pi_1(\mathcal{Q})$ with $\gamma_n(0) = 1$ (the root of Γ). We are interested in describing all possible pairs (s, w) of limiting speeds

$$s = \lim_{n\to\infty} T_n^{-1} \cdot \mathrm{dist}(\gamma_n(T_n), e)$$

and directions $e \in \mathrm{Ends}(\Gamma)$ in which the curves γ_n go to infinity in Γ. Here dist denotes the word distance in the group $\pi_1(\mathcal{Q})$ (or, equivalently, the graph distance in Γ), $0 \le s < \infty$, and w is an element of the Cantor set $\mathrm{Ends}(\Gamma)$ of all ends of the hyperbolic group $\pi_1(\mathcal{Q})$, see [2]. So the natural habitat of the (set of) limiting homotopical "rotation vectors" (s, e) is the infinite cone

$$C = [0, \infty) \times \mathrm{Ends}(\Gamma)/\{0\} \times \mathrm{Ends}(\Gamma)$$

erected upon the Cantor set $\mathrm{Ends}(\Gamma)$. For convenience, we identify all homotopical rotation vectors $(0, e)$ with zero speed. The arising set of all achievable homotopical rotation vectors $(s, e) \in C$ will be called the (full) rotation set and denoted by R.

2.1. **Principles for the design of admissible trajectories.** The trajectories are constructed following the principles enumerated below, see also Fig. 2.1.1. Throughout this construction of length minimizing curves (using the variational method) we will be using an analogy from mechanics as follows: The orbit segments under construction are being thought of as made by a spanned, perfectly elastic rubber band that tries to shrink itself as much as possible, subjected to the side conditions that it needs to touch the boundaries of some scatterers in an order prescribed by their symbolic collision itinerary. The somewhat "misterious" concepts of "past force" and in the upcoming text refer to the forces arising in this tightened rubber band that the already constructed piece (the "past") exerts on the current (present) piece under construction. The "future force" is defined analogously.

(1) In our construction of admissible trajectory segments, using the arc length minimizing variational method, we will be assuming that the scatterers have radius $r_0 = 0$. This does not yield an actual toroidal (cylindrical) billiard trajectory. However, we can overcome this problem by continuously swelling the radii to some small positive r_0.
(2) A trajectory enters a cell through one of eight entry faces and exits through a different exit face.
(3) Each passage into or out of a cell C occurs via a collision with a scatterer, bounding the corresponding entry- or exit face of the cell, meaning that the trajectory does not pass through the tubular r_0-neighborhood of the crossed face without bouncing back from a scatterer bounding that face.
(4) During its time within C, the trajectory visits several scatterers.
(5) For any two consecutively visited scatterers, corresponding edges E_1 and E_2 (which are necessarily different), it should be true that the intersection of the convex hull $\mathrm{conv}(E_1, E_2)$ of the edges and the interior of the fundamental cell should be nonempty.
(6) In the construction of an admissible trajectory segment, at a point of collision there is a predetermined force from the past which pulls the point of contact between the trajectory and the scatterer toward one end of the corresponding edge E. The construction should be such that the future force pulls the point of contact to the opposite end of E, thus balancing the point of contact so that it stays in the interior of the edge at the equilibrium.

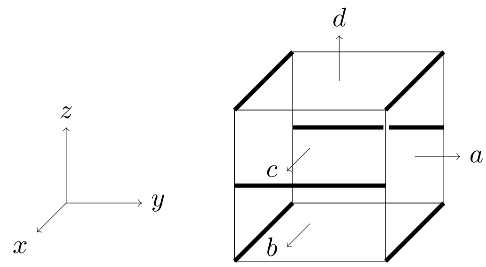

Figure 2.1.1: The fundamental domain, two perpendicular and disjoint toroidal scatterers removed

(7) If the admissible trajectory under construction is to exit the cell via one of the four face crossings b, b^{-1}, c, c^{-1} and the past force is pulling the point of contact P with the exit edge upward in the x_3 direction (i.e the previous edge of contact is on the top of the cell to exit from), then we make sure that the future force also pulls the point P upward. The same is required concerning forces pulling P downward in the x_3 direction. This convention is necessary to obtain proper billiard trajectories and to keep track of face crossings.

These principles, together with the arc-length minimizing variational method, yield a trajectory whose point of contact with a scatterer occurs away from the ends of the scatterer in the fundamental cell, i.e. away from the faces perpendicular to the scatterer, thus the arising shortest curce will indeed be a segment of a billiard trajectory. Furthermore, any word in Γ can serve as the guiding symbolic itinerary of the trajectory segment to be constructed this way. This guarantees that a trajectory of a prescribed homotopy type can be constructed via the variational method for arc length minimizing curves.

3. The admissible rotation set

First we describe the fundamental group of the configuration space.

Theorem 3.1. *The fundamental group $\pi_1(\mathcal{Q})$ is the group finitely presented as follows:*

$$\pi_1(\mathcal{Q}) \cong \langle a, b, c, d \mid ab = ba,\ ac = ca,\ dbd^{-1} = c\rangle.$$

Proof. First, we add back the three extra edges

$$E_1 = \left\{(x_1, x_2, x_3)\big|\ x_1 = x_3 = 0\right\},$$

$$E_2 = \left\{(x_1, x_2, x_3)\big|\ x_1 = x_2 = 0,\ 0 \le x_3 \le \frac{1}{2}\right\},$$

$$E_3 = \left\{(x_1, x_2, x_3)\big|\ x_1 = x_2 = 0,\ \frac{1}{2} \le x_3 \le 1\right\}$$

modulo $\mathbb{Z}^3$. The arising modified configuration space $\mathcal{Q}'$ is homotopically equivalent to the bouquet of 4 loops, corresponding to the face crossings a, b, c, d, so the fundamental group of $\mathcal{Q}'$ is freely generated by a, b, c, d. Removing the added three edges E_1, E_2, E_3 means that a small loop around them is collapsed to the identity 1 of the group, i.e. $aba^{-1}b^{-1} = 1$ (corresponding to the removal of E_2), $aca^{-1}c^{-1} = 1$ (corresponding to the removal of E_3), and $dbd^{-1}c^{-1} = 1$ (corresponding to the removal of E_1). $\square$

As discussed in §2.1, the admissible trajectories are constructed in such a way that each point of contact p_n with a scatterer has a force coming from the preceding point of contact p_{n-1} and the succeeding point of contact p_{n+1} which pulls p_n toward opposite ends of the scatterer. Recall that the two scatterers S_1 and S_2 are perpendicular and disjoint. The trajectories are immediately lifted to the covering space $\mathbb{R}^3$ of $\mathbb{T}^3$.

3.1. **Admissible turns.** The requirements from §2.1 give rise to 17 admissible turns which are unique up to time-reversal and geometric symmetry. These admissible turns, occuring within a fundamental cell, are given in the table below along with a corresponding upper bound for the maximum amount of time each trajectory piece remains within the fundamental cell.

turn	upper bound	turn	upper bound	turn	upper bound
aa	$\sqrt{6}$	ad^{-1}	$\sqrt{6}$	bd^{-1}	$\frac{3}{2}$
ab	$\sqrt{6}+\frac{3\sqrt{3}}{2}$	ba	$\frac{3}{2}$	da	$\sqrt{6}$
ac	$\frac{3}{2}$	bb	$\sqrt{6}$	db	$\frac{3}{2}$
ad	$\sqrt{6}$	bc	$\sqrt{6}+2\sqrt{3}$	dc	$\sqrt{6}+\frac{3\sqrt{3}}{2}$
ab^{-1}	$\sqrt{6}+\frac{\sqrt{3}}{2}$	bd	$\sqrt{6}+\frac{\sqrt{3}}{2}$	dd	$\sqrt{6}$
ac^{-1}	$\sqrt{6}+\frac{\sqrt{3}}{2}$	bc^{-1}	$\sqrt{6}+\sqrt{3}$		

These upper bounds are found using the extreme points of scatterers on the entry and exit faces (most adverse situation concerning the time spent in the investigated cell), and the midpoints of scatterers visited between entry and exit scatterers. It is clear that of these admissible turns, the turn bc requires the longest amount of time within a single fundamental cell: The turn bc enters the fundamental cell via the lower half of the scatterer located at $\{0\}\times[0,1]\times\{\frac{1}{2}\}$. By symmetry, we may assume that the force predetermined by the past pulls the point of contact toward the point $(0,0,\frac{1}{2})$. Admissibility and length minimization require the trajectory to visit these scatterers in the order listed: $\{0\}\times[0,1]\times\{\frac{1}{2}\}$, $[0,1]\times\{1\}\times\{0\}$, $\{1\}\times[0,1]\times\{\frac{1}{2}\}$, $[0,1]\times\{0\}\times\{1\}$, $\{0\}\times[0,1]\times\{\frac{1}{2}\}$, $[0,1]\times\{1\}\times\{1\}$, and finally $\{1\}\times[0,1]\times\{\frac{1}{2}\}$. Observe that the time spent within the fundamental cell is no more than

$$\sqrt{\frac{3}{2}}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\sqrt{\frac{3}{2}}=\sqrt{6}+2\sqrt{3},$$

and the final point of contact with the last scatterer is pulled toward the endpoint $(1,0,\frac{1}{2})$. A similar elementary inspection of the remaining turns admitted by the principles in §2.1 shows that none spend more time in the fundamental cell than does a bc turn.

Note that, in the construction of admissible billiard orbits using the arc-length minimizing variational method, these upper bounds are used to show the existence of billiard orbits whose liftings to the fundamental group $\pi_1(\mathcal{Q})$ escape to infinity as fast as possible. While proving the lower radial bound for AR in Theorem 3.2, we need to slow down the speed of escape by injecting idle runs, i. e. pieces of the trajectory spending a long time in a single elementary cell.

3.2. **Anchoring.** In constructing an admissible orbit segment corresponding to an irreducible finite word $\mathbf{w}=w_1w_2\cdots w_k$, the above construction works only if two additional toroidal scatterers are added to the trajectory segment. The first edge to be added is to the initial piece of the trajectory segment whereas the second is to the final piece of the trajectory segment constructed using the principles enumerated above. These two edges are added in such a way that neither lengthens the irreducible finite word $\mathbf{w}$ and that each balances the force acting on the initial or terminal point of contact coming from the already constructed trajectory segment. Finally, we call the midpoints of the the newly-constructed initial and final edges of the trajectory segment the anchor points, and we call these newly-added initial and final edges anchor edges.

We construct here the initial anchor point using the method just described. The construction of the final (terminal) anchor point is analogous. We construct the initial anchor point for only one possible situation — all other situations can be constructed using the same method. Assume that the constructed trajectory segment exits the fundamental cell through the upper face via the edge $\{1\} \times [0, 1] \times \{1\}$ and that the point coming from the already-constructed forward segment pulls the point of contact $(1, x_2, 1)$ towards the point $(1, 0, 1)$. Then the initial anchor point would be the point $\left(\frac{1}{2}, 1, \frac{1}{2}\right)$.

3.3. **Radial estimates of the rotation set.**

Theorem 3.2. *The admissible rotation set, and thus the full rotations set, contains the ball centered at 0 with radius $1/(\sqrt{6} + 2\sqrt{3})$. That is, we have*

$$B\left(0, \frac{1}{\sqrt{6} + 2\sqrt{3}}\right) \subset AR \subset R.$$

We note that the number $\sqrt{6} + 2\sqrt{3}$ in the denominator of the radius is the weakest upper bound that we obtained in the table at the beginning of Subsection 3.1 for the maximum amount of time an admissible orbit segment can spend in each visited fundamental cell.

Proof. Let $\mathbf{w} = w_1 w_2 w_3 \dots$ be an (irreducible) infinite word corresponding to an end of the hyperbolic fundamental group of the configuration space $\pi_1(\mathcal{Q})$. The results of this section enable the construction of an infinitely long admissible trajectory $S^{[0,\infty)}x_0$ which has the itinerary prescribed by $\mathbf{w}$, yielding

$$\liminf_{n \to \infty} \frac{n}{T_n} \geq \frac{1}{\sqrt{6} + 2\sqrt{3}} = 0.169101979 \dots$$

Here T_n is the time spent by the trajectory $S^{[0,\infty)}x_0$ in the first n fundamental cells. Now, since any trajectory can be slowed down by injecting an appropriate amount of idle collisions, i.e. a lot of consecutive collisions in the same fundamental cell, we see that every homotopical rotation number $(s, e) \in B\left(0, \frac{1}{\sqrt{6}+2\sqrt{3}}\right)$ can be obtained by an admissible trajectory.

□

Theorem 3.3. *The full rotation set, and thus the admissible rotation set, is contained within a ball centered at 0 with radius $\sqrt{3} = 1.732050808\dots$.*

Proof. The proof is similar to the upper radial estimate for the full rotation set for the billiard model discussed in [12]. However, we need to consider more face crossings in our current model.

Let $S^{[0,T]}x_0$ be a trajectory segment with foot point $x_0 = x(0)$ and T large, $S^t(x_0) = x(t) = (q(t), v(t))$, $v(t) = (v_1(t), v_2(t), v_3(t))$. Eventually, we'll take $T \to \infty$ and get asymptotic estimates. Now, denote by n_a, n_b, n_c, n_d the number of face crossings the trajectory segment $S^{[0,T]}x_0$ makes within the fundamental cell, where a face crossing means a crossing of the face of the fundamental cell associated with each generator a, b, c, or d of the fundamental group. Hence, n_a would be the number crossings of the $y - z$ face (irrespective of direction), n_b the number of crossings of the half of the $x - z$ face below the scatterer on that face, n_c the number of crossing of the half of the $x - z$ face above the scatterer on that face, and n_d the number of crossings of the $x - y$ face. Now, because the integral of $|v_i|$ between any two face crossings in the i^{th} direction is at least one, we have the following inequalities:

$$\int_0^T |v_1(t)|dt \geq n_b + n_c - 1,$$

$$\int_0^T |v_2(t)|dt \geq n_a - 1,$$

and

$$\int_0^T |v_3(t)|dt \geq n_d - 1.$$

Adding the inequalities, we have

$$T = \int_0^T |v(t)|dt \geq \frac{1}{\sqrt{3}}\int_0^T (|v_1(t)| + |v_2(t)| + |v_3(t)|)dt \geq \frac{n_a + n_b + n_c + n_d - 3}{\sqrt{3}}.$$

And thus we have

$$\limsup_{T\to\infty} \frac{n_a + n_b + n_c + n_d}{T} \leq \sqrt{3}.$$

□

Theorem 3.4 (Convexity of the Admissible Rotation Set)**.** *The admissible rotation set AR is a convex subset of the cone C.*

Proof. The cone C is a totally disconnected, Cantor set-type family of infinite rays that are glued together at their common endpoint, the vertex of the cone. Therefore, the convexity of AR means that for any $(s, e) \in AR$ and for any t with $0 \leq t \leq s$ we have $(t, e) \in AR$. However, this immediately follows from our construction, since we can always insert a suitable amount of idle runs into an admissible orbit segment to be constructed, hence slowing it down to the asymptotic speed t, as required. □

Theorem 3.5 (Periodic Rotation Vectors are Dense in AR)**.** *All the rotation vectors $(s, e) \in AR$ that correspond to periodic admissible trajectories form a dense subset of AR.*

Proof. The following statement immediately follows from the flexibility of our construction: Given any finite, admissible trajectory segment $S^{[0,T]}x_0$, with the approximative prescribed rotation vector $(s, e) \in AR$, one can always append a bounded initial and terminal segment to $S^{[0,T]}x_0$, so that after this expansion the following properties hold:

(1) The initial and the terminal compartments of $S^{[0,T]}x_0$ differ by the same integer translation vector $\vec{v} \in \mathbb{Z}^3$ by which the initial and terminal anchor edges differ;
(2) The future force acting on the point of contact with the initial anchor is opposite to the past force acting on the point of contact with the terminal anchor edge.

These two properties gurantee that, by releasing the midpoints as the points of contact and just requiring that they differ by the integer vector $\vec{v}$, one constructs a periodic admissible orbit with the approximative rotation vector (s, e). □

4. Topological entropy of the flow

In this section we use the classic result in dynamics which relates the growth rate of volume in the universal covering of our fundamental domain to the topological entropy of the flow. In particular, we will use the result as stated in [12]:

$$h_{\text{top}}(r_0) = \lim_{T\to\infty} \frac{1}{T} \log n_T(x,y),$$

where $h_{\text{top}(r_0)}$ is the topological entropy of the flow of our billiard model and $n_T(x,y)$ is the number of homotopically distinct trajectories joining x and y in our fundamental domain having arc length no more than T. This calculation is independent of the choice of x and y. Note: The statement of the above fact in [12] is made in terms of geodesic flows on closed, connected C^∞ manifolds having non-positive sectional curvature. This result applies to our billiard flow. For further reference, [13]. See [18] for an exposition on the relationship between our billiard model and geodesic flows.

By using the result just stated, we may obtain upper and lower estimates for the topological entropy our billiard flow by estimating $n_T(x,y)$ from above and below, which we do in the proof of the following theorem.

Theorem 4.1. *The topological entropy of our billiard flow is bounded below by* $\frac{1}{\sqrt{6+2\sqrt{3}}}\log 3$ *and above by* $\sqrt{3}\log 7$*:*

$$0.185777512\dots = \frac{1}{\sqrt{6+2\sqrt{3}}}\log 3 \le h_{top} \le \sqrt{3}\log 7 = 3.370415245\dots$$

Proof. According to the first theorem of the previous section the fundamental group $\pi_1(\mathcal{Q})$ is the group finitely presented as follows:

$$\pi_1(\mathcal{Q}) \cong \langle a,b,c,d \mid ab=ba,\ ac=ca,\ dbd^{-1}=c\rangle.$$

Since this group contains the subgroup $F_2(a,d)$ freely generated by a and d and the Cayley graph of $F_2(a,d)$ is a 4-regular tree branching into 3 directions at each vertex, we get that the exponential growth rate

$$\lambda = \lim_{\rho\to\infty} \frac{1}{\rho} \log \mathrm{Vol}(B(\rho))$$

of the number of vertices of the Cayley graph Γ of the group $\pi_1(\mathcal{Q})$ is at least $\log 3$. Furthermore, since the group $\pi_1(\mathcal{Q})$ is generated by 4 elements, the growth rate of the number of words in Γ of length n is at most $\mathrm{const}\cdot 7^n$, the growth rate λ cannot exceed $\log 7$.

According to Theorem 3.3, the linear growth rate $\rho(T)/T$ of the radius ρ is at most $\sqrt{3}$. On the other hand, by Theorem 3.2, in time T all paths in Γ with lengths not exceeding $\frac{1}{\sqrt{6+2\sqrt{3}}}$ are realizable as billiard trajectories of length $\le T$. Combining the above results we obtain the desired bounds

$$\frac{1}{\sqrt{6+2\sqrt{3}}}\log 3 \le h_{\text{top}} \le \sqrt{3}\log 7.$$

□

References

[1] P. Boyland, *New dynamical invariants on hyperbolic manifolds*, Israel J. Math. **119**, 253–289 (2000).
[2] M. R. Bridson, and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Matematischen Wissenschaften **319**, Springer Verlag, Berlin-Heidelberg 1999.
[3] A. Blokh, M. Misiurewicz, *Entropy and over-rotation numbers for interval maps*, Proc. Steklov Inst. Math. **(1) 216**, 229–235 (1997).
[4] A. Blokh, M. Misiurewicz, and N. Simanyi, *Rotation sets of billiards with one obstacle*, Commun. Math. Phys. **266**, 239–265 (2006).
[5] M. Coornaert, and A. Papadopoulos, *Symbolic dynamics and hyperbolic groups*, Springer-Verlag, New York, 1993.
[6] J. Franks, *Realizing rotation vectors for torus homeomorphisms*, Trans. Amer. Math. Soc. **311, Number 1**, January 1989, pp. 107–115.
[7] J. Franks, and M. Misiurewicz, *Rotation sets of toral flows*, Proc. Amer. Math. Soc. **109, Number 1**, May 1990, pp. 243–249.
[8] W. Geller, M. Misiurewicz, *Rotation and entropy*, Trans. Amer. Math. Soc. **(7) 351**, 2927–2948 (1999).
[9] O. Jenkinson, *Directional entropy of rotation sets*, C. R. Acad. Sci. Paris Sér I Math. **(10) 332**, 921–926 (2001).
[10] O. Jenkinson, *Rotation, entropy, and equilibrium states*, Trans. Amer. Math. Soc. **(9) 353**, 3713–3739 (2001).
[11] J. Kwapisz, *Every convex polygon with rational vertices is a rotation set*, Ergodic Theory Dynam. Systems **(2) 12**, 333–339 (1992).
[12] R. Mañé, *On the topological entropy of geodesic flows*, J. Differential Geom., **45, Number 1**, 74–93 (1997).
[13] A. Manning, *Topological entropy for geodesic flows*, Annals of Mathematics, **110, Number 3**, 567–573 (1979).
[14] M. Misiurewicz, *Rotation intervals for a class of maps of the real line into itself*, Ergodic Theory Dynam. Systems, **(1) 6**, 117–132 (1986).
[15] M. Misiurewicz, *Persistent rotation intervals of old maps*, Banach Center Publ., **23**, Panstwowe Wydawnictwo Naukowe., Warsaw (1989).
[16] M. Misiurewicz, and K. Ziemian, *Rotation sets for maps of tori*, J. London Math. Soc. **(2) 40**, 490–506 (1989).
[17] M. Morse, *A fundamental class of geodesics on any closed surface of genius greater than one*, Trans. Amer. Math. Soc. **26**, 25–60 (1924).
[18] C. Moxley, and N. Simanyi, *Homotopical complexity of a 3D billiard flow.* To appear in the Proceedings of the Conference on Dynamical Systems, Ergodic Theory, and Probability. Edited by: A. Blokh, I. Bunimovich, P. Jung, L. Oversteegen, and Ya. Sinai. Contemporary Mathematics, (2017).
[19] A. Passeggi, *Rational polygons as rotation sets of generic homeomorphisms of the two torus*, J. London Math. Soc. **(2) 89**, no. 1, 235–254 (2014).
[20] H. Poincaré, *Oeuvres completes*, **tome 1**, Gauthier–Villars, Paris, 137–158 (1952).
[21] S. Schwartzman, *Asymptotic cycles*, Annals of Math. **66**, 270–284 (1957).
[22] K. Ziemian, *Rotation sets of subshifts of finite type*, Fund. Math. **(2) 146**, 189–201 (1995).

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