PROOF OF WOJTKOWSKI'S FALLING PARTICLE CONJECTURE

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ABSTRACT. In this paper we present an unconditional proof of Wojtkowski's Ergodicity Conjecture for almost every system of $1D$ perfectly elastic balls falling down in a half line under constant gravitational acceleration, [W1985], [W1986], [W1990a], [W1990b], [W1998]. Namely, by introducing a new algebraic approach, we prove that almost every such system is (completely hyperbolic and) ergodic.

1. INTRODUCTION/PREREQUISITES

In order to introduce the subject of our investigation, the system of $1D$ falling balls subjected to constant gravitation, along with the employed technicalities, we will be closely following the first two sections of [S2024]. In order to make this presentation self-contained and easier to read, we quote below two passages of those two sections of [S2024], essentially verbatim.

In his paper [W1990a] M. Wojtkowski introduced the following Hamiltonian dynamical system with discontinuities: There is a vertical half line $\{q|\, q \geq 0\}$ given and n (≥ 2) point particles with masses $m_1 \geq m_2 \geq \cdots \geq m_n > 0$ and positions $0 \leq q_1 \leq q_2 \leq \cdots \leq q_n$ are moving on this half line so that they are subjected to a constant gravitational acceleration $a = -1$ (they fall down), they collide elastically with each other, and the first (lowest) particle also collides elastically with the hard floor $q = 0$. We fix the total energy

$$H = \sum_{i=1}^{n} \left(m_i q_i + \frac{1}{2} m_i v_i^2 \right)$$

by taking $H = 1$. The arising Hamiltonian flow with collisions $(\mathbf{M}, \{\psi^t\}, \mu)$ (μ is the Liouville measure) is the studied model of this paper.

Before formulating the result of this article, however, it is worth mentioning here three important facts:

1. Since the phase space $\mathbf{M}$ is compact, the Liouville measure μ is finite.
2. The phase points $x \in \mathbf{M}$ for which the trajectory $\{\psi^t(x)|, t \in \mathbb{R}\}$ hits at least one singularity (i. e. a multiple collision) are contained in a countable union of proper, smooth submanifolds of $\mathbf{M}$ and, therefore, such points form a set of μ measure zero.
3. For μ-almost every phase point $x \in \mathbf{M}$ the collision times of the trajectory $\{\psi^t(x)|, t \in \mathbb{R}\}$ do not have any finite accumulation point, see Proposition A.1 of [S1996].

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In the paper [W1990a] Wojtkowski formulated his main conjecture pertaining to the dynamical system $(\mathbf{M}, \{\psi^t\}, \mu)$:

Conjecture 1.1 (Wojtkowski's Conjecture)**.** If $m_1 \geq m_2 \geq \cdots \geq m_n > 0$ and $m_1 \neq m_n$, then all but one characteristic (Lyapunov) exponents of the flow $(\mathbf{M}, \{\psi^t\}, \mu)$ are nonzero. Futhermore, the system is ergodic.

Remark 1.1. 1. The only exceptional exponent zero must correspond to the flow direction.

2. The condition of nonincreasing masses (as above) is essential for establishing the invariance of the symplectic cone field — an important condition for obtaining nonzero characteristic exponents. As Wojtkowski pointed out in Proposition 4 of [W1990a], if $n = 2$ and $m_1 < m_2$, then there exists a linearly stable periodic orbit, thus dimming the chances of proving ergodicity.

In the paper [S2024] we proved Wojtkowski's Ergodicity Conjecture 1.1 for almost every selection of masses $m_1 > m_2 > \cdots > m_n$, provided that the Transversality Conditions (Claim 3.1 of [S2024]) is verified, i. e. singularities of different order are transversal to each other and, analogously, the stable and unstable local invariant manifolds are transversal to all singularities.

Here our main result is to prove the above Transversality Conditions and, as the main corollary, we obtain our

Theorem 1.2 (Main Theorem)**.** *For almost every selection of masses $m_1 > m_2 > \cdots > m_n$ the falling ball flow $(M, \{\psi^t\}, \mu)$ is (completely hyperbolic and) ergodic.*

We recall that the corresponding billiard map (Poincaré section) $(\partial M, T, \nu)$ is an invertible dynamical system T mapping the boundary

$$\partial M = \left\{ (q, v) \in M \big| \; q \in \partial Q \right\}$$

of the phase space M onto itself and preserving the finite measure ν on ∂M that can be obtained by projecting along the flow the invariant measure μ of the flow onto ∂M. Also, as usual, in ∂M one identifies the pre-collision phase point $(q, v^-) \in \partial M$ with the post-collision phase point $(q, v^+) \in \partial M$.

Finally, denote by

$$\Pi : \partial M \to \partial Q$$

the natural projection of ∂M onto ∂Q, i. e. $\Pi(q, v) = q$.

2. The Geometry of the Symplectic Flow

We will be working with the symplectic coordinates $(\delta h, \delta v)$ for the tangent vectors of the reduced phase space $\mathbf{M}$ satisfying the usual reduction equations $\sum_{i=1}^n \delta h_i = 0 = \sum_{i=1}^n \delta v_i$.

Remark 2.1. The coordinates δh_i and δv_i $(i = 1, 2, \ldots, n)$ serve as suitable symplectic coordinates in the codimension-one subspace $\mathcal{T}_x$ of the full tangent space $\mathcal{T}_x\mathbf{M}$ of $\mathbf{M}$ at x. Recall that the $(2n-2)$-dimensional vector space $\mathcal{T}_x$ is transversal to the flow direction, and the restriction of the canonical symplectic form

$$\omega = \sum_{i=1}^{n} \delta q_i \wedge \delta p_i = \sum_{i=1}^{n} \delta h_i \wedge \delta v_i$$

of $\mathbf{M}$ is non-degenerate on $\mathcal{T}_x$, see [W1990a]. We also recall that

$$\delta h_i = m_i \delta q_i + m_i v_i \delta v_i = m_i \delta q_i + v_i \delta p_i.$$

Corresponding to the above choice of symplectic coordinates, the considered monotone Q-form will be

$$Q_1(\delta h, \delta v) = \sum_{i=1}^{n} \delta h_i \delta v_i. \tag{2.1}$$

It is clear that the evolution of $DS^t(\delta h(0), \delta v(0)) = (\delta h(t), \delta v(t))$ between collisions is

$$\frac{d}{dt}\left(\delta h(t), \delta v(t)\right) = (0, 0). \tag{2.2}$$

If a collision of type $(i, i+1)$ $(i = 1, 2, \ldots, n-1)$ takes place at time t_k, then the derivative of the flow at the collision $\delta h^-(t_k) \mapsto \delta h^+(t_k)$, $\delta v^-(t_k) \mapsto \delta v^+(t_k)$ is given by the matrices

$$\begin{aligned} \delta h^+(t_k) &= R_i^* \left[\delta h^-(t_k) + S_i \delta v^-(t_k)\right] \\ \delta v^+(t_k) &= R_i \delta v^-(t_k), \end{aligned} \tag{2.3}$$

where the matrix R_i is the $n \times n$ identity matrix, except that its 2×2 submatrix at the crossings of the i-th and $(i+1)$-st rows and columns is

$$R_i^{(i,i+1)} = \begin{bmatrix} \gamma_i & 1-\gamma_i \\ 1+\gamma_i & -\gamma_i \end{bmatrix}$$

with $\gamma_i = \dfrac{m_i - m_{i+1}}{m_i + m_{i+1}}$. The matrix S_i is, similarly, the $n \times n$ zero matrix, except its 2×2 submatrix at the crossings of the i-th and $(i+1)$-st rows and columns, which takes the form of

$$S_i^{(i,i+1)} = \begin{bmatrix} \alpha_i & -\alpha_i \\ -\alpha_i & \alpha_i \end{bmatrix}$$

with

$$\alpha_i = \frac{2 m_i m_{i+1}(m_i - m_{i+1})}{(m_i + m_{i+1})^2}(v_i^- - v_{i+1}^-) > 0. \tag{2.4}$$

These formulas can be found, for example, in Sention 4 of [W1990a].

Concerning a floor collision $(0, 1)$ at time t_k, the transformations are

$$\begin{aligned} \delta h_1^+(t_k) &= \delta h_1^-(t_k) \\ \delta v_1^+(t_k) &= \delta v_1^-(t_k) + \frac{2\delta h_1^-(t_k)}{m_1 v_1^+(t_k)}, \end{aligned} \tag{2.5}$$

see, for instance, Section 4 of [W1990a] or [W1998].

3. Proof of the Transversality Conditions

If one closely studies the ergodicity proofs based upon the Birkhoff-Sinai Zig-zag Method, like the one in [L-W1995], one realizes that, whenever a property is needed to be proved for singular phase points $x_0 = (q_0, v_0) \in \mathcal{S}_0$ (like the transversality of $\mathcal{S}_k$ to $\mathcal{S}_0$ at x_0, or the transversality of the local stable manifold $\gamma^s(x_0)$ to $\mathcal{S}_0$), it is always enough to estabslish the required property for almost every point x_0 of $\mathcal{S}_0$ with respect to the hypersurface measure of $\mathcal{S}_0$. This is what we do in this section.

We will be focusing on the billiard map $(\partial M, T, \nu)$. Our first result asserts that the system $(\partial M, T, \nu)$ has no focal points almost surely almost everywhere.

Proposition 3.1. For almost every selection of the masses $m_1 > m_2 > \cdots > m_n$ it is true that for almost every phase point $(q_0, v_0) \in \partial M$, for every positive integer k, and for every small enough $\epsilon > 0$ the map

$$\Pi \circ T^k : \; C_\epsilon(x_0) \to \partial Q \tag{3.2}$$

is of full rank (i. e. locally onto) at x_0, where

$$C_\epsilon(x_0) = \left\{ (q_0, v) \in \partial M \big| \; \|v - v_0\| < \epsilon \right\} \tag{3.3}$$

is the so called "candle manifold", and

$$\Pi : \; \partial M \to \partial Q \tag{3.4}$$

is the natural projection, taking $\Pi(q, v) = q$ for $(q, v) \in \partial M$.

Proof. It is enough to prove the proposition for a given k and a given symbolic collision sequence $\Sigma = (\sigma_0, \sigma_1, \dots, \sigma_k)$, where $\sigma_l(i_l, i_l + 1)$ tells that the collision $(i_l, i_l + 1)$ takes place at $T^l(x_0)$, where $i_l = 0$ indicates a floor collision.

According to (9) of [W1990a], the time evolution (abrupt change) at a ball-to-ball collision $(i, i + 1)$ is given by

$$\begin{aligned} v_i^+ &= \gamma_i v_i^- + (1 - \gamma_i) v_{i+1}^- \\ v_{i+1}^+ &= (1 + \gamma_i) v_i^- - \gamma_i v_{i+1}^-, \end{aligned} \tag{3.5}$$

where

$$\gamma_i = \frac{m_i - m_{i+1}}{m_i + m_{i+1}},$$

and

$$v_1^+ = -v_1^- \tag{3.6}$$

for any floor collision. It is obvious that the time evolution between collisions is given by

$$\frac{d}{dt} q_t = v_t, \; \frac{d}{dt} v_t = (-1, -1, \dots, -1). \tag{3.7}$$

This means that the entire trajectory segment $\{T^l(x_0) \big| \, l = 0, 1, \dots, k\}$ is governed by rational functions of the initial data $(q_0, v_0, \vec{m})$, including the moments t_l of the collisions σ_l.

Furthermore, by taking derivative of the flow 3.5–3.7, one obtains that for any tangent vector

$$\tau_0 = (\delta q_0, \delta v_0) \in \mathcal{T}_{x_0}\partial M$$

the images $D\psi^t[\tau_0] = \tau_t$ are evolving in time as follows:

$$\begin{aligned} \delta v_i^+ &= \gamma_i \delta v_i^- + (1-\gamma_i)\delta v_{i+1}^- \\ \delta v_{i+1}^+ &= (1+\gamma_i)\delta v_i^- - \gamma_i \delta v_{i+1}^-, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \delta q_i^+ &= \gamma_i \delta q_i^- + (1-\gamma_i)\delta q_{i+1}^- \\ \delta q_{i+1}^+ &= (1+\gamma_i)\delta q_i^- - \gamma_i \delta q_{i+1}^-, \end{aligned} \tag{3.9}$$

for $i > 0$, where $\delta v_j^\pm = (\delta v^\pm)_{t_j}$, $\delta q_j^\pm = (\delta q^\pm)_{t_j}$, for $j = i, i+1$, see (5) in [W1990b].

At a floor collision $(0, 1)$ we clearly have

$$\delta q_1^+ = -\delta q_1^-, \ \delta v_1^+ = -\delta v_1^-. \tag{3.10}$$

It is also clear that the time evolution of the image tangent vector $\tau_t = D\psi^t[\tau_0] = (\delta q_t, \delta v_t)$ between collisions is given by

$$\frac{d}{dt}\delta q_t = \delta v_t, \ \frac{d}{dt}\delta v_t = 0. \tag{3.11}$$

It follows from 3.8–3.11 that the time evolution of the tangent vectors $\tau_t = D\psi^t[\tau_0] = (\delta q_t, \delta v_t)$ is also fully governed by finitely many rational functions of the initial data $(q_0, v_0, \vec{m})$.

Finally, the negation of the assertion of the Proposition for a particular phase point $x_0 = (q_0, v_0)$ means that the system of homogeneous linear equations

$$\delta q_{t_k} = 0 \tag{3.12}$$

has a nontrivial solution $\tau_0 = (0, \delta v_0)$. This, in turn, means that cetain minors of this system vanish, i.e.

$$R_j(q_0, v_0, \vec{m}) = 0, \ j = 1, 2, \dots, m \tag{3.13}$$

for certain rational functions $R_1, R_2, \dots, R_m$ of the initial variables.

Consider now the limiting system with

$$m_1 = m_2 = \cdots = m_n > 0. \tag{3.14}$$

This system may not possess a trajectory segment with our prescribed symbolic collision sequence $\Sigma = (\sigma_0, \sigma_1, \dots, \sigma_k)$, yet all the rational functions R_j above are well defined, and they correspond to the time evolution of this limiting system, so that the unwanted collisions are annihilated in such a way that any two particles, about to making an unwanted collision, are let to penetrate through each other without interaction. This limiting system is essentially integrable, after the dynamic change of labels at collisions, as if the particles just penetrate through each other without interaction, see also the paragraph right after Corollary 2.23 in [S2024].

In this limiting case, however, we have that $\delta q_t = t\delta v_0$, after the mentioned dynamic change of labels. This means, in turn, that the system of homogeneous linear equations 3.12 only has the trivial solution $\tau_0 = 0$, therefore, not all rational functions R_j are identically zero, even if we assume $m_1 = m_2 = \cdots = m_n$.

This completes the proof of Proposition 3.1.

□

Our next result is analogous to the statement of Proposition 3.1, claimed here for almost every singular phase point $x_0 = (q_0, v_0) \in \mathcal{S}_0$.

Proposition 3.15. For almost every selection of the masses $m_1 > m_2 > \cdots > m_n$ and for every positive integer k it is true that for almost every singular phase point $x_0 = (q_0, v_0) \in \mathcal{S}_0$ and for all small enough $\epsilon > 0$ the map in 3.2 is of maximum rank (i. e. locally onto) at x_0.

Proof. In Proposition 3.1, q_0 possessed $n-1$ independent coordinates. (Keep in mind, that $q_0 \in \partial Q$, so the q coordinates of two colliding particles are equal, or $q_1 = 0$.) When a singular collision takes place at time zero, i. e. $x_0 = (q_0, v_0) \in \mathcal{S}_0$, then one more equation holds true for the configuration coordinates, so one more coordinate should be eliminated to work with independent configuration coordinates. The arising vector with $n-2$ coordinates is denoted by $\tilde{q}_0$. After this, the proof of this proposition is verbatim the same as that of Proposition 3.1, except that q_0 needs to be replaced everywhere by $\tilde{q}_0$.

□

Corollary 3.16 ((Transversality of singularities of different order))**.** For almost every selection of the masses $m_1 > m_2 > \cdots > m_n$ and for every positive integer k it is true that for almost every singular phase point $x_0 = (q_0, v_0) \in \mathcal{S}_0$ if $x_k = T^k(x_0)$ happens to belong to $\mathcal{S}_0$, then $T^k(\mathcal{S}_0)$ and $\mathcal{S}_0$ are transversal at x_k.

Proof. The statement immediately follows from the facts that

(i) $C_\epsilon(x_0) \subset \mathcal{S}_0$,
(ii) $\Pi \circ T^k : C_\epsilon(x_0) \to \partial Q$ is locally onto,
(iii) $\mathcal{S}_0$ is defined purely in terms of the q coordinates.

□

Corollary 3.17. For almost every selection of the masses $m_1 > m_2 > \cdots > m_n$ and for almost every singular phase point $x_0 = (q_0, v_0) \in \mathcal{S}_0$ the local stable manifold $\gamma^s(x_0)$ of x_0 is transversal to the singularity $\mathcal{S}_0$.

Proof. This statement follows from the fact that $\gamma^s(x_0)$ is the C^1-uniform limit (locally, near x_0) in the following way:

$$\gamma^s(x_0) = \lim_{k\to\infty} T^{-k}\left[C_\epsilon(T^k(x_0))\right],$$

the inverse images $T^{-k}\left[C_\epsilon(T^k(x_0))\right]$ project locally onto ∂Q at x_0, and, finally, the maximum rank property is a C^1 open property. Hence the projection

$$\Pi : \gamma^s(x_0) \to \partial Q$$

is locally onto, and $\mathcal{S}_0$ is defined purely in terms of the q coordinates in ∂Q.
□

This completes the proof of our Main Theorem.

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