A conjecture for Kingman partition structures, invariant under normalized Bernoulli-p thinning

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What's a "partition"?

If $n \in \mathbb{N}_+ = \{1, 2, 3, \ldots\}$, then a partition is a way of writing n as a sum of positive integers, if one does not care about the order of the summands:

$$\mathrm{Par}(4) = \{(4)\,,\ (3,1)\,,\ (2,2)\,,\ (2,1,1)\,,\ (1,1,1,1)\}$$

One sometimes abbreviate by putting the multiplicity of an entry in the exponent.
E.g.,

$$\mathrm{Par}(4) = \{(4)\,,\ (3,1)\,,\ (2^2)\,,\ (2,1^2)\,,\ (1^4)\}$$

One sometime writes $\lambda \vdash n$ for $\lambda \in \mathrm{Par}(n)$.

One usually writes $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$ for the "parts".

The length of the partition is $l = l(\lambda)$.

In case it is not otherwise clear, the norm of the partition is written $n = n(\lambda)$.

One makes a Young diagram as follow

$$\mathrm{Par}(4) = \Big\{ (4) \;\; \bullet\bullet\bullet\bullet \;,\; (3,1) \;\; \begin{matrix}\bullet&\bullet&\bullet\\ \bullet&&\end{matrix} \;,$$

$$(2^2) \;\; \begin{matrix}\bullet&\bullet\\ \bullet&\bullet\end{matrix} \;,\; (2,1^2) \;\; \begin{matrix}\bullet&\bullet\\ \bullet&\\ \bullet&\end{matrix} \;,\; (1^4) \;\; \begin{matrix}\bullet\\ \bullet\\ \bullet\\ \bullet\end{matrix} \Big\}$$

A little about set partitions

Define the set $X_n = \{1, \ldots, n\}$.
A partition of X_n is an (unordered) tuple of subsets $Y_1, \ldots, Y_l$, for some $l \le n$, with

$$
\begin{aligned}
Y_i &\neq \emptyset \quad \text{for} \quad 1 \le i \le l\,; \\
Y_i \cap Y_j &= \emptyset \quad \text{for} \quad 1 \le i < j \le l\,; \\
\text{and } X_n &= Y_1 \cup \cdots \cup Y_l\,.
\end{aligned}
$$

This is equivalent to an equivalence relation $\sim$ on X_n, where the Y_i's are equivalence classes.
Ranking the cardinalities $|Y_1|, \ldots, |Y_l|$ in descending order gives a partition $\lambda \in \mathrm{Par}(n)$.

Random subpartitions

J.F.C. Kingman was motivated to define an exchangeable "partition structure" by applications in population genetics.

Let $\lambda \in \mathrm{Par}(n)$, and let $\{Y_1, \ldots, Y_l\}$ be any set partition of X_n having ranked cardinalities equal to $\lambda_1, \ldots, \lambda_l$.
Choose $i \in X_n$, randomly, uniformly.
Consider the induced set partition $\{Y'_1, \ldots, Y'_{l'}\}$ of $X_n \setminus \{i\}$.

Let $\mu \in \mathrm{Par}(n-1)$ be the *random* partition corresponding to the ranked cardinalities of this induced set partition.

Let

$$S^{\,n}_{n-1}\binom{\lambda}{\mu}$$

be the conditional probability of $\mu \in \mathrm{Par}(n-1)$, given $\lambda \in \mathrm{Par}(n)$. E.g.

$$\lambda \;=\; \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & & \end{array} \;=\; (3,2,2,1) \in \mathrm{Par}(8)\,,$$

Then

$$S^{\,8}_{7}\binom{\lambda}{\mu} \;=\; \begin{cases} 3/8\,, & \mu = \begin{array}{ll} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array} = (2,2,2,1)\,; \\ 4/8\,, & \mu = \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \\ \bullet & & \end{array} = (3,2,1,1)\,; \\ 1/8\,, & \mu = \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \end{array} = (3,2,2)\,; \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Kingman's "exchangeable partition structure"

Let $\mathcal{M}_1(\mathrm{Par}(n))$ be the set (simplex) of probability measures on the finite set $\mathrm{Par}(n)$.

Our convention is $P_n \in \mathcal{M}_1(\mathrm{Par}(n))$ iff $P_n : \mathrm{Par}(n) \to \mathbb{R}$ such that

$$\forall \lambda \in \mathrm{Par}(n)\,, \quad P_n(\lambda) \geq 0\,; \quad \text{and} \quad \sum_{\lambda \in \mathrm{Par}(n)} P_n(\lambda) = 1\,.$$

Define a linear map $\pi^n_{n-1} : \mathcal{M}_1(\mathrm{Par}(n)) \to \mathcal{M}_1(\mathrm{Par}(n-1))$, such that, if $P_n \in \mathcal{M}_1(\mathrm{Par}(n))$, then

$$\forall \mu \in \mathrm{Par}(k)\,, \qquad (\pi^n_{n-1} P_n)(\mu) = \sum_{\lambda \in \mathrm{Par}(n)} P_n(\lambda) S^{\,n}_{n-1}\binom{\lambda}{\mu}\,.$$

Extend the definition to π^n_k for $1 \leq k \leq n$ as

$$\pi^n_n = \mathrm{Id}_{\mathcal{M}_1(\mathrm{Par}(n))}\,, \quad \text{and} \quad \pi^n_k = \pi^{k+1}_k \pi^{k+2}_{k+1} \cdots \pi^n_{n-1} \quad \text{for } k < n\,. \qquad (1)$$

A **Kingman partition structure** is a sequence of probability measures

$$P = (P_n \in \mathcal{M}_1(\mathrm{Par}(n)) \,:\, n \in \mathbb{N}_+)$$

such that

$$\forall k \leq n\,, \qquad P_k = \pi^n_k P_n\,.$$

These form a projective system, with projections π^n_{n-1}.
By the Daniell-Kolmogorov extension theorem, there exists the projective limit $\mathbb{P}^P \in \mathcal{M}_1(\bigtimes_{n\in\mathbb{N}_+} \mathrm{Par}(n))$, satisfying

- the marginal of $\mathbb{P}^P$ on Par(n) is P_n, and
- $\mathbb{P}^P[\lambda^{(n-1)}|\lambda^{(n)}, \lambda^{(n+1)}, \ldots] = S^{\,n}_{n-1}\binom{\lambda^{(n)}}{\lambda^{(n-1)}}$.

Note, it is easy to make a finite sequence satisfying the consistency conditions. For any $n \in \mathbb{N}_+$ and any $P_n \in \mathcal{M}_1(\mathrm{Par}(n))$, define

$$P_k = \pi^n_k P_n \quad \text{for } k = 1, \ldots, n-1\,.$$

Then $(P_k \,:\, k \leq n)$ satisfies $P_j = \pi^k_j P_k$ for $j \leq k \leq n$.

Main examples: Kingman's "paintbox partitions"

Given a discrete (pure point / purely atomic) probability measure, label the atoms in nonincreasing order

$$\mathcal{X} = (\xi_n : n \in \mathbb{N}_+), \quad \xi_1 \geq \xi_2 \geq \cdots \geq 0, \quad \sum_{n \in \mathbb{N}_+} \xi_n = 1.$$

All such sequences make a convex subset of $[0,1]^{\mathbb{N}_+}$ (in fact a Choquet simplex) which we call Δ_1.

For each $\mathcal{X} \in \Delta_1$, define $\mathbb{P}^{\mathcal{X}} \in \mathcal{M}_1(\mathbb{N}_+)$ such that $\mathbb{P}^{\mathcal{X}}(\{n\}) = \xi_n$.
Let $(a_1, \ldots, a_n)$, be i.i.d., $\mathbb{P}^{\mathcal{X}}$-distributed, and define a random equivalence relation on X_n by $i \sim j$ iff $a_i = a_j$.
Let $\lambda \in \mathrm{Par}(n)$ be the ranked cardinalities of the equivalence classes $Y_1, \ldots, Y_l \subset X_n$.
Define $P_n^{\mathcal{X}}$ to be the distribution of such random partitions λ.
This is called a "paintbox partition structure".

The closure of Δ_1 in $[0,1]^{\mathbb{N}}$ is the set of all sequences

$$\mathcal{X} = (\xi_n : n \in \mathbb{N}_+), \quad \xi_1 \geq \xi_2 \geq \cdots \geq 0, \quad \sum_{n \in \mathbb{N}_+} \xi_n \leq 1.$$

We call this set $\Delta_{\leq 1}$.

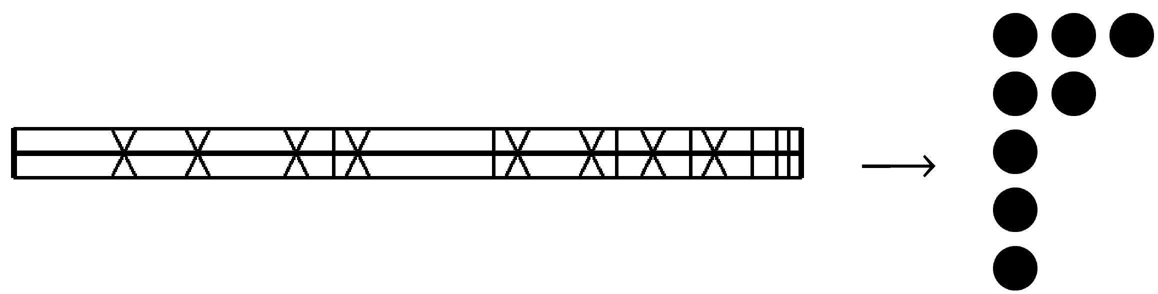
It is a compact, convex set (in fact a Bauer simplex).

One can extend the map $\mathcal{X} \mapsto P_n^{\mathcal{X}}$, continuously to $\Delta_{\leq 1}$.

E.g., for a purely continuous measure, $\xi_n = 0$ for all $n \in \mathbb{N}_+$, and $P_n^{\mathcal{X}} = \delta_{(1^n)}$.

We interpret each $\mathcal{X} \in \Delta_{\leq 1}$ as a Gibbs measure, with a continuous component iff $\sum_n \xi_n < 1$, in which case the total mass of the continuum is $1 - \sum_n \xi_n$.

Eg.2)



Kingman's representation theorem

Kingman showed that the set of all partition structures,

$$\mathcal{D} \;=\; \{P = (P_n \in \mathcal{M}_1(\mathrm{Par}(n)) \;:\; n \in \mathbb{N}_+) \;:\; \forall k \le n\,,\; P_k = \pi^n_k P_n\}\,,$$

is in correspondence to the set $\mathcal{M}_1(\Delta_{\le 1})$.
Particularly, every $P \in \mathcal{D}$ can be decomposed as P^ρ, for a unique $\rho \in \mathcal{M}_1(\Delta_{\le 1})$, where

$$\forall n \in \mathbb{N}_+\,, \qquad P^\rho_n \;:=\; \int_{\Delta \le 1} \rho(d\mathcal{X})\, P^{\mathcal{X}}_n\,.$$

In other words, Kingman showed that $\mathcal{D}$ is a Choquet simplex, and identified the extreme points, $\mathcal{E}(\mathcal{D}) = \{P^{\mathcal{X}} \;:\; \mathcal{X} \in \Delta_{\le 1}\}$.
Note, the decomposition is unique because of the LLN or Glivenko-Cantelli theorem:

$$\forall k \in \mathbb{N}_+\,,\; \forall \epsilon > 0\,,\; \lim_{n\to\infty} P^{\mathcal{X}}_n[|n^{-1}\,\lambda_k - \xi_k| > \epsilon] = 0\,.$$

Kingman's exchangeable partitions of $\mathbb{N}_+$

Define the set of all set partitions of $\mathbb{N}_+$ to be $\mathrm{Par}(\mathbb{N}_+)$.
A probability distribution on $\mathrm{Par}(\mathbb{N}_+)$, is said to be *exchangeable* if, for each $n \in \mathbb{N}_+$, the induced distribution of set-partitions $\Pi_n \in \mathrm{Par}(X_n)$ is invariant under S_n.
Given, Π, let λ_n be the ranked cardinalities of Π_n, so that $(\lambda_n \ : \ n \in \mathbb{N}_+) \in \bigtimes_{n\in\mathbb{N}_+} \mathrm{Par}(n)$.
Given P, there is a unique exchangeable distribution $\mathbb{P}^P_{\mathbb{N}_+}$ on $\mathrm{Par}(\mathbb{N}_+)$ such that the marginal on $\bigtimes_{n\in\mathbb{N}_+} \mathrm{Par}(n)$ is given by $\mathbb{P}^P$.

An example of a part of a set partition of $\mathbb{N}_+$:

$$\Pi = \{Y_n \ : \ n \in \mathbb{N}_+\}\,,$$

$$\begin{array}{lcl}
Y_1 & = & \{1, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots\} \\
Y_2 & = & \{ \quad 2, \qquad\qquad\qquad\qquad\qquad 8, \qquad\qquad\qquad\qquad\qquad \ldots\} \\
Y_3 & = & \{ \qquad\quad 3, \qquad\qquad\qquad\qquad\qquad\qquad\qquad 11, \qquad\quad \ldots\} \\
Y_4 & = & \{ \qquad\qquad\quad 4, \qquad\qquad 7, \qquad\qquad 10, \qquad\qquad\qquad \ldots\} \\
Y_5 & = & \{ \qquad\qquad\qquad\quad 5, \qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots\} \\
Y_6 & = & \{ \qquad\qquad\qquad\qquad\quad 6, \qquad\qquad\qquad\qquad\qquad\qquad \ldots\} \\
Y_7 & = & \{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 9, \qquad\qquad 12, \quad \ldots\} \\
\ldots & &
\end{array}$$

Normalized Bernoulli-p thinning

Suppose P is a partition structure, such that $\mathbb{P}^P_{\mathbb{N}_+}$ is supported on partitions $\Pi = (Y_n \ : \ n \in \mathbb{N}_+)$ with an infinite number of nonempty parts.

For each $n \in \mathbb{N}_+$, define an independent random variable $\mathfrak{t}_n$, uniformly distributed in $[0, 1]$.

Given $p \in (0, 1]$, consider the subset

$$\mathrm{N}_p = \bigcup_{n\in\mathbb{N}_+ \, \mathfrak{t}_n \leq p} Y_n \, . \tag{2}$$

With probability 1, the number of $n \in \mathbb{N}_+$ with $\mathfrak{t}_n \leq p$ is infinite.

Therefore, N_p is infinite, almost surely. Let $f_p : \mathrm{N}_p \to \mathbb{N}_+$ be the order-preserving bijection, and let $\widetilde{\Pi}^p$ be the induced set partition of $\mathbb{N}_+$, given by $(f_p(Y_n) \ : \ \mathfrak{t}_n \leq p)$.

It is trivial to see that the law of $\widetilde{\Pi}^p$ is also an exchangeable distribution of set partitions.
Moreover, there is an induced measure on $\bigtimes_{n\in\mathbb{N}_+} \mathrm{Par}(n)$ given by the ranked cardinalities of parts of $(\widetilde{\Pi}^p)_n$.
This corresponds to a new partition structure $\widetilde{P}^p$.
One can calculate that, for any $\lambda \in \mathrm{Par}(n)$, with $l = l(\lambda)$,
$(q = 1 - p \in [0, 1))$

$$\widetilde{P}^p_n(\lambda) = \sum_{m=n}^{\infty} \frac{n}{m} \sum_{k=l}^{\infty} (1-q)^l q^{k-l} \sum_{\mu \in \mathrm{Par}(m,k)} C^m_n \binom{\mu}{\lambda} P_m(\mu)\,, \tag{3}$$

where $C^m_n\binom{\mu}{\lambda}$ is a combinatorial factor

$$C^m_n\binom{\mu}{\lambda} = \prod_{i=1}^{\infty} \binom{a_i(\mu)}{a_i(\lambda)}, \tag{4}$$

where we write $\lambda = (\lambda_1, \ldots, \lambda_l) = (1^{a_1}, 2^{a_2}, \ldots)$.

We call a partition structure $P \in \mathcal{D}$, "continuously thinning invariant" if the distribution $\widetilde{P}^p$ equals P for all $p \in (0, 1]$.

It is clear that this family of partition structures forms a simplex.

The map "normalized Bernoulli-p thinning" has not been defined on partition structures with only finitely many parts.

But one can extend the definition, not so as to be continuous, but so as to be the boundary value of a family of continuous maps: essentially, if in the process of thinning, you remove all parts, then you put all parts back.

Easy to see that with this extension, any continuously thinning invariant process with a positive probability to have only finitely many parts has conditional probability, conditioned on having only finitely many parts, equal identically to $\Pi = \{\mathbb{N}_+\}$.

I.e., $P_n = \delta_{(n)}$ for all n, if P_n is the conditional distribution of λ_n.

Then it is true that the invariant simplex is compact, i.e., a Bauer simplex.

The Conjecture

The Bauer simplex of continuously-thinning-invariant partition structures $\mathcal{D}^I \subset \mathcal{D}$ is spanned by the second branch of Poisson-Dirichlet processes, defined by Pitman and Yor, $\mathcal{E}(\mathcal{D}^I) = \{\mathrm{PD}(\alpha, 0) \,:\, \alpha \in [0, 1]\}$.

These are the partition structures which (are rigorously proved to) describe the limiting random Gibbs distributions of Bernard Derrida's random energy model.

Kingman defined a one-parameter family of partition structures which are ubiquitous in genetical models, $\mathrm{PD}(\theta)$, $\theta \geq 0$.
Pitman and Yor later defined a two-parameter generalization $\mathrm{PD}(\alpha, \theta)$, defined for $\alpha \in [0, 1]$ and $\theta \geq -\alpha$.
In this notation, $\mathrm{PD}(\theta) = \mathrm{PD}(0, \theta)$.

Following Pitman, we will construct $\mathrm{PD}(\alpha, 0)$ for $0 < \alpha < 1$ as the "Poisson-Kingman" distributions derived from an α-stable subordinator.
The partition structure $\mathrm{PD}(0, 0)$ is the process such that, almost surely, $\Pi = \{\mathbb{N}_+\}$.
The partition structure $\mathrm{PD}(1, 0)$ is the process such that, almost surely, $\Pi = \{\{n\} : n \in \mathbb{N}_+\}$.

What's a "Poisson-Kingman" process?

One can make a big family of partition structures in the following way. Start with a *Levy measure* Λ on $\mathbb{R}_+ = (0, \infty)$, satisfying that

A1. for any $M < \infty$, $\int_0^M x\,\Lambda(dx) < \infty$;

A2. for any $\epsilon > 0$, $\Lambda((\epsilon, \infty)) < \infty$;

A3. $\Lambda(\mathbb{R}_+) = \infty$.

(All the requirements to be a Levy measure are subsumed by the stronger requirements of A1-A3.)

Next, make a Poisson point process $X = (x_1 \geq x_2 \geq \cdots > 0)$ which is a random point set (possibly with multiplicity) in $\mathbb{R}_+$, satisfying that for any Borel set E,

$$\mathbb{P}[\#\{n \in \mathbb{N}_+ \,:\, x_n \in E\} = k] \;=\; e^{-\Lambda(E)} \frac{\Lambda(E)^n}{n!} \,.$$

Because of the assumptions, A1-A3, it is the case that, almost surely,

1. there are an infinite number of $n \in \mathbb{N}_+$ with $x_n \geq 0$,
2. the sum, $Z = \sum_{n=1}^{\infty} x_n$, is finite,
3. the expectation of Z is infinite.

Of course, as long as at least one $x_1 > 0$, we know $Z > 0$.

A Poisson-Kingman process is define by taking $\mathcal{X} = (\xi_n \ : \ n \in \mathbb{N}_+)$, where $\xi_n = x_n/Z$ for all $n \in \mathbb{N}_+$.

Many things can be calculated in terms of the function $\psi : \mathbb{R}_+ \to [0, \infty]$ defined as

$$\psi(y) \;=\; \int_0^\infty (1 - e^{-xy})\, \Lambda(dx)\,.$$

Particularly, one can see that for $\lambda \in \mathrm{Par}(n, l)$,

$$P_n(\lambda) \;=\; \frac{(-1)^{n+l}}{(n-1)!l!} \binom{n}{\lambda_1, \ldots, \lambda_l} \binom{l}{a_1(\lambda), a_2(\lambda), \ldots} \int_0^\infty \exp(-\psi(y)) \prod_{j=1}^{l} \left[y^{\lambda_j} \psi^{(\lambda_j)}(y) \right] \frac{dy}{y}\,.$$

The Poisson-Dirichlet processes $\mathrm{PD}(\alpha, 0)$

The Poisson-Dirichlet processes, $\mathrm{PD}(\alpha, 0)$, $0 < \alpha < 1$ are obtained by taking $\Lambda_\alpha(dx) = \Gamma(1-\alpha)^{-1} \frac{d}{dx}(-x^{-\alpha})\, dx$.
It is easy to see that this satisfies A1-A3 precisely when $0 < \alpha < 1$.
It is also easy to see that then

$$\psi_\alpha(y) \;=\; y^\alpha \,.$$

This implies that for $P = \mathrm{PD}(\alpha, 0)$,

$$P_n(\lambda) \;=\; \frac{1}{\alpha}\,\frac{n}{l}\binom{l}{a_1(\lambda), a_2(\lambda), \ldots}\prod_{j=1}^{l}(-1)^{\lambda_j - 1}\binom{\alpha}{\lambda_j}.$$

It is also easy to see that this is continuously thinning invariant. Indeed, the infinitesimal version of the condition of thinning invariance for these processes is the identity

$$\sum_{k=1}^{\infty}(-1)^{k-1}\binom{\alpha}{k} = 1\,,$$

which is true (and the series is absolutely summable) as long $\mathrm{Re}(\alpha) > 0$.

Also, using the characterization of P for arbitrary Poisson-Kingman processes, one can determine that for Poisson-Kingman processes, the requirement to be continuously thinning invariant is that $\psi(y) \propto y^{\alpha}$ for some α. Since $0 < \alpha < 1$ are the only possible values for a Levy measure satisfying A1–A3, this proves that the only possibly Poisson-Kingman processes which are continuously thinning invariant are $\mathrm{PD}(\alpha, 0)$, $0 < \alpha < 1$.