Short Course on the Sherrington Kirkpatrick Model : Lecture 2

Shannon Starr

3 November 2003

Material which is not otherwise attributed in these lectures is motivated by joint work with Michael Aizenman and Robert Sims, or, in other cases, is material communicated to me in private conversations by Michael Aizenman.

1 Parisi's Ansatz

1.1 Replica Symmetry Breaking

To review, we are considering the Sherrington-Kirkpatrick model whose Hamiltonian function is

$$H_N(\sigma) = \frac{-1}{\sqrt{2N}} \sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j - h\sum_{i=1}^{N} \sigma_i\,. \tag{1}$$

Here $H_N : \mathbb{Z}_2^N \to \mathbb{R}$ is a *random* function because all the $(J_{ij} : 1 \leq i,j \leq N)$ are iid $\mathcal{N}(0,1)$ random variables. We are somewhat interested in the distribution of the partition function

$$Z_N(\beta,h) = \sum_{\sigma\in\Omega_N} e^{-\beta H_N(\sigma,h)}\,. \tag{2}$$

But we are most interested in the quantity called the finite volume *quenched pressure*

$$p_N(\beta,h) = \mathbb{E}\left\{\frac{1}{N}\log\, Z_N(\beta,h)\right\}\,, \tag{3}$$

and the limiting *quenched pressure*

$$p(\beta,h) = \lim_{N\to\infty} p_N(\beta,h)\,, \tag{4}$$

when the limit exists. (It is now known that the limit always exists, and we will go over Guerra and Toninelli's proof of this fact [5] in one or two lectures from now.)

In order to approach the value of $p(\beta,h)$ we considered the real moments of $Z_N(\beta,h)$ as

$$\phi_N(\beta,h;\lambda) := \frac{1}{N\lambda}\log\,\mathbb{E}\{Z_N(\beta,h)^\lambda\}\,, \tag{5}$$

and their limiting values

$$\phi(\beta,h;\lambda) = \lim_{N\to\infty}\phi_N(\beta,h;\lambda)\,, \tag{6}$$

when the limits exist. (Using Guerra and Toninelli's method, one can show that the limit always does exist, and this is one of the homework problems from last week, though it will be easier to prove in one or two weeks.) Neglecting the question of determining $\phi(\beta,h;\lambda)$ for all $\lambda\in\mathbb{R}\setminus\{0\}$, we followed [6], van Hemmen and Palmer's proof and simplified formula for the limit $\phi(\beta,h;n)$ when $n\in\mathbb{N}^*=\{1,2,\dots\}$. Their formula is the following

$$\phi(\beta,h;n) = \frac{\beta^2}{4} + \max_{\boldsymbol{Q}\in\mathbb{R}^{n(n-1)/2}}\mathcal{F}_n(\boldsymbol{Q})\,, \tag{7}$$

where $\boldsymbol{Q} = (Q_{\alpha,\alpha'} : 1 \leq \alpha < \alpha' \leq n)$ will be called the order parameter for a while, and

$$\mathcal{F}_n(\boldsymbol{Q}) = -\frac{\beta^2}{2} \sum_{\alpha<\alpha'} Q_{\alpha,\alpha'}^2 + \log \sum_{S\in\mathbb{Z}_2^N} e^{-\widetilde{H}_n(\beta h,\beta^2 \boldsymbol{Q},S)}\, , \tag{8}$$

$$\widetilde{H}_n(\beta h, \beta^2\boldsymbol{Q}, S) = -\beta h \sum_{\alpha=1}^{n} S_\alpha - \sum_{\alpha<\alpha'} \beta^2 Q_{\alpha,\alpha'} S_\alpha S_{\alpha'}\, . \tag{9}$$

This result is derived just using the Hubbard-Stratonovich formula and the saddle point method. Alternatively to the saddle point method, one can derive large deviation estimates. A reference for the saddle point method (also known as the method of the stationary phase) is [10].

A result we didn't prove in class, but which we did mention is van Hemmen, Palmer and Lieb's proof that for $h = 0$ there is always an optimizer such that there is a $q \in [0, 1]$ with $Q_{\alpha,\alpha'} = q$ for all $\alpha < \alpha'$. Their proof relies on reflection positivity, and would handle the case $h = 0$ except for a technicality (common in applications of reflection positivity). Sherrington and Kirkpatrick simply assumed that the optimizer is always of this form, and used that result to obtain a formula for $\phi(\beta, h; n)$ in which n appears as a parameter, not necessarily $n \in \mathbb{N}^*$, as is required by the formula above. They then *extrapolated* their answer to 0 to obtain a formula for $p(\beta, h)$. But, aside from being completely nonrigorous, Sherrington and Kirkpatrick's result gave nonphysical answers when β was too large. In particular, as they pointed out, themselves, when $\beta \to \infty$, their formula gave a negative ground state entropy, which is impossible in a discrete system. (The ground state entropy is the logarithm of the number of ground states in a discrete system, and there is always at least one ground state.)

The replica trick, the idea to calculate the $n \to 0$ limit of $\phi(\beta, h; n)$ by taking $n \to 0$ through $\mathbb{N}^*$ may seem crazy. But Parisi's ansatz showed that perhaps it wasn't crazy enough, because Parisi's ansatz adds another level of completely heuristic manipulations, and yet somehow gets a more reasonable answer from the physical point-of-view. For example, the ground state entropy becomes nonnegative when the full ansatz that Parisi proposed is used. Parisi called his approach *replica symmetry breaking*.

Suppose n is a very large integer and $m|n$; i.e., $n/m \in \mathbb{N}^*$. Parisi considered the order parameter $\boldsymbol{Q}$ given by the formula

$$Q_{\alpha,\alpha'} = \begin{cases} q_1 & \text{if } \alpha \equiv \alpha' \mod n/m\, , \\ q_0 & \text{if } \alpha \not\equiv \alpha' \mod n/m\, , \end{cases} \tag{10}$$

where $q_1 > q_0$. Then

$$-\frac{\beta^2}{2} \sum_{\alpha<\alpha'} Q_{\alpha,\alpha'}^2 = -\frac{n\beta^2}{4}[(m-1)q_1^2 + (n-m)q_0^2]\, . \tag{11}$$

It may seem that if n is large enough then either $m - 1$ can be replaced by m or $n - m$ could be replaced by n without much loss. But this is not our purpose. Our eventual purpose is to follow Sherrington and Kirkpatrick in taking $n \to 0$, and then m will be a number in $(0, 1)$. This is part of Parisi's extra craziness. One also has

$$\sum_{S\in\mathbb{Z}_2^N} e^{-\widetilde{H}_n(\beta h,\beta^2\boldsymbol{Q},S)} = \sum_{S\in\mathbb{Z}_2^N} e^{\beta h \sum_\alpha S_\alpha} e^{\beta^2 q_0 \sum_{\alpha<\alpha'} S_\alpha S_{\alpha'}} e^{\beta^2(q_1-q_0)\sum_{k=1}^{n/m} \sum_{\alpha<\alpha'\in[k]} S_\alpha S_{\alpha'}}\, . \tag{12}$$

Here we use the notation $[k]$ for the congruence class modulo n/m. To simplify further, we use the Hubbard-Stratonovich transformation, which yields a hefty calculation, but which turns out nice in the end. It is important that we will introduce $(n/m) + 1$ Gaussian random variables. One for the single sum in the exponent with coefficient q_0 and n/m for each sum with coefficient $(q_1 - q_0)$. Recall the standard notation

$\mu(z) = e^{-z^2/2}\, dz/\sqrt{2\pi}$. Continuing the last display we obtain

$$
\begin{aligned}
&= \int \cdots \int d\mu(z_0) \left[\prod_{k=1}^{n/m} d\mu(z_k)\right] \sum_{S\in\mathbb{Z}_2^N} e^{\beta h \sum_{\alpha=1}^n S_\alpha} e^{-\beta^2 q_0 n/2} e^{\beta\sqrt{q_0}\, z_0 \sum_{\alpha=1}^n S_\alpha} e^{-\beta^2(q_1-q_0)n/2} e^{\beta\sqrt{q_1-q_0}\sum_{k=1}^{n/m} z_k \sum_{\alpha\in[k]} S_\alpha} \\
&= e^{-\beta^2 q_1 n/2} \int \cdot \int d\mu(z_0) \left[\prod_{k=1}^{n/m} d\mu(z_k)\right] \prod_{k=1}^{n/m} \prod_{\alpha\in[k]} \left(2\cosh(\beta[h+\sqrt{q_0}\, z_0 + \sqrt{q_1-q_0}\, z_k])\right) \\
&= 2^n e^{-\beta^2 q_1 n/2} \int \cdot \int d\mu(z_0) \left[\prod_{k=1}^{n/m} d\mu(z_k)\right] \prod_{k=1}^{n/m} \left(2\cosh(\beta[h+\sqrt{q_0}\, z_0 + \sqrt{q_1-q_0}\, z_k])\right)^m \\
&= 2^n e^{-\beta^2 q_1 n/2} \int d\mu(z_0) \left[\int d\mu(z_1)\, \cosh^m(\beta[h+\sqrt{q_0}\, z_0 + \sqrt{q_1-q_0}\, z_1])\right]^{n/m} .
\end{aligned} \tag{13}
$$

This leads to a variational calculation for $\phi(\beta, h; n)$, which one knows is wrong, but which one can at least write down, as

$$
\begin{aligned}
\Phi(\beta, h; n; m; q_0, q_1) &= \log(2) + \frac{\beta^2}{4}(1 - 2q_1 - (n-m)q_0^2 - (m-1)q_1^2) \\
&+ \frac{1}{n}\log \int d\mu(z_0)\, \exp\left(\frac{n}{m}\log \int d\mu(z_1)\, \exp\left(m \log\cosh[\beta(h+\sqrt{q_0}\, z_0 + \sqrt{q_1-q_0}\, z_1)]\right)\right) .
\end{aligned} \tag{14}
$$

We have written it in this funny way to facilitate the extrapolation to $n = 0$. Indeed, one obtains (completely formally)

$$
\begin{aligned}
\Phi(\beta, h; 0; m; q_0, q_1) &= \log(2) + \frac{\beta^2}{4}(1 - 2q_1 + mq_0^2 + (1-m)q_1^2) \\
&+ \int d\mu(z_0)\, \frac{1}{m}\log\left(\int d\mu(z_1)\, e^{m\log\cosh[\beta(h+\sqrt{q_0}\, z_0+\sqrt{q_1-q_0}\, z_1)]}\right) .
\end{aligned} \tag{15}
$$

Here it is assumed that $0 < m < 1$. This formula is called 1-level replica symmetry breaking.

1.2 The Hopf-Cole Transform

The Hopf-Cole transform is a method for solving certain nonlinear differential equation. The simplest example of a problem where the Hopf-Cole transform works is the famous Ricatti equation, which is a nonlinear ODE.

1.2.1 The Ricatti Equation

The Ricatti equation with constant coefficients is

$$
u'(t) = a + bu(t) + cu(t)^2 . \tag{16}
$$

This equation plays a central role in two separate fields of mathematics: control theory, and integrable systems. We will not attempt to explain what the precise role is since we will not do it justice. For this simple case of constant coefficients, the solution can be obtained by making a transformation

$$
\begin{aligned}
y(t) &= e^{-c\int_0^t u(s)\, ds} , \\
y'(t) &= -ce^{-c\int_0^t u(s)\, ds} u(t) = -cy(t)u(t) , \\
y''(t) &= -cy'(t)u(t) - cy(t)u'(t) = -cy(t)(u'(t) - cu(t)^2) .
\end{aligned} \tag{17}
$$

Because of the differential equation for u, we obtain the following equation for y,

$$y''(t) - by'(t) + acy(t) = 0\,. \tag{18}$$

This is a linear equation, in fact the equation for damped harmonic oscillation, which is easily solved. For example, if $a = -1$, $b = 0$, $c = 1$, we obtain

$$u = -\tanh[t - t_0]\,, \tag{19}$$

assuming that $-1 < u(0) < 1$.

It is interesting to note that there is a second, seemingly independent "solution" of this problem using the Legendre transform [2]. Let us again restrict to the case $a = -1$, $b = 0$, $c = 1$, only because one needs to decide on the sign of c. Then

$$u'(t) = u(t)^2 - 1\,. \tag{20}$$

Observe that for any $t \geq 0$,

$$u(t)^2 = \max_{w(t)\in\mathbb{R}} [2u(t)w(t) - w(t)^2]\,, \tag{21}$$

therefore, the equation for $u(t)$ is

$$u'(t) = \max_{w(t)} 2u(t)w(t) - w(t)^2 - 1\,. \tag{22}$$

If one knew the choice of $w(t)$ independent of $u(t)$, this would be a linear equation in $u(t)$ which is easily solved. For each choice of $w : \mathbb{R}_+ \to \mathbb{R}$, one can define a solution $U(t; w)$ to the equation

$$U'(t) = 2U(t)w(t) - w(t)^2 - 1\,. \tag{23}$$

Incidentally $U(T)$ depends only on w up to time T. Moreover, there is an important property, called the optimality condition. If $w^* : [0, T_1] \to \mathbb{R}$ gives the maximum value for $U(T_1; w^*)$, then for any time $T_2 \in (0, T_1)$, one knows that $w^{*}{*} = w^*|[0, T_2]$ gives the maximum value for $U(T_2; w^{*}{*})$. This is a notion generalizing the property of geodesics in the calculus of variations and the min-cut/max-flow principle in graph theory. It is familiar in statistical mechanics, because the DLR equations are a statement of optimality with respect to the Gibbs variational principle. See for example [15, 17]. (Often times one must optimize subject to boundary conditions, as in the DLR equation. For the Ricatti equation, that is not the case because the linear equation requires only one condition, the initial condition.) Some other pretty examples of the optimality condition in statistical mechanics can be seen in tiling problems [19, 18].

All that remains is to actually solve

$$U'(t) = 2U(t)w(t) - w(t)^2 - 1\,. \tag{24}$$

This is easily done using integrating factors, and one obtains, assuming $U(0) = 0$,

$$U(t) = -\int_0^t (1 + w(s)^2)e^{-2\int_s^t w(r)\,dr}\,ds\,. \tag{25}$$

From this one deduces the rather funny identity

$$\tanh(t) = \min_{w:[0,t]\to\mathbb{R}} \int_0^t (1 + w(s)^2)e^{-2\int_s^t w(r)\,dr}\,ds\,. \tag{26}$$

The reason for this seemingly unmotivated digression into the Ricatti equation is threefold. First, the principle of optimality is absolutely essential in many areas of mathematics, but relatively little known in statistical mechanics, although statistical mechanics might be one of the biggest applications, since the DLR equations are absolutely fundamental [15, 17]. Second, one would like to know if there is a principle of optimality for the equations which we will discuss momentarily. It should be noted that except for a rather brief analysis in [3], there has not been any real attempt at analyzing the PDE arising in Parisi's ansatz. Third, as we will see, Parisi's ansatz leads to a hybrid, which seems to be halfway between an optimization problem and a nonlinear PDE. (There is a control parameter, called the order parameter, to optimize, but one also has to optimize subject to a nonlinear PDE.) If one could transform either half of the problem, this would likely result in new insights.

1.3 The Nonlinear Backwards Heat Equation

Consider the PDE

$$\begin{cases} f_t(t,x) + \frac{1}{2}(f_{xx}(t,x) + m f_x(t,x)^2) = 0 & (t,x) \in (0,1) \times \mathbb{R}\,, \\ f(1,x) = F(x)\,. \end{cases} \tag{27}$$

This time, one has the transformation

$$\begin{aligned} y(t,x) &= \frac{1}{m}(e^{mf(t,x)} - 1)\,, \\ y_t(t,x) &= (my(t,x)+1) f_t(t,x)\,, \\ y_x(t,x) &= (my(t,x)+1) f_x(t,x)\,, \\ y_{xx}(t,x) &= m y_x(t,x) f_x(t,x) + (my(t,x)+1) f_{xx}(t,x)\,, \\ &= (my(t,x)+1)(f_{xx}(t,x) + m f_x(t,x)^2)\,. \end{aligned} \tag{28}$$

Because of the differential equation for f, one has

$$\begin{cases} y_t(t,x) + \frac{1}{2} y_{xx}(t,x) = 0 & (t,x) \in (0,1) \times \mathbb{R}\,, \\ y(1,x) = \frac{1}{m}(e^{mF(x)} - 1)\,. \end{cases} \tag{29}$$

Note that this must be a final-value problem because it is unstable forward in time.

The heat kernel is the generator for the Brownian motion. This means that $f(t,x)$ can be seen as a functional of Brownian motion. (And for example, the basic equation is somewhat similar to the KPP equation for branching Brownian motion [13, 4].) Without entering into this very interesting side issue, we merely note that using Wick's formula, one can easily verify that a solution is given by

$$y(t,x) = \int d\mu(z)\, y(1, x + \sqrt{1-t}\, z)\,, \tag{30}$$

for $t < 0$. One can prove uniqueness of the solution of the heat equation, and indeed it is one of the nicest situations, because the heat kernel is actually infinitely smoothing. We will not enter into this for one simple reason: Questions of uniqueness of differential equations are rather sensitive to transformations. Since we do not know if uniqueness of the solution to the nonlinear heat equation follows from uniqueness of the transformed heat equation, we will not discuss it. However, it is an interesting open problem to prove uniqueness for the nonlinear backward heat equation, which we encourage someone to solve.

Homework 1. *Verify that the formula above for y does solve the backward heat equation, using Wick's formula. The secret is always to remove any extra factor of z which arises from a first differentiation by performing a second differentiation á la Wick's rule.*

Therefore, one easily finds the solution $f(t,x) = \frac{1}{m}\log y(t,x)$.

$$f(t,x) = \frac{1}{m} \log \int d\mu(z)\, \exp(mF(x + \sqrt{1-t}\, z))\,. \tag{31}$$

For reasons of our own, we now change variables. We let $t \to q$ and $x \to h$. The differential equation for f becomes

$$\begin{cases} f_q(q,h) + \frac{1}{2}(f_{hh}(q,h) + m f_h(q,h)^2) = 0 & (q,h) \in (0,1) \times \mathbb{R}\,, \\ f(1,h) = F(h)\,, \end{cases} \tag{32}$$

And the solution becomes

$$f(q,h) = \frac{1}{m} \log \int d\mu(z)\, \exp(mF(h + \sqrt{1-q}\, z))\,. \tag{33}$$

Here we assume that $m \neq 0$, but one can make essentially the same definition for $m = 0$. Then f itself solves the backwards heat equation. Note that as $m \to 0$, one recovers $y(q,h) = f(q,h)$. Also,

$$\lim_{m \to 0} \frac{1}{m} \log \int d\mu(z)\, \exp(mF(h + \sqrt{1-q}\, z)) = \int d\mu(z)\, F(h + \sqrt{1-q}\, z)\,. \tag{34}$$

So everything is consistent there. Due to the special way that this equation arises for us, there is one more question we address before moving on.

1.3.1 Consistency at $m = 1$

Here we do not mean consistency of the formula for f. We mean something else which will be clear in the next section, consistency of Parisi's ansatz. Note that the equation for f is nonlinear. But it has a "zero mode", meaning that the linearized differential operator has a kernel, which is precisely the constant functions. In other words, if f is a solution satisfying $f(1,h) = F(h)$. Then $\tilde{f} = f - c$ is a solution satisfying $\tilde{f}(1,h) = F(h) - c$. Suppose now that $m = 1$ and $F(h) = \log\cosh(\beta h)$. Note that this converges to $|\beta h|$ for h very large (and therefore does not fall under the somewhat restricted analysis of [3]). Now note two things. Firstly,

$$F''(h) + F(h)^2 = \beta^2\,\mathrm{sech}^2(\beta h) + \beta^2\tanh^2(\beta h) = \beta^2\,. \tag{35}$$

Secondly, that

$$\begin{aligned}
\log\left(\int d\mu(z)\,\exp(F(h+\sqrt{1-q}\,z))\right) &= \log\left(\int d\mu(z)\,\cosh(\beta(h+\sqrt{1-q}\,z))\right)\\
&= \log\left(\int d\mu(z)\,\frac{1}{2}(e^{\beta h}e^{\beta\sqrt{1-q}\,z} + e^{-\beta h}e^{-\beta\sqrt{1-q}\,z})\right)\\
&= \log\left(\frac{1}{2}(e^{\beta h}e^{\beta^2(1-q)/2} + e^{-\beta h}e^{\beta^2(1-q)/2}\right)\\
&= \frac{\beta^2(1-q)}{2} + \log\cosh(\beta h)\,.
\end{aligned} \tag{36}$$

What this is telling us is that for $m = 1$, our final condition, which arises from purely physical considerations, $F(h) = \log(\cosh(\beta h))$ is an eigenvector (in some generalized sense) of the differential operator $[AF](h) = F_{hh} + F_h^2$. This is really not surprising, because if we call $u = F_h$, we see precisely the Ricatti operator $u' - u^2$, and then $u = \beta\tanh(\beta h)$. This will be important next.

1.4 The Parisi Ansatz Differential Equation

In this section, we will not desribe the Parisi ansatz, just the nonlinear differential equation which arises there.

Let us define a function $X : [0,1] \to [0,1]$ which is piecewise constant

$$X(q) = \begin{cases} 0 & 0 \le q < q_0\,,\\ m & q_0 \le q < q_1\,,\\ 1 & q_1 \le q < 1\,. \end{cases} \tag{37}$$

We assume, as in the first section, $0 < m < 1$. Now consider the differential equation for $f(q,h) : [0,1]\times\mathbb{R} \to \mathbb{R}$ given by

$$\begin{cases} f_q(q,h) + \frac{1}{2}(f_{hh}(q,h) + X(q)f_h(q,h)^2) = 0 \quad (q,h) \in (0,1)\times\mathbb{R}\,,\\ f(1,h) = \log\cosh(\beta h)\,, \end{cases} \tag{38}$$

From the discussion above, we know that for $q_1 \le q \le 1$,

$$f(q,h) = \log\cosh(\beta h) + \frac{\beta^2(1-q)}{2}\,. \tag{39}$$

For $q \le q_1$, let us define $\tilde{f}(q,h) = f(q,h) - \frac{1}{2}(1-q_1)\beta^2$. Then we have

$$\begin{cases} \tilde{f}_q(q,h) + \frac{1}{2}(\tilde{f}_{hh}(q,h) + X(q)\tilde{f}_h(q,h)^2) = 0 \quad (q,h) \in (0,q_1)\times\mathbb{R}\,,\\ \tilde{f}(q_1,h) = \log\cosh(\beta h)\,, \end{cases} \tag{40}$$

We also know from the discussion above that for $q_0 \le q \le q_1$, the solution is given by

$$\tilde{f}(q,h) = \frac{1}{m}\log\int d\mu(z)\,\exp(m\log\cosh(h+\sqrt{q_1-q}\,z))\,. \tag{41}$$

In particular, at $q = q_0$, we have

$$\tilde{f}(q_0, h) = \frac{1}{m} \log \int d\mu(z)\, \exp(m \log\cosh(h + \sqrt{q_1 - q_0}\, z))\,. \tag{42}$$

For $0 \le q \le q_0$, we obtain

$$\tilde{f}(q, h) = \int d\mu(z)\, \tilde{f}(q_0, h + \sqrt{q_0 - q}\, z)\,. \tag{43}$$

So, in particular,

$$\begin{aligned} \tilde{f}(0, h) &= \int d\mu(z)\, \tilde{f}(q_0, h + \sqrt{q_0}\, z) \\ &= \int d\mu(z_0)\, \frac{1}{m} \log \left(\int d\mu(z_1)\, \exp(m \log\cosh(h + \sqrt{q_0}\, z_0 + \sqrt{q_1 - q_0}\, z_1)) \right)\,. \end{aligned} \tag{44}$$

Finally, recalling that $f = \tilde{f} + \frac{1}{2}(1 - q_1)\beta^2$, we have

$$f(0, h) = \frac{(1 - q_1)\beta^2}{2} + \int d\mu(z_0)\, \frac{1}{m} \log \left(\int d\mu(z_1)\, \exp(m \log\cosh(h + \sqrt{q_0}\, z_0 + \sqrt{q_1 - q_0}\, z_1)) \right)\,. \tag{45}$$

1.5 Completing Parisi's Ansatz

One now sees that the solution $f(0, h)$ from the equation in the last section is very similar in form to Parisi's variational function for pressure. Observe that for $X(q)$ as defined above,

$$\int_0^1 qX(q)\, dq = \frac{1}{2} - \frac{m}{2} q_0^2 - \frac{1 - m}{2} q_1^2\,. \tag{46}$$

Therefore, we can rewrite the Parisi formula as

$$\Phi(\beta, h; m; q_0, q_1) = \log(2) + f(0, h; \beta; m; q_0, q_1) - \frac{\beta^2}{2} \int_0^1 qX(q)\, dq\,. \tag{47}$$

This is one level of replica symmetry breaking because there is only one value of m. The k-level replica symmetry breaking would be for $X(q)$ to take on $k + 2$ values, with the first and last equal to 0 and 1 respectively. The full replica symmetry breaking functional is

$$\Phi(h; \beta; X) = \log(2) + f(0, h; beta; X) - \frac{\beta^2}{2} \int_0^1 qX(q)\, dq\,, \tag{48}$$

where one restricts that $X : [0, 1] \to [0, 1]$ is nondecreasing and such that $X(0) = 0$, $X(1) = 1$, and $f(q, h; \beta; X)$ is defined by the differential equation

$$\begin{cases} f_q(q, h) + \frac{1}{2}(f_{hh}(q, h) + X(q) f_h(q, h)^2) = 0 & (q, h) \in (0, 1) \times \mathbb{R}\,, \\ f(1, h) = \log\cosh(\beta h)\,, \end{cases} \tag{49}$$

If X is piecewise constat, we can always find at least one solution using the Hopf-Cole transform. While it has not yet been proved that this is the unique solution, we will always assume that it is the case. This amounts to assuming that we choose this solution of the differential equation. Somebody should do the analysis to prove that this is the case, but most researchers are busy instead trying to verify Parisi's ansatz.

Conjecture 1.1 *(Parisi's Ansatz) The pressure for the Sherrington-Kirkpatrick model is obtained as*

$$p(\beta, h) = \inf_{\substack{X:[0,1]\to[0,1] \\ \textit{nondecr., piecewise const.}}} \Phi(h; \beta; X)\,. \tag{50}$$

At this point we will leave the Sherrington-Kirkpatrick model to consider the Curie-Weiss model. We do this in order to gain a better understanding of the Sherrington-Kirkpatrick model and Parisi's ansatz. We will eventually return to these, our main topics.

2 The Curie-Weiss Model

2.1 Definition and Pure States

The Curie-Weiss model is a deterministic mean-field Ising model, given by the Hamiltonian

$$H_N(\sigma) = -h \sum_{i=1}^{N} \sigma_i - \frac{J}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j \,. \tag{51}$$

Now both h and J are fixed parameters in $\mathbb{R}$. If $J \geq 0$, then the model is ferromagnetic. If $J \leq 0$ the model is antiferromagnetic. For the Curie-Weiss model, the ferromagnetic regime is more interesting because there is a possibility of a phase transition, as manifested through spontaneous magnetization, whereas no such feature exists for the antiferromagnet. We will prove this analytically shortly.

First, before going any further, let us determine the set of possible pure states of this model. As is well known, it is difficult to do statistical mechanics on the sequence of complete graphs, because quasilocal observables do not really exist. However, there is some remnant of statistical mechanics left. In statistical mechanics, one defines a convex set of limiting Gibbs states, which are characterized as satisfying the DLR equations, in classical statistical mechanics, or satisfying the β-KMS condition for quantum statistical mechanics. In both cases, it turns out that the equilibrium states form a simplex, and the question becomes the classification of the extremal points, called the pure states. The extremal points of a convex domain are those points which cannot be written as a nontrivial convex combination of any other points in the domain. Since the space of states is weak-$*$ compact, the Krein-Milman theorem says that every point can be expressed as a suitable convex combination of extreme points. But a simplex (or more precisely a Choquet simplex) is a domain such that this representation is unique. Thus the property of being a simplex expresses the feature that the extreme points are sufficiently linearly independent, in some sense. The first question, however, in statistical mechanics is the classification of all extreme points.

It is often the case that the Hamiltonian, and hence the Gibbs states, have a symmetry. It may happen that the symmetry is broken in the thermodynamic limit, for example the spin-flip symmetry is broken for the Ising model. This is because the DLR equations allow the addition of small symmetry-breaking perturbations which are localized at the boundary, therefore vanish in the thermodynamic limit. It is important to allow the symmetry breaking to understand the role of phase transitions in the thermodynamic limit. But there are other examples of symmetries which are not broken. For example, translation invariance is often times not broken, though there are some models where it is. In this case, one can restrict the search for states to those which preserve the symmetry. One can even identify the extreme points of all those states preserving the symmetry, which should amount to finding irreducible representations of the symmetry group. One then knows that whatever the pure states for the Hamiltonian are, they must be a convex combination of these "pure states for the symmetry". That is what we will do now.

Observe that H_N has a symmetry under the permutation group S_N. Since this is not a symmetry under a single group, it is important to observe that S_N can be embedded in S_M for any $M > N$. For example, let us consider the representation of S_N on $\mathbb{Z}_2^N$. Given a permutation π, and a configuration σ, one defines $(\pi\sigma)_i = \sigma_{\pi^{-1}(i)}$. One can also construct the representation at the level of the (commutative in this case) observable algebra. Given $f : \mathbb{Z}_2^N \to \mathbb{R}$, define $\pi f(\sigma) = f(\pi\sigma)$. (Note that $(Z_2^N)^* \equiv \mathbb{C}[\mathbb{Z}_2^N]$, the group algebra.) This automorphism of the observable algebra extends in the usual way to the observable algebra for $\mathbb{Z}_2^M$ with $M > N$. At the level of the configurations, one simply defines

$$(\pi\sigma)_i = \begin{cases} \sigma_{\pi^{-1}(i)} & i \leq N \\ \sigma_i & i > N \end{cases} \tag{52}$$

This is equivalent to defining an embedding $j_N^M : S_N \to S_M$ by

$$[j_N^M \pi](i) = \begin{cases} \pi(i) & i \leq N \\ i & i > N \end{cases} \tag{53}$$

Just as one can take the inductive limit of the observable algebra, one can take the inductive limit of the permutation groups to obtain the infinite symmetric group $S_\infty = \underleftarrow{\lim} S_N$. We will now consider an important result regarding this group in statistical mechanics, called de Finetti's theorem.

2.2 de Finetti's theorem

Consider any weak-$*$ limit of Gibbs states for H_N, with $N \to \infty$. Since every such Gibbs state is invariant under the action of S_n as soon as $N \geq n$, we know that the limiting state is invariant under the action of S_∞. In this case, that the observable algebra is commutative, the limiting state is simply a measure on $\mathbb{Z}_2^\infty = \underleftarrow{\lim} \mathbb{Z}_2^N$. Any measure on an infinite product space which is invariant under S_∞ is called *exchangeable.* Observe that if ρ is any probability measure on $\mathbb{Z}_2$, then the infinite product $\rho^{\otimes\infty}$ is defined such that for any $n \in \mathbb{N}^*$ and any cylinder set $C(n; A_1, \dots, A_n) = A_1 \times \cdots \times A_n \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$, one has $\rho^{\otimes\infty}(C(n; A_1, \dots, A_n)) = \prod_{i=1}^n \rho(A_i)$. Such a product measure is automatically exhchangeable. Note that the family of exchangeable probability measures is convex. So any convex combination of such product measures is also exchangeable. De Finetti's theorem states that this is all that can happen. Instead of stating de Finetti's theorem as he did, we state the slightly later version due to Hewitt and Savage [7]. Their theorem has the right level of generality, and also enjoys the advantage that it is proved using Choquet theory. Choquet theory is the general topic of identifying extreme points for a Choquet simplex. This always gives a representation for a class of objects. (For example, the spectral theorem.) Often times Choquet theory proofs are clever in the sense that they can be understood much more quickly than they can be derived. For this reason, we highly recommend [7].

Theorem 2.1 *Hewitt and Savage Suppose X is a locally compact Hausdorff space and μ is an exchangeable Borel probability measure on $\tilde{X} = \prod_{n=1}^\infty X_n$ (where all $X_n \equiv X$). Then there is a measure α on $\mathcal{M}_1(X)$, the set of probability measures on X, called the* director measure*, such that for any Borel measurable set $A \in \tilde{X}$, one has*

$$\mu(A) = \int d\alpha(\rho) \rho^{\otimes\infty}(A) . \tag{54}$$

The representation is unique by the law of large numbers. This identifies the convex set of all exchangeable measures as a Choquet simplex (in fact a Bauer simplex because the set of extreme points is closed), and identifies the extreme points as the collection of product measures.

Hewitt and Savage's proof is very clever. The main trick could be described in a couple of sentences. But I don't want to take away the enjoyment of thinking about the problem, and (as in my case) eventually reading their proof if one cannot solve the problem oneself. The pleasure one finds is equal to the pleasure one finds in the Choquet theory proof of Bochner's theorem, for example.

Homework 2. *Think about how to prove de Finetti's theorem, and look up Hewitt and Savage's paper to see how they did it. The main issue in a Choquet theory proof is to find a way to write a non-product state as a convex combination of two distinct states. (This is always the trick in a Choquet theory proof.)*

Returning to our case of $X = \mathbb{Z}_2$, we see that any probability measure is necessarily a Bernoulli. Therefore, $\mathcal{M}_1(\mathbb{Z}_2) \equiv [0, 1]$ where $\rho_p(+) = p$ and $\rho_p(-) = 1 - p$. Coming from a physics background, I prefer to specify the Bernoullis through an external field instead. Then we observe that $\mathcal{M}_1(\mathbb{Z}_2) \equiv \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, where $\rho_\eta(+) = e^\eta / 2\cosh(\eta)$ and $\rho_\eta(-) = e^{-\eta}/2\cosh(\eta)$. Then this is exactly equal to the Gibbs measure on $\mathbb{Z}_2$ at temperature 1, given by the Hamiltonian $H_1 = \eta\sigma$. Moreover, the product measure $\rho_\eta^{\otimes\infty}$ is the Gibbs measure on $\mathbb{Z}_2^\infty$ at temperature 1 for the ultralocal Hamiltonian $H_\infty = \sum_{n=1}^\infty \eta\sigma_n$.

Let us return to the relationship of these states to the pure states of the Curie-Weiss model. It is not that for a particular choice of h and J, all of these measures will be pure states, i.e., extreme points of the family of limiting measures. Indeed they will not. If one restricts to weak-$*$ limits of the finite volume Gibbs states, then there will always be a unique limit point. If one allows small perturbations in an appropriate norm, then there will be two pure states precisely when $h = 0$, $J > 0$ and $\beta J > 1$. What is true, however, is that the pure states will always be a subset of this family of extreme points in the simplex of exchangeable measures. The identification of a small family of potential pure states is crucial for developing a good variational principle. Also, it motivates our conjecture about pure states of the SK model, where, just as for the CW model, the definition is not entirely clear.

2.3 Solution of the CW Model by Steepest Descents

We observe that, using the permutation symmetry, and introducing the parameter

$$m_\sigma = \frac{1}{N}\sum_{i=1}^{N}\sigma_i\,, \tag{55}$$

we have

$$\begin{aligned} p_N(\beta,h) &= \sum_{\sigma\in\mathbb{Z}_2^N}\exp\left(N\left[\beta h m_\sigma + \frac{1}{2}\beta J m_\sigma^2\right]\right) \\ &= \sum_{\substack{m=-1\\ \Delta m=2/M}}^{1}\sum_{\substack{\sigma\in\mathbb{Z}_2^N\\ m_\sigma=m}}\exp\left(N\left[\beta h m + \frac{1}{2}\beta J m^2\right]\right) \\ &= \sum_{\substack{m=-1\\ \Delta m=2/M}}^{1}\binom{N}{(1+m)N/2}\exp\left(N\left[\beta h m + \frac{1}{2}\beta J m^2\right]\right)\,. \end{aligned} \tag{56}$$

Using Stirling's approximation $n! \sim e^{n\log n - n}\sqrt{2\pi n}$ one obtains an asymptotic formula for the binomial coefficients

$$I(m) := \lim_{N\to\infty}\frac{1}{N}\log 2^{-N}\binom{N}{(1+m)N/2} = -\frac{1}{2}[(1+m)\log(1+m)+(1-m)\log(1-m)]\,. \tag{57}$$

Note that this is the large deviation rate function for the sum of independent symmetric Bernoulli's

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{P}[m_\sigma > m] = -\frac{1}{2}[(1+m)\log(1+m)+(1-m)\log(1-m)]\,, \tag{58}$$

for $m>0$. Plugging this into the formula for pressure gives

$$\begin{aligned} p(\beta,h) &:= \lim_{N\to\infty}p_N(\beta,h) \\ &= \log(2) + \lim_{N\to\infty}\frac{1}{N}\log\sum_{\substack{m=-1\\ \Delta m=2/M}}^{1}\exp\left(N\left[\beta h m + \frac{1}{2}\beta J m^2 + I(m)\right]\right) \\ &= \max_{m\in[-1,1]}\mathcal{F}(m) \end{aligned} \tag{59}$$

where

$$\mathcal{F}(m) = \beta h m + \frac{1}{2}\beta J m^2 + I(m)\,. \tag{60}$$

We are being a little loose with the derivation because we will have a second, completely rigorous derivation of this result next lecture. One might introduce the parameters $\alpha=\beta h$ and $\gamma=\beta J$ to obtain

$$p(\alpha,\gamma) = \max_{m\in[-1,1]}\mathcal{F}(\alpha,\gamma,m)\,,\quad \mathcal{F}(\alpha,\gamma,m) = \alpha m + \frac{1}{2}\gamma m^2 + I(m)\,. \tag{61}$$

Note that $I(m)$ has the property that $I'(m) = \frac{1}{2}\log\left(\frac{1-m}{1+m}\right) = -\tanh^{-1}(m)$. Therefore $\partial_m\mathcal{F}(\alpha,\gamma,m)|_{m=1} = -\infty$ for any finite α,γ, while $\partial_m\mathcal{F}(\alpha,\gamma,n)|_{m=-1} = +\infty$. This implies that the global optimizers of $\mathcal{F}(\alpha,\gamma,m)$ with respect to $m\in[-1,1]$ are always local optimizers. For m a local optimizer, or even just a critical point of $\mathcal{F}(\alpha,\beta,m)$, one has

$$\alpha + \gamma m = \tanh^{-1}(m)\,. \tag{62}$$

To be a local maximizer, one requires

$$\gamma \le \frac{1}{1-m^2}\,. \tag{63}$$

Defining $\eta = \tanh^{-1}(m)$, one has

$$\frac{\eta - \alpha}{\gamma} = \tanh(\eta) \tag{64}$$

and the condition for local stability is that

$$\frac{1}{\gamma} \geq \operatorname{sech}^2(\eta)\,. \tag{65}$$

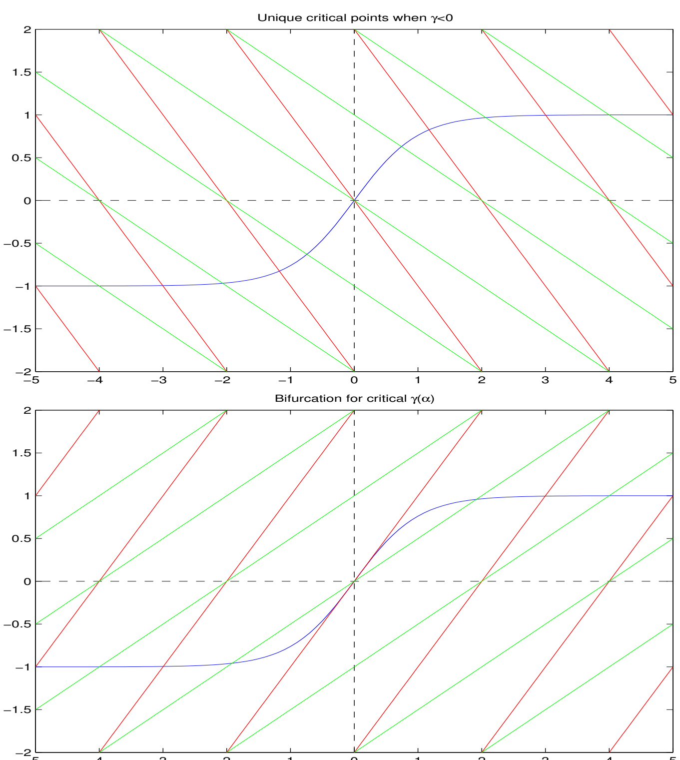
Defining $\eta(\alpha, \gamma)$ to be the optimizer, whenever the optimizer is unique, it is clear that local stability becomes critical when $\gamma^{-1} = \operatorname{sech}^2(\eta) = 1 - \tanh^2(\eta)$. Note that this implies $\gamma \geq 1$. At this point there will be a bifurcation, which we interpret as a phase transition. There are two local maximizers bracketing a local minimizer. But, unless $\alpha = 0$, there is only one global minimizer. Local maximizers are often called "metastable states" because in the thermodynamic limit it would take an infinite amount of time to leave a metastable state by a stochastic monte carlo dynamics. (In continuous systems such as van der Waal's, there is a continuous dynamics associated.) With the right scaling of time with N, one may see the system stay in the metastable state for a finite amount of time.Suppose $\alpha \geq 0$ so that $\eta \geq 0$ (this is easily seen), then $\tanh(\eta) = \sqrt{1 - \gamma^{-1}}$. So one has

$$\eta = \alpha + \sqrt{\gamma(\gamma - 1)}\,, \tag{66}$$

which shows that the critical $\gamma(\alpha)$ is never less than 1. Finally plugging back into the equation $\gamma^{-1} = \operatorname{sech}^2(\eta)$ gives an implicit equation for the critical interaction strength, γ:

$$\gamma(\alpha) = \cosh^2(\alpha + \sqrt{\gamma(\alpha)(\gamma(\alpha) - 1)})\,. \tag{67}$$

One can deduce the behavior of the choice of η, including possible bifurcations by graphically solving $(\eta - \alpha)/\gamma = \tanh(\eta)$. Here are two pictures for two choices each of $\gamma < 0$ and $\gamma > 0$.



There are other relevant pictures one can look at. For these, see [1].

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