

Conditional Proof of the Ergodic Conjecture for Falling Ball Systems

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ABSTRACT. In this paper we present a conditional proof of Wojtkowski's Ergodicity Conjecture for the system of $1D$ perfectly elastic balls falling down in a half line under constant gravitational acceleration, [W85], [W86], [W90a], [W90b], [W98]. Namely, we prove that almost every such system is (completely hyperbolic and) ergodic, by assuming the transversality between different singularities and between singularities and stable (unstable) invariant manifolds. This transversality condition will be removed in my upcoming paper [S2023].

1. Introduction/Prerequisites

In his paper [W90a] M. Wojtkowski introduced the following Hamiltonian dynamical system with discontinuities: There is a vertical half line $\{q | q \geq 0\}$ given and n (≥ 2) point particles with masses $m_1 \geq m_2 \geq \dots \geq m_n > 0$ and positions $0 \leq q_1 \leq q_2 \leq \dots \leq q_n$ are moving on this half line so that they are subjected to a constant gravitational acceleration $a = -1$ (they fall down), they collide elastically with each other, and the first (lowest) particle also collides elastically with the hard floor $q = 0$. We fix the total energy

$$H = \sum_{i=1}^n \left(m_i q_i + \frac{1}{2m_i} p_i^2 \right)$$

by taking $H = 1$. The arising Hamiltonian flow with collisions $(\mathbf{M}, \{\psi^t\}, \mu)$ (μ is the Liouville measure) is the studied model of this paper.

Before formulating the result of this article, however, it is worth mentioning here three important facts:

- (1) Since the phase space \mathbf{M} is compact, the Liouville measure μ is finite.
- (2) The phase points $x \in \mathbf{M}$ for which the trajectory $\{\psi^t(x) | t \in \mathbb{R}\}$ hits at least one singularity (i. e. a multiple collision) are contained in a countable union of proper, smooth submanifolds of \mathbf{M} and, therefore, such points form a set of μ measure zero.

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- (3) For μ -almost every phase point $x \in \mathbf{M}$ the collision times of the trajectory $\{\psi^t(x) | t \in \mathbb{R}\}$ do not have any finite accumulation point, see Proposition A.1 of [S96].

In the paper [W90a] Wojtkowski formulated his main conjecture pertaining to the dynamical system $(\mathbf{M}, \{\psi^t\}, \mu)$:

CONJECTURE 1.1 (Wojtkowski's Conjecture). If $m_1 \geq m_2 \geq \dots \geq m_n > 0$ and $m_1 \neq m_n$, then all but one characteristic (Lyapunov) exponents of the flow $(\mathbf{M}, \{\psi^t\}, \mu)$ are nonzero. Furthermore, the system is ergodic.

REMARK 1.1. 1. The only exceptional exponent zero must correspond to the flow direction.

2. The condition of nonincreasing masses (as above) is essential for establishing the invariance of the symplectic cone field — an important condition for obtaining nonzero characteristic exponents. As Wojtkowski pointed out in Proposition 4 of [W90a], if $n = 2$ and $m_1 < m_2$, then there exists a linearly stable periodic orbit, thus dimming the chances of proving ergodicity.

Our paper will very closely follow all the notations and definitions of the article [W90a], so the reader is warmly recommended to be familiar with that paper.

2. Eventually Strict Cone Invariance

We will be working with the symplectic coordinates $(\delta h, \delta v)$ for the tangent vectors of the reduced phase space \mathbf{M} satisfying the usual reduction equations $\sum_{i=1}^n \delta h_i = 0 = \sum_{i=1}^n \delta v_i$.

REMARK 2.1. The coordinates δh_i and δv_i ($i = 1, 2, \dots, n$) serve as suitable symplectic coordinates in the codimension-one subspace \mathcal{T}_x of the full tangent space $\mathcal{T}_x \mathbf{M}$ of \mathbf{M} at x . Recall that the $(2n - 2)$ -dimensional vector space \mathcal{T}_x is transversal to the flow direction, and the restriction of the canonical symplectic form

$$\omega = \sum_{i=1}^n \delta q_i \wedge \delta p_i = \sum_{i=1}^n \delta h_i \wedge \delta v_i$$

of \mathbf{M} is non-degenerate on \mathcal{T}_x , see [W90a]. We also recall that

$$\delta h_i = m_i \delta q_i + m_i v_i \delta v_i = m_i \delta q_i + v_i \delta p_i.$$

Corresponding to the above choice of symplectic coordinates, the considered monotone Q-form will be

$$(2.1) \quad Q_1(\delta h, \delta v) = \sum_{i=1}^n \delta h_i \delta v_i.$$

It is clear that the evolution of $DS^t(\delta h(0), \delta v(0)) = (\delta h(t), \delta v(t))$ between collisions is

$$(2.2) \quad \frac{d}{dt} (\delta h(t), \delta v(t)) = (0, 0).$$

If a collision of type $(i, i + 1)$ ($i = 1, 2, \dots, n - 1$) takes place at time t_k , then the derivative of the flow at the collision $\delta h^-(t_k) \mapsto \delta h^+(t_k)$, $\delta v^-(t_k) \mapsto \delta v^+(t_k)$ is given by the matrices

$$(2.3) \quad \begin{aligned} \delta h^+(t_k) &= R_i^* [\delta h^-(t_k) + S_i \delta v^-(t_k)] \\ \delta v^+(t_k) &= R_i \delta v^-(t_k), \end{aligned}$$

where the matrix R_i is the $n \times n$ identity matrix, except that its 2×2 submatrix at the crossings of the i -th and $(i + 1)$ -st rows and columns is

$$R_i^{(i,i+1)} = \begin{bmatrix} \gamma_i & 1 - \gamma_i \\ 1 + \gamma_i & -\gamma_i \end{bmatrix}$$

with $\gamma_i = \frac{m_i - m_{i+1}}{m_i + m_{i+1}}$. The matrix S_i is, similarly, the $n \times n$ zero matrix, except its 2×2 submatrix at the crossings of the i -th and $(i + 1)$ -st rows and columns, which takes the form of

$$S_i^{(i,i+1)} = \begin{bmatrix} \alpha_i & -\alpha_i \\ -\alpha_i & \alpha_i \end{bmatrix}$$

with

$$(2.4) \quad \alpha_i = \frac{2m_i m_{i+1} (m_i - m_{i+1})}{(m_i + m_{i+1})^2} (v_i^- - v_{i+1}^-) > 0.$$

These formulas can be found, for example, in Section 4 of **[W90a]**. Concerning a floor collision $(0, 1)$ at time t_k , the transformations are

$$(2.5) \quad \begin{aligned} \delta h_1^+(t_k) &= \delta h_1^-(t_k) \\ \delta v_1^+(t_k) &= \delta v_1^-(t_k) + \frac{2\delta h_1^-(t_k)}{m_1 v_1^+(t_k)}, \end{aligned}$$

see, for instance, Section 4 of **[W90a]** or **[W98]**.

In this section we will be studying *non-singular* trajectory segments

$$S^{[0,T]}x_0 = \{x_t = S^t x_0 \mid 0 \leq T \leq T\}$$

of the flow $\{S^t\}$ with the symbolic collision sequence $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$, where $\sigma_k = (i_k, i_{k+1})$, $0 \leq i_k \leq n-1$, $k = 1, 2, \dots, N$. The collision graph $\mathcal{G} = \mathcal{G}(\Sigma)$ of Σ has the set $\{0, 1, \dots, n\}$ as its vertex set, and the unoriented edges of \mathcal{G} are the unordered pairs $\{i_k, i_{k+1}\}$, $k = 1, 2, \dots, N$, counted without multiplicity.

According to Wojtkowski's arguments between the Theorem and Proposition I of Section 5 of **[W90a]**, in order to prove the strict invariance of the cone field $C_1 = \{Q_1 \geq 0\}$ along the considered trajectory segment

$$S^{[0,T]}x_0 = \{x_t = S^t x_0 \mid 0 \leq T \leq T\},$$

it is enough to prove that

- (A) for every non-zero vector $(0, \delta v) \in \mathcal{T}_{x_0}$ it is true that $Q_1(DS^T(0, \delta v)) > 0$, and
- (B) for every non-zero vector $(\delta h, 0) \in \mathcal{T}_{x_0}$ it is true that $Q_1(DS^T(\delta h, 0)) > 0$.

Moreover, Wojtkowski's mentioned arguments in Section 5 of [W90a] actually prove property (A) above in the case when the collision graph $\mathcal{G}(\Sigma)$ restricted to the vertex set $\{1, 2, \dots, n\}$ is connected, i. e. all collisions $(i, i+1)$ with $i > 0$ occur. Here we briefly rephrase his ideas:

Formula 2.5 shows that a tangent vector of the form $(0, \delta v)$ is unchanged at any floor collision. Suppose now that a collision $(i, i+1)$ ($1 \leq i \leq n$) occurs at time t_k and $\delta h^-(t_k) = 0$. Equation 2.3 shows that, after pushing the tangent vector $(0, \delta v^-(t_k))$ through the collision $(i, i+1)$, either

$$(2.6) \quad Q_1(\delta h^+(t_k), \delta v^+(t_k)) = \alpha_i(\delta v_i^-(t_k) - \delta v_{i+1}^-(t_k))^2 > 0,$$

or

$$(2.7) \quad \delta v_i^-(t_k) = \delta v_{i+1}^-(t_k) = \delta v_i^+(t_k) = \delta v_{i+1}^+(t_k).$$

Thus, as long as $Q_1(DS^T(0, \delta v(0))) = 0$ and the restriction of the collision graph \mathcal{G} on the vertex set $\{1, 2, \dots, n\}$ is connected, we have that

$$(2.8) \quad \delta v_i(t) = \delta v_i(0)$$

for all t , $0 \leq t \leq T$ and all $i = 1, 2, \dots, n$. Therefore, $\delta v_i(t) = 0$ for all t , $0 \leq t \leq T$ and all $i = 1, 2, \dots, n$, thanks to the usual reduction equation $\sum_{i=1}^n \delta v_i = 0$. This finishes the proof of Property (A) in the case when the collision graph $\mathcal{G}(\Sigma)$ contains all edges $(i, i+1)$ with $1 \leq i \leq n-1$.

REMARK 2.2. From now on, we will always assume that the collision graph \mathcal{G} contains all collisions $(i, i+1)$ with $i > 0$.

Checking Property (B) for the non-singular trajectory segment $S^{[0, T]}x_0$ is, however, a bit harder and it requires a bit more on the side of the collision graph $\mathcal{G}(\Sigma)$ than simply containing all edges $(i, i+1)$, $i = 1, 2, \dots, n-1$.

First of all, we define the linear space of all *neutral vectors* as follows:

$$(2.9) \quad \mathcal{N}_{x_0}^T = \{(\delta h(0), \delta v(0)) \in \mathcal{T}_{x_0} \mid Q_1(\delta h(t), \delta v(t)) = 0, \text{ for } 0 \leq t \leq T\}.$$

Recall that $(\delta h(t), \delta v(t)) = DS^t((\delta h(0), \delta v(0)))$.

It flows from the time-evolution equations 2.2, 2.3, and 2.5 that

$$(2.10) \quad DS^t((\delta h(0), \delta v(0))) = (\delta h(t), 0)$$

for $0 \leq t \leq T$, i.e. $\delta v(t) = 0$. Indeed, at any floor collision the form Q_1 increases by the amount of

$$(2.11) \quad \frac{2(\delta h_1^-(t_k))^2}{m_1 v_1^+(t_k)},$$

which forces $\delta h_1^-(t_k) = \delta h_1^+(t_k) = 0$ and, as a corollary, $\delta v_1^-(t_k) = \delta v_1^+(t_k)$. According to 2.3, at an $(i, i+1)$ collision ($i \geq 1$), occurring at time t_k , the form Q_1 increases by the amount

$$(2.12) \quad \alpha_i (\delta v_i^-(t_k) - \delta v_{i+1}^-(t_k))^2,$$

and this forces $\delta v_i^-(t_k) = \delta v_{i+1}^-(t_k) = \delta v_i^+(t_k) = \delta v_{i+1}^+(t_k)$, thus $\delta v(t) = 0$ for $0 \leq t \leq T$, as claimed.

So, the neutral vector takes the form $DS^t(\delta h(0), 0) = (\delta h(t), 0)$ for $0 \leq t \leq T$, and $\delta h(t)$ only changes at ball-to-ball collisions $(i, i+1)$ (at time t_k) according to the law

$$(2.13) \quad \delta h^+(t_k) = R_i^* \delta h^-(t_k),$$

whereas at a floor collision, occurring at time t_k , the law

$$(2.14) \quad \delta h^-(t_k) = \delta h^+(t_k) = 0$$

is forced by the neutrality requirement $Q_1(\delta h(t), \delta v(t)) = 0$. Accordingly, we define $R_0 = I = R_0^*$, the $n \times n$ identity matrix.

Out of the considered collisions $\sigma_1, \sigma_2, \dots, \sigma_N$, let the floor collisions be

$$\sigma_{k_1}, \dots, \sigma_{k_m},$$

$1 \leq k_1 < \dots < k_m \leq N$. The above argument shows that the neutral space $\mathcal{N}_{x_0}^T$ can be defined by the following system of homogeneous linear equations:

$$(2.15) \quad \mathcal{N}_{x_0}^T = \mathcal{N}(\Sigma, \vec{m}) = \left\{ (\delta h(0), 0) \in \mathcal{T}_{x_0} \mid \Pi_1 R_{i_{k_l}}^* \dots R_{i_2}^* R_{i_1}^* \delta h(0) = 0, \text{ for } l = 1, \dots, m \right\},$$

where $\Pi_1(\delta h_1, \delta h_2, \dots, \delta h_n) = \delta h_1$ is the projection onto the first component.

We observe that the neutral space $\mathcal{N}_{x_0}^T = \mathcal{N}(\Sigma, \vec{m})$ only depends on the n-tuple of masses $m_1 > m_2 > \dots > m_n > 0$ and on the symbolic collision sequence $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$.

DEFINITION 2.3. The non-singular trajectory segment $S^{[0,T]}x_0$ or, equivalently, the corresponding pair (Σ, \vec{m}) is said to be *sufficient* iff

$$\mathcal{N}_{x_0}^T = \mathcal{N}(\Sigma, \vec{m}) = \{0\}.$$

Otherwise, these objects are said to be *insufficient*.

REMARK 2.4. A consequence of the previous definition is that the sufficiency of (Σ, \vec{m}) implies that Σ must contain all n types of collisions. Indeed, a missing $(0, 1)$ collision would mean that the system of homogeneous linear equations 2.15 contains no equations, whereas, a missing collision $(i_0, i_0 + 1)$ ($i_0 = 1, 2, \dots, n-1$) would establish no connection between the subsystems of particles $\{1, 2, \dots, i_0\}$ and $\{i_0 + 1, i_0 + 2, \dots, n\}$, thus preventing sufficiency.

Summarizing what we have seen so far, we have

PROPOSITION 2.16. The sufficiency of the trajectory segment $S^{[0,T]}x_0$ (or, equivalently, the sufficiency of (Σ, \vec{m})) is equivalent to the *strict invariance* of the cone field

$$C_1(x_t) = \left\{ (\delta h, \delta v) \in \mathcal{T}_{x_t} \mid \sum_{i=1}^n \delta h_i \delta v_i \geq 0 \right\}$$

along

$$S^{[0,T]}x_0 = \{x_t = S^t x_0 \mid 0 \leq t \leq T\},$$

that is, this sufficiency exactly means that

$$DS^T(C_1(x_0)) \subset \text{int}(C_1(x_T)).$$

According to the time evolution equations 2.2, 2.3, 2.5, in order for the derivative DS^T of the flow to be defined on the tangent space \mathcal{T}_{x_0} , it is not enough to know the pair (Σ, \vec{m}) , but one needs to know the relative velocities

$$(2.17) \quad \rho_k = \rho_k(\sigma_k) = v_{i_k}^-(t_k) - v_{i_k+1}^-(t_k) > 0$$

for all ball-to-ball collisions $\sigma_k = (i_k, i_k + 1)$ (they play a role in 2.3 as a part of α_{i_k}) and the velocities

$$(2.18) \quad \rho_k = \rho_k(\sigma_k) = v_1^+(t_k)$$

for any floor collision $\sigma_k = (0, 1)$ that play a role in 2.5, $k = 1, 2, \dots, N$.

Therefore, the *strict cone invariance* formulated in 2.16 is a property of the *extended symbolic sequence*

$$(2.19) \quad \tilde{\Sigma} = ((\sigma_1, \rho_1), (\sigma_2, \rho_2), \dots, (\sigma_N, \rho_N))$$

and the vector of the masses \vec{m} .

The characterization of sufficiency with the strict cone invariance 2.16 has the big advantage that it appears to be *combinatorially monotone*, i.e. if one inserts an additional collision (σ^*, ρ^*) into the extended symbolic sequence

$$\tilde{\Sigma} = ((\sigma_1, \rho_1), (\sigma_2, \rho_2), \dots, (\sigma_N, \rho_N))$$

between (σ_k, ρ_k) and $(\sigma_{k+1}, \rho_{k+1})$, then the sufficiency will not be lost by this insertion:

PROPOSITION 2.20 (Combinatorial Monotonicity Property, CMP). Assume that, for a given mass distribution $m_1 > m_2 > \dots > m_n$, the extended symbolic sequence

$$\tilde{\Sigma} = ((\sigma_1, \rho_1), (\sigma_2, \rho_2), \dots, (\sigma_N, \rho_N))$$

is sufficient. Let the extended symbolic sequence

$$\tilde{\Sigma}^* = ((\sigma_1, \rho_1), \dots, (\sigma_k, \rho_k), (\sigma^*, \rho^*), (\sigma_{k+1}, \rho_{k+1}), \dots, (\sigma_N, \rho_N))$$

be obtained from $\tilde{\Sigma}$ by the insertion of (σ^*, ρ^*) , as indicated above. Then $\tilde{\Sigma}^*$ also satisfies the strict cone invariance property formulated in 2.16.

PROOF. For any number l , $l = 1, 2, \dots, N$, let D_l be the derivative of the flow (in terms of $(\delta h, \delta v)$, as always) resulting from the collision (σ_l, ρ_l) , and D^* be the derivative of the flow resulting from the collision (σ^*, ρ^*) . We have

$$\begin{aligned} D_N D_{N-1} \dots D_1(C_1(x_0)) &\subset \text{int}(C_1(x_T)), \\ D^* D_k D_{k-1} \dots D_1(C_1(x_0)) &\subset D_k D_{k-1} \dots D_1(C_1(x_0)), \end{aligned}$$

thus

$$D_N \dots D_{k+1} D^* D_k \dots D_1(C_1(x_0)) \subset D_N D_{N-1} \dots D_1(C_1(x_0)) \subset \text{int}(C_1(x_T)).$$

□

Since the coefficients of the system of homogeneous linear equations 2.15 are given rational functions of the masses m_1, m_2, \dots, m_n , we immediately obtain

PROPOSITION 2.21 (Dichotomy). For any given symbolic sequence

$$\Sigma = (\sigma_1, \dots, \sigma_N)$$

either

- (D₁) for almost every n-tuple of masses \vec{m} ($m_1 > m_2 > \dots > m_n > 0$, the exceptional set being a proper algebraic subset of \mathbb{R}^n) it is true that $\mathcal{N}(\Sigma, \vec{m}) = \{0\}$, or
- (D₂) for every mass vector \vec{m} we have $\mathcal{N}(\Sigma, \vec{m}) \neq \{0\}$.

DEFINITION 2.5. In the case (D₁) above the symbolic sequence Σ itself is said to be sufficient, otherwise it is said to be insufficient.

Thanks to the characterization result 2.16 and the Combinatorial Monotonicity Property 2.20 we get

COROLLARY 2.22. For any given mass distribution \vec{m} , for any symbolic collision sequence Σ , and for any (not necessarily contiguous) subsequence Σ_1 of Σ , the sufficiency of the pair (Σ_1, \vec{m}) implies the sufficiency of (Σ, \vec{m}) .

Thanks to 2.21, we obtain the typical, i.e. the mass distribution-free version of 2.22:

COROLLARY 2.23. For any symbolic collision sequence Σ and for any (not necessarily contiguous) subsequence Σ_1 of Σ , the sufficiency of Σ_1 implies the sufficiency of Σ .

Next, we take a quick look at the limiting case $m_1 = m_2 = \dots = m_n$. In this case, when a ball-to-ball collision $(i, i + 1)$ occurs, we can simply swap the labels i and $i + 1$ of the involved particles. In this way, by performing all these dynamic label changes, the entire flow becomes the independent motion of the n particles, each of them falling at unit acceleration and bouncing back elastically from the floor. (Doing so, they freely pass through each other.)

Observe that, in the case of equal masses, the transformation matrices R_i^* in 2.13 are the reflections

$$R_i = R_i^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

thus the components $\delta h_i(t)$ will be independent of the time t after the dynamic re-labeling.

The above makes it clear that the trajectory segment $S^{[0,T]}x_0$ is sufficient in the sense of 2.3, provided that all particles hit the floor. (After the dynamic re-labeling of the particles, of course.) This means that, for the limiting system $m_1 = m_2 = \dots = m_n$ there exists a uniform time threshold $\tau_0 > 0$ such that $S^{[0,T]}x_0$ is sufficient, provided that $T \geq \tau_0$. By continuity, this property extends to every falling ball system $m_1 > m_2 > \dots > m_n > 0$ with $m_n/m_1 \geq 1 - \epsilon_0$ for some fixed $\epsilon_0 > 0$. However, this property, along with the CMP 2.23, implies that there is a large enough $K \in \mathbb{Z}_+$ such that every symbolic sequence $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$ is sufficient if Σ contains at least K consecutive, contiguous, connected subsequences. (Recall that a subsequence is connected when its collision graph is connected or,

equivalently, the subsequence contains all types of collisions.) However, the system of falling balls obviously possesses the property that every trajectory $S^{[0,\infty)}x_0$ contains infinitely many appearances of each collision type. Thus we obtain the main result of this section:

THEOREM 2.6 (Main Result). *For almost every n -tuple of masses $m_1 > m_2 > \dots > m_n > 0$ (the exceptional set being a countable union of proper algebraic subsets of \mathbb{R}^n) the following is true: Every non-singular positive orbit $S^{[0,\infty)}x_0$ is sufficient, hence possesses the property of eventually strict cone invariance.*

3. Conclusions

Here we summarize the corollaries of our Theorem 2.6. These corollaries will be *conditional* on the condition that the following property is possessed by the falling ball system:

CLAIM 3.1 (Transversality Condition). Singularities of different order are transversal to each other. Analogously, stable and unstable invariant manifolds are transversal to all singularities.

Our goal is to prove the Ergodic Conjecture of Wojtkowski [**W90a**] for almost every falling ball system, assuming the above transversality condition. The exceptional systems are the ones for which our main result 2.6 is false, so they form a countable union a proper algebraic sets in terms of the masses \vec{m} . For this purpose we are going to use the celebrated Local Ergodicity Theorem (LET, for short) of [**L-W95**], so we investigate the first return map $T : U_0 \rightarrow U_0$ of the billiard map to a suitably small, open neighborhood of a hyperbolic phase point x_0 with at most one singularity on its trajectory.

First of all, the second part of 3.1 is the so called Chernov-Sinai Ansatz, a condition of the LET of [**L-W95**].

Second, the transversality of singularities guarantees that the set of double singularities is a countable union of smooth, codimension-two (≥ 2) submanifolds, hence a slim set, negligible in dynamical considerations, see Section 2 of [**K-S-Sz92**], esp. Lemma 2.12 there. Thus, we may safely assume that the considered hyperbolic phase point x_0 has at most one singularity on its trajectory. Hyperbolicity means, of course, that the cone field C_1 is strictly invariant along the trajectory of x_0 .

Third, the second part of 3.1 is Property 5' of [**Ch93**], thus the result of [**Ch93**] guarantees the Proper Alignment of Singularities, another condition of the LET in [**L-W95**].

Since the first return map $T : U_0 \rightarrow U_0$ enjoys strict C_1 -cone invariance, according to Proposition 6.1 of [**L-W95**], the minimum Q_1 -expansion rate σ of T is uniformly bigger than 1. This establishes the Non-Contraction Property and the Strict Unboundedness Property for the first return map $T : U_0 \rightarrow U_0$, two more conditions of the LET.

Fourth, the Regularity of the Singular Set condition of [**L-W95**] is obtained as follows:

- (1) The phase points with two singularities on their trajectory form a *slim* set, as stated before.
- (2) The smooth components of different singularities do not accumulate at any point of the phase space, since there are no infinitely many collisions

in finite time (Proposition A.1 of the Appendix of [S96]), and the horizon is finite.

- (3) The derivatives of the singularities do not “explode”, since there are no grazing (tangential) singularities in the falling ball system, only of corner type.

Finally, the Local Ergodicity Theorem of [L-W95] states that the entire open neighborhood of the hyperbolic phase point x_0 (with at most one singularity on its trajectory) belongs to a single ergodic component of the falling ball system. For almost every mass vector \vec{m} , the set of phase points x_0 not having the two properties required above (strict cone invariance and at most one singularity along its trajectory) is a slim set. According to Lemma 2.12 of [K-S-Sz92], such slim sets are unable to cut the phase space into different, open ergodic components. Thus, we conclude:

THEOREM 3.1 (Main Corollary). *Assuming the condition 3.1, for almost every mass vector $m_1 > m_2 > \dots > m_n > 0$ the falling ball system is ergodic.*

REMARK 3.2. In my upcoming paper [S2023] I will be proving the Wojtkowski Conjecture unconditionally for almost every falling ball system by employing, again, algebraic methods. Namely, the following two, geometric properties will be proved there for almost every falling ball flow:

- (A) For every non-collision phase point $x_0 = (q_0, v_0) \in \mathbf{M}$ of the flow it is true that if x_0 belongs to the interior (smooth part) of a singularity manifold \mathcal{S} , then the tangent space $\mathcal{T}_{x_0}\mathcal{S}$ of \mathcal{S} at x_0 contains the full tangent space $\mathcal{T}_{x_0}\mathbf{Q}$ of the configuration space \mathbf{Q} at x_0 . (The so-called “ \mathbf{Q} -onto property” of the singularities.)
- (B) For every non-collision phase point $x_0 = (q_0, v_0) \in \mathbf{M}$ of the flow, if x_0 is the interior point of a local stable (unstable) invariant manifold γ , then the tangent space $\mathcal{T}_{x_0}\gamma$ contains the orthogonal complement of the velocity vector v_0 in $\mathcal{T}\mathbf{Q}$.

We note that these two properties together obviously imply the transversality condition 3.1 of this paper.

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