Short Course on the Sherrington Kirkpatrick Model : Lecture 3 (short lecture)

Shannon Starr

5 November 2003

Material which is not otherwise attributed in these lectures is motivated by joint work with Michael Aizenman and Robert Sims, or, in other cases, is material communicated to me in private conversations by Michael Aizenman.

1 Curie Weiss, Redux

We want to return to the Curie-Weiss model to rederive the formula for the pressure. Before beginning we wish to make a correction. In the last lecture we dropped a log(2) in the final formula for the pressure based on the method of steepest descents. The actual pressure, for a system with $\beta h = \eta$ and $\beta J = \gamma$ is

$$p(\eta, \gamma) = \log(2) + \max_{m \in [-1,1]} \eta m + \frac{\gamma m^2}{2} + I(m) \tag{1}$$

where $I(m) = -\frac{1+m}{2} \log(1+m) - \frac{1-m}{2} \log(1-m)$. We will adopt the parameters η and γ in place of β and h for the rest of this lecture, because they are more convenient.

Now, forget about the last paragraph! We are going to rederive the solution of the Curie-Weiss model in a more elegant way. This approach is entirely motivated by recent activity in the study of the Sherrington-Kirkpatrick model, but could have been (and maybe should have been) done fifty years ago. The motivation to study the CW model is as a simpler example where some of the recent ideas can be applied.

We use the parameters $\eta = \beta h$ and $\gamma = \beta J$, mostly to decouple the effect of the external magnetic field from the temperature. (It is a nuisance that they were ever coupled to begin with.) We also change the sign of the Hamiltonian, to allow for it to appear with a positive sign in the exponent:

$$H_N(\eta, \gamma; \sigma) = N \left(\eta m_\sigma + \frac{\gamma m_{\sigma^2}}{2} \right) , \tag{2}$$

for $\sigma \in \mathbb{Z}_2^N$ where $m_\sigma = \frac{1}{N} \sum_{i=1}^N \sigma_i$. The fundamental object of study is the pressure. The finite-volume pressure is given by the formula

$$p_N(\eta, \gamma) = \frac{1}{N} \log \left(\sum_{\sigma \in \mathbb{Z}_2^N} e^{H_N(\eta,\gamma;\sigma)} \right) . \tag{3}$$

The first result is the following elementary, but important, fact. (It is the analogue of Guerra and Toninelli's recent result for the SK model [3].)

Lemma 1.1 *If $\gamma \geq 0$, then the sequence $(Np_N(\eta, \gamma) : N \in \mathbb{N}^* = \{1, 2, \dots\})$ is subadditive, meaning for $M, N \in \mathbb{N}^*$,*

$$(M + N)p_{M+N}(\eta, \gamma) \leq Mp_M(\eta, \gamma) + Np_N(\eta, \gamma) . \tag{4}$$

If $\gamma \leq 0$, then the sequence is superadditive.

To prove the lemma, we will use a deterministic analogue of Slepian's lemma.

Lemma 1.2 *Suppose that $H : \mathcal{A} \to \mathbb{R}$ and $K : \mathcal{A} \to \mathbb{R}$ are two functions such that $K(\alpha) \geq H(\alpha)$ for all $\alpha \in \mathcal{A}$. Then*

$$\log\left(\sum_{\alpha\in\mathcal{A}} e^{H(\alpha)}\right) \leq \log\left(\sum_{\alpha\in\mathcal{A}} e^{K(\alpha)}\right). \tag{5}$$

This is evident because exp and log are both increasing functions. It can also be proved in a way which is more parallel to the proof of Slepian's lemma. Introduce the interpolating Hamiltonian, $L(\alpha;t) = (1-t)H(\alpha) + tK(\alpha)$, and observe that the interpolating pressure is increasing

$$\partial_t \log\left(\sum_{\alpha} e^{L(\alpha,t)}\right) = \langle \partial_t L(\cdot,t)\rangle_{L(\cdot,t)} = \langle K - H_{L(\cdot,t)} \geq 0\,. \tag{6}$$

This proves it.

Proof of Lemma 1.1. Compare the Hamiltonian $H_{M+N}(\eta,\gamma,(\sigma,\tau))$ to the Hamiltonian $H_M(\eta,\gamma,\sigma) + H_N(\eta,\gamma,\tau)$ where $\sigma \in \mathbb{Z}_2^M$, $\tau \in \mathbb{Z}_2^N$ and $(\sigma,\tau) \in \mathbb{Z}_2^M \times \mathbb{Z}_2^N \equiv \mathbb{Z}_2^{M+N}$. Using the fact that

$$m_{(\sigma,\tau)} = \frac{M}{M+N} m_\sigma + \frac{N}{M+N}\sigma_\tau\,, \tag{7}$$

one has

$$H_{M+N}(\alpha,\gamma,(\sigma,\tau)) = \eta M m_\sigma + \eta N m_\tau + \frac{\gamma(M+N)}{2}\left(\frac{M}{M+N}m_\sigma + \frac{N}{M+N}m_\tau\right)^2. \tag{8}$$

Due to convexity of the map $m \mapsto m^2$, and supposing $\gamma \geq 0$,

$$\begin{aligned} H_{M+N}(\alpha,\gamma,(\sigma,\tau)) &\leq \eta M m_\sigma + \eta N m_\tau + \frac{\gamma(M+N)}{2}\cdot\frac{M}{M+N}m_\sigma^2 + \frac{\gamma(M+N)}{2}\cdot\frac{N}{M+N}m_\tau^2 \\ &= H_M(\eta,\gamma,\sigma) + H_M(\eta,\gamma,\tau)\,. \end{aligned} \tag{9}$$

If $\gamma \leq 0$, clearly the opposite inequality holds. Then the proof follows from 1.2.□

We will show momentarily that Lemma 1.1 implies the existence of $\lim_{N\to\infty} p_N(\beta,h)$, although possibly in $-\infty$ or $+\infty$ if $\gamma \geq 0$ or $\gamma \leq 0$, respectively. Recall that $\gamma \geq 0$ is a ferromagnetic interaction because, in the absence of an external magnetic field, the Gibbs weight is largest when $m_\sigma^2 = 1$; i.e., all spins aligned. Likewise $\gamma \leq 0$ is antiferromagnetic. The next simple result is a bound, which we would call the cavity field bound. (This is the analogue of a generalization of Guerra's RSB bound [2], the generalization being half of the result of Aizenman, Sims and myself [1].)

Definition 1.3 *Suppose $\mu \in \mathcal{M}_1([-1,1])$, which is a notation for the set of Borel probability measures on $[-1,1]$. Define the cavity field functional*

$$G_N(\eta,\gamma;\mu) = \log\left(\frac{\int_{-1}^1 d\mu(m)\,\sum_{\sigma\in\mathbb{Z}_2^N} e^{N(\eta+\gamma m)m_\sigma}}{\int_{-1}^1 d\mu(m)\,e^{\gamma N m^2/2}}\right). \tag{10}$$

The motivation for this precise definition will be explained shortly. For now observe that, superficially at least, it is simpler than $p_N(\eta,\gamma)$ because the effective Hamiltonian for σ is linear in m_σ. This means we know how to sum on σ, and also, as we will see shortly, it means we can easily find the measure μ which optimizes $G_N(\eta,\beta;\mu)$.

Lemma 1.4 *If $\gamma \geq 0$, then for any $N \in \mathbb{N}^*$,*

$$p_N(\eta,\gamma) \geq \sup_{\mu\in\mathcal{M}_1([-1,1])} G_N(\eta,\gamma;\mu)\,. \tag{11}$$

If $\gamma \leq 0$ the opposite bound holds

$$p_N(\eta,\gamma) \leq \inf_{\mu\in\mathcal{M}_1([-1,1])} G_N(\eta,\gamma;\mu)\,. \tag{12}$$

Proof. The proof will again use Lemma 1.2. Define $\mathcal{A} = \mathbb{Z}_2^N \times [-1,1]$, and replace the sum over α by the finite measure $\xi = \rho_0^{\otimes N} \otimes \mu$, where $\rho_0(\{+\}) = \rho_0(\{-\}) = 1$. Compare the Hamiltonians

$$H(\alpha) = H(\sigma, m) = H_N(\eta, \gamma; \sigma) + N\gamma m^2/2 = N\left(\eta m_\sigma + \frac{\gamma}{2}(m^2 + m_\sigma^2)\right), \tag{13}$$

and

$$K(\alpha) = K(\sigma, m) = (\eta + \gamma m)m_\sigma\,. \tag{14}$$

Note that, assuming $\gamma \geq 0$,

$$H(\alpha) - K(\alpha) = \frac{\gamma}{2}(m - m_\sigma)^2 \geq 0\,. \tag{15}$$

If $\gamma \leq 0$ the opposite inequality is true. □

An important note is that all we have really used is the convexity of the map $m \mapsto m^2$, as with the proof of Lemma 1.1. We want to establish a variational principle, which requires that the bound is actually an equality. It is not quite true, rather what is true is that the bound becomes an equality in the limit $N \to \infty$. To state this correctly, we should first explain why $p(\eta,\gamma) = \lim_{N\to\infty} p_N(\eta,\gamma)$ exists.

Lemma 1.5 *If $(Q_N : N \in \mathbb{N}^*)$ is a subadditive sequence, then the following two limits exist and are equal*

$$\lim_{N\to\infty} \frac{Q_N}{N} = \lim_{N\to\infty} \limsup_{M\to\infty} \frac{Q_{M+N} - Q_M}{N}\,. \tag{16}$$

They may both equal $-\infty$. By taking negatives one sees that if $(Q_N : N \in \mathbb{N}^*)$ is a superadditive sequence then

$$\lim_{N\to\infty} \frac{Q_N}{N} = \lim_{N\to\infty} \liminf_{M\to\infty} \frac{Q_{M+N} - Q_M}{N}\,. \tag{17}$$

The limit may equal $+\infty$.

Proof. There is a rather unfortunate use of symbols in this proof, which I cannot set right. But everything is logically correct. Suppose that $M = kN + r$ where $k, r \in \mathbb{N}$. Then

$$\begin{aligned} \frac{Q_M - Q_r}{kN} &= \frac{Q_{kN+r} - Q_r}{kN} \\ &= \frac{1}{k}\sum_{j=1}^{k} \frac{Q_{jN+r} - Q_{(j-1)N+r}}{N} \\ &\leq \sup_{s\geq r} \frac{Q_{N+s} - Q_s}{N}\,. \end{aligned} \tag{18}$$

Now for any $M \in \mathbb{N}^*$ (and N fixed) decompose all integers $M > n$ as $M = k(M)N + r(M)$ with $r(M) \in \{n, \dots, n+N-1\}$, which uniquely specifies the decomposition. Then $Q_{r(M)} \in \{Q_n, \cdots, Q_{n+N-1}\}$ is clearly bounded independent of M. Since $|M - k(M)N|$ is also bounded, we have

$$\limsup_{M\to\infty} \frac{Q_M - Q_{r(M)}}{k(M)N} = \limsup_{M\to\infty} \frac{Q_M}{M}\,. \tag{19}$$

But also, since

$$\frac{Q_M - Q_{r(M)}}{k(M)N} = \frac{Q_{k(M)N+r(M)} - Q_{r(M)}}{k(M)N} \leq \sup_{s\geq r(M)} \frac{Q_{N+s} - Q_s}{N} \leq \sup_{s\geq n} \frac{Q_{N+s} - Q_s}{N}\,, \tag{20}$$

we obtain

$$\limsup_{M\to\infty} \frac{Q_M}{M} \leq \sup_{s\geq n} \frac{Q_{N+s} - Q_s}{N}\,. \tag{21}$$

Since this is true for any $n \in \mathbb{N}^*$ it is also true taking $n \to \infty$. I.e.,

$$\limsup_{M\to\infty} \frac{Q_M}{M} \leq \limsup_{n\to\infty} \frac{Q_{N+n} - Q_n}{N}\,. \tag{22}$$

Taking the $\liminf$ with respect to N, and rewriting things a little, gives We can rewrite this as

$$\boxed{\limsup_{N\to\infty} \frac{Q_N}{N} \leq \liminf_{N\to\infty} \limsup_{M\to\infty} \frac{Q_{M+N} - Q_M}{N}\,.} \tag{23}$$

By subadditivity, for any M,

$$\frac{Q_{M+N} - Q_M}{N} \leq \frac{Q_N}{N}\,; \tag{24}$$

hence,

$$\limsup_{M\to\infty} \frac{Q_{M+N} - Q_M}{N} \leq \frac{Q_N}{N}\,. \tag{25}$$

Finally, taking the $\liminf$ with respect to N gives

$$\boxed{\liminf_{N\to\infty} \limsup_{M\to\infty} \frac{Q_{M+N} - Q_M}{N} \leq \liminf_{N\to\infty} \frac{Q_N}{N}\,.} \tag{26}$$

The two boxed equations together prove that $\lim_{N\to\infty} Q_N/N$ exists in the extended sense that it may equal $\pm\infty$. But since every Q_N is finite, and by subadditivity, it cannot equal $+\infty$. Now, returning to equations (22) and (25), and using the fact that Q_N/N converges, we have

$$\lim_{N\to\infty} \frac{Q_N}{N} \leq \limsup_{M\to\infty} \frac{Q_{M+N} - Q_M}{N} \leq \frac{Q_N}{N}\,. \tag{27}$$

Therefore, by the sandwich theorem, the following limit exists and equals

$$\lim_{N\to\infty} \limsup_{M\to\infty} \frac{Q_{M+N} - Q_M}{N} = \lim_{N\to\infty} \frac{Q_N}{N}\,, \tag{28}$$

as was to be proved.□

With this we know that $p(\eta,\gamma) = \lim_{N\to\infty} p_N(\eta,\gamma)$ exists. By the lemma, we have the alternative formula

$$\gamma \geq 0 \;\Rightarrow\; p(\eta,\gamma) = \lim_{N\to\infty} \limsup_{M\to\infty} \frac{(M+N)p_{M+N}(\eta,\gamma) - Mp_M(\eta,\gamma)}{N}\,, \tag{29}$$

$$\gamma \leq 0 \;\Rightarrow\; p(\eta,\gamma) = \lim_{N\to\infty} \liminf_{M\to\infty} \frac{(M+N)p_{M+N}(\eta,\gamma) - Mp_M(\eta,\gamma)}{N}\,. \tag{30}$$

This brings in the possibility of using perturbation theory, when $M \gg N$. By perturbation theory I mean nothing more complicated than to use Taylor's theorem. This simplistic application of perturbation theory to a mean field model such as this one is typically called the *cavity field approach* by physicists. We will use this to prove the following:

Theorem 1.6 *If* $\gamma \geq 0$*, then*

$$p(\eta,\gamma) = \lim_{N\to\infty} \sup_{\mu\in\mathcal{M}_1([-1,1])} G_N(\eta,\gamma;\mu)\,. \tag{31}$$

If $\gamma \leq 0$*, then*

$$p(\eta,\gamma) = \lim_{N\to\infty} \inf_{\mu\in\mathcal{M}_1([-1,1])} G_N(\eta,\gamma;\mu)\,. \tag{32}$$

Proof. We will just treat the case $\gamma \geq 0$, leaving the other case to the reader. By Lemma 1.4, we know that

$$p(\eta,\gamma) \geq \lim_{N\to\infty} \sup_{\mu\in\mathcal{M}_1([-1,1])} G_N(\eta,\gamma;\mu)\,. \tag{33}$$

Because of equation (29), we can establish the opposite bound if we prove

$$\limsup_{M\to\infty} \frac{(M+N)p_{M+N}(\eta,\gamma) - Mp_M(\eta,\gamma)}{N} \leq \sup_{\mu\in\mathcal{M}_1([-1,1])} G_N(\eta,\gamma;\mu)\,. \tag{34}$$

This is exactly what we shall do, variationally.

For each $M \in \mathbb{N}^*$, define a measure $\mu \in \mathcal{M}_1([-1,1])$ by

$$\mu_M = \sum_{\substack{m=-1\\ \Delta m = 2/M}}^{1} \delta_m \binom{M}{(1-m)M/2} \exp\left(M\left[\eta m + \frac{\gamma}{2}\cdot\frac{M}{M+N}m^2\right]\right) . \tag{35}$$

Here δ_m is the Dirac measure at m. This is not a probability measure, but it could be made one by scaling because it is a finite measure. Since $G_N(\eta,\gamma;\mu) = G_N(\eta,\gamma;r\mu)$ for any $r>0$, the overall normalization is irrelevant. (We say that $G_N(\eta,\gamma;\cdot)$ is homogeneous of degree zero.) Note the strong resemblance, although not exact, to the distribution of m_σ with respect to the Boltzmann-Gibbs measure for $\sigma \in \mathbb{Z}_2^M$ distributed according to $H_M(\eta,\gamma;\sigma)$. But this is not exactly the measure. Note that

$$G_N(\eta,\gamma;\mu_M) = \frac{1}{N}\log\left(\int_{-1}^{1} d\mu_M(m) \sum_{\sigma\in\mathbb{Z}_2^N} e^{N(\eta+\gamma m)m_\sigma}\right) - \frac{1}{N}\log\left(\int_{-1}^{1} d\mu_M(m)\, e^{\gamma N m^2/2}\right) . \tag{36}$$

We call the first summand the numerator and the second summand the denominator, since that is the way they appear in the argument of the logarithm.

The denominator term satisfies the identity

$$\begin{aligned}\frac{1}{N}\log\int_{-1}^{1} d\mu_M(m) e^{\gamma N m^2/2} &= \frac{1}{N}\log \sum_{\substack{m=-1\\ \Delta m = 2/M}}^{1} \delta_m \binom{M}{(1-m)M/2} \exp\left(M\left[\eta m + \frac{\gamma}{2}\left(\frac{M}{M+N}+\frac{N}{M}\right)m^2\right]\right)\\ &= \frac{1}{N}\log \sum_{\substack{m=-1\\ \Delta m = 2/M}}^{1} \delta_m \binom{M}{(1-m)M/2} \exp\left(M\left[\eta m + \frac{\gamma}{2}\left(1+\frac{N^2}{M(M+N)}\right)m^2\right]\right)\end{aligned} \tag{37}$$

Because of this,

$$\begin{aligned}\left|\frac{1}{N}\int_{-1}^{1} d\mu_M(m)\, e^{\gamma N m^2/2} - \frac{M p_M(\eta,\gamma)}{N}\right| &\le \max_{-1\le m\le 1}\frac{1}{N}\left|\log e^{-\gamma M \frac{N^2}{M(M+N)}m^2/2}\right|\\ &= \frac{\gamma}{2}\cdot\frac{N}{M+N} ,\end{aligned} \tag{38}$$

and this error approaches zero as $M \to \infty$ with N fixed.

Similarly, the numerator is

$$\begin{aligned}\frac{1}{N}\log\int_{-1}^{1} d\mu_M(m) \sum_{\sigma\in\mathbb{Z}_2^N} e^{N(\eta+\gamma m)m_\sigma} &= \frac{1}{N}\log\sum_{\tau\in\mathbb{Z}_2^M}\sum_{\sigma\in\mathbb{Z}_2^N} e^{\eta(Mm_\tau+Nm_\sigma)}\exp\left(\frac{\gamma}{2}\left[\frac{M^2}{M+N}m_\tau^2 + 2Nm_\tau m_\sigma\right]\right)\\ &= \frac{1}{N}\log\sum_{\tau\in\mathbb{Z}_2^M}\sum_{\sigma\in\mathbb{Z}_2^N} e^{(M+N)[\eta m_{(\tau,\sigma)}+\frac{\gamma}{2}m_{(\tau,\sigma)}^2]}\exp\left(\frac{\gamma}{2}\cdot\frac{N^2}{M+N}(2m_\sigma m_\tau - m_\sigma^2)\right) .\end{aligned} \tag{39}$$

Here

$$m_{(\tau,\sigma)} = \frac{M}{M+N}m_\tau + \frac{N}{M+N}m_\sigma , \quad \text{so that} \quad m_{(\tau,\sigma)}^2 = \frac{M^2}{(M+N)^2}m_\tau^2 + \frac{2MN}{(M+N)^2}m_\sigma m_\tau + \frac{N^2}{(M+N)^2}m_\sigma^2 . \tag{40}$$

Therefore, one observes that

$$\begin{aligned}\left|\frac{1}{N}\int_{-1}^{1} d\mu_M(m) \sum_{\sigma\in\mathbb{Z}_2^N} e^{N(\eta+\gamma m)m_\sigma} - \frac{(M+N)p_{M+N}(\eta,\gamma)}{N}\right| &\le \max_{-1\le m\le 1}\frac{1}{N}\left|\log\exp\left(\frac{\gamma}{2}\cdot\frac{N^2}{M+N}(2m_\sigma m_\tau - m_\sigma^2)\right)\right|\\ &= \frac{3\gamma}{2}\cdot\frac{N}{M+N} ,\end{aligned} \tag{41}$$

Once again this approaches zero as $M \to \infty$ with N fixed. Therefore, we actually have

$$\lim_{M\to\infty} \left[G_N(\eta,\gamma;\mu_M) - \frac{(M+N)p_{M+N}(\eta,\gamma) - Mp_M(\eta,\gamma)}{N} \right] = 0\,. \tag{42}$$

This certainly implies that

$$\limsup_{M\to\infty} \frac{(M+N)p_{M+N}(\eta,\gamma) - Mp_M(\eta,\gamma)}{N} \le \sup_{\mu\in\mathcal{M}_1([-1,1])} G_N(\eta,\gamma;\mu)\,, \tag{43}$$

as was required to show. □

Homework 1. *Establish the other half of the theorem, for the antiferromagnetic case $\gamma \le 0$.*

1.1 Euler-Lagrange equation for $G_N(\eta,\gamma;\mu)$

We now consider the optimizer μ of $G_N(\eta,\gamma;\mu)$. Is there an optimizer? There is certainly an optimizing sequence of Borel probability measure $\mu_i \in \mathcal{M}_1([-1,1])$. Since the space $\mathcal{M}_1([-1,1])$ is weak-$*$ compact (i.e., compact in the vague topology), one can extract a convergent subsequence. But it is easy to verify that $\mu \mapsto G_N(\eta,\gamma;\mu)$ is bounded and continuous because the functions

$$m \mapsto \sum_{\sigma\in\mathbb{Z}_2^N} e^{N(\eta+\gamma m)m_\sigma} \quad \text{and} \quad m \mapsto e^{\eta N m^2/2} \tag{44}$$

are both continuous on $[-1,1]$ and their range is compactly supported in $(0,\infty)$, where the logarithm is continuous. Therefore, the limit of the convergent subsequence is an optimizer.

Now suppose that $\nu \in \mathcal{M}([-1,1])$ is any other finite, positive measure such that $\nu \ll \mu$, i.e., ν is absolutely continuous with respect to μ. Also suppose that the Radon-Nikodym derivative $f = d\nu/d\mu$ is uniformly bounded. If one prefers not to think about absolute continuity and Radon-Nikodym derivatives, just consider those ν obtained from μ by taking positive continuous functions $f \in \mathbb{C}([-1,1])$ and letting $\nu = f\cdot\mu$. This suffices for our purposes. Then for all $\varepsilon \in \mathbb{R}$ with $|\varepsilon|$ small enough, the measure $\mu+\varepsilon\nu$ is a finite, positive measure. Since $G_N(\eta,\gamma;\cdot)$ is homogeneous of degree zero, it does not matter that $\mu+\varepsilon\nu$ is not normalized (only that it could be normalized). Now, as always with the Euler-Lagrange equation, we consider ε small and Taylor expand. One can check that $G_N(\eta,\gamma;\mu+\varepsilon\nu)$ is analytic in ε for ε small enough. Moreover,

Homework 2. *Check that*

$$G_N(\eta,\gamma;\mu+\varepsilon\nu) = G_N(\eta,\gamma;\mu) + \frac{\varepsilon}{N}\cdot\frac{\int d\nu(m)\,\sum_\sigma e^{N(\eta+\gamma m)m_\sigma}}{\int d\mu(m)\,\sum_\sigma e^{N(\eta+\gamma m)m_\sigma}} - \frac{\varepsilon}{N}\cdot\frac{\int d\nu(m)\,e^{\gamma N m^2/2}}{\int d\mu(m)\,e^{\gamma N m^2/2}} + O(\varepsilon^2)\,. \tag{45}$$

Deduce that if μ is an optimizer, then for any $\nu \ll \mu$, one has $G_N(\eta,\gamma;\nu) = G_N(\eta,\gamma;\mu)$.

Because of this identity and continuity of $G_N(\eta,\gamma;\cdot)$ in the weak-$*$ topology, one deduces that for any $m \in \operatorname{supp}\mu$, $G_N(\eta,\gamma;\delta_m) = G_N(\eta,\gamma;m)$. This implies that (for $\gamma \ge 0$)

$$p(\gamma,\eta) = \lim_{N\to\infty}\min_{m\in[-1,1]} G_N(\eta,\gamma;\delta_m)\,. \tag{46}$$

But clearly

$$G_N(\eta,\gamma;\delta_m) = \log\cosh(\eta+\gamma m) \,-\, \frac{\gamma m^2}{2}\,. \tag{47}$$

Therefore, we have established the following result (and the analogue for $\gamma \le 0$ follows analogously)

Corollary 1.7 *If $\gamma \ge 0$, then*

$$p(\gamma,\eta) = \max_{m\in[-1,1]} \log\cosh(\eta+\gamma m) \,-\, \frac{\gamma m^2}{2}\,. \tag{48}$$

If $\gamma \le 0$, then

$$p(\gamma,\eta) = \min_{m\in[-1,1]} \log\cosh(\eta+\gamma m) \,-\, \frac{\gamma m^2}{2}\,. \tag{49}$$

References

[1] M Aizenman, R. Sims and S. Starr, An Extended Variational Principle for the Sherrington-Kirkpatrick Model to appear in *Phys. Rev. B*, submitted June 2003.

[2] F. Guerra, Broken Replica Symmetry Bounds in the Mean Field Spin Glass Model. *Commun. Math. Phys.* **233** (2003), 1–12.

[3] F. Guerra and F. Toninelli. The thermodynamic limit in mean field spin glass models. *Comm. Math. Phys.*, 230(1):71–79, 2002.