MATH 585 – TOPICS IN MATHEMATICAL PHYSICS – FALL 2006
MATHEMATICS OF MEAN FIELD SPIN GLASSES AND THE REPLICA METHOD
LECTURE 12': PARISI'S ANSATZ (REWRITE)

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CONTENTS

Let us begin to write matrices using boldfaced symbols such as $\boldsymbol{q} = (q_{ij})_{i,j=1}^n$.

1. SHERRINGTON AND KIRKPATRICK'S ANSATZ

Last time we used the letters a and b to indicate indices running from 1 to n, and we put such indices in the superscript. This was to avoid confusion with indices running from 1 to N, which we put in the subscript. Those latter are no longer present, so let us now write i and j for the indices from 1 to n and put them in the subscript.

Last time we defined

$$\mathcal{P}(\beta, x; n) \;=\; \lim_{N\to\infty} \frac{1}{N} \log\left(\mathbb{E}\left[\left(\mathsf{Z}_N(\beta,x)\right)^n\right]\right),$$

and we proved that

$$\mathcal{P}(\beta, x; n) \;=\; \sup_{\boldsymbol{q}\in \mathrm{Ov}(n)} \left[-\frac{\beta^2}{4}\sum_{i,j=1}^n q_{ij}^2 + \log\left(\sum_{\sigma\in\Omega^n} e^{x\sum_{i=1}^n \sigma_i + \frac{\beta^2}{2}\sum_{i,j=1}^n q_{ij}\sigma_i\sigma_j}\right)\right],$$

where $\mathrm{Ov}(n)$ consists of those $n\times n$ real matrices $\boldsymbol{q} = (q_{ij})_{i,j=1}^n$ which are positive semi-definite, and have $q_{i,i} = 1$ for all $i = 1, \dots, n$. (It is better not to think of the matrix $\mathbf{q}$ as an algebraic object with the usual rules of matrix multiplication, but as an $n\times n$ array of numbers, or as a bilinear form.) Let us define

$$F_n(\beta, x; \boldsymbol{q}) \;=\; F_n^{(1)}(\beta, x; \boldsymbol{q}) - F_n^{(2)}(\beta; \boldsymbol{q}),$$

where

$$F_n^{(1)}(\beta, x; \boldsymbol{q}) \;=\; \log\left(\sum_{\sigma\in\Omega^n} e^{x\sum_{i=1}^n \sigma_i + \frac{\beta^2}{2}\sum_{i,j=1}^n q_{ij}\sigma_i\sigma_j}\right),$$

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and

$$F_n^{(2)}(\beta; \boldsymbol{q}) \;=\; \frac{\beta^2}{4} \sum_{i,j=1}^n q_{ij}^2 \,.$$

Then

$$\mathcal{P}(\beta, x; n) \;=\; \sup_{\boldsymbol{q} \in \mathrm{Ov}(n)} F_n(\beta, x; \boldsymbol{q}) \,.$$

Sherrington and Kirkpatrick knew this formula, although they arrived at it by different means than we did. They then considered the ansatz that

$$q_{ij} \;=\; \begin{cases} 1 & \text{if } i = j; \\ q_0 & \text{if } i \neq j; \end{cases}$$

for some real number $q_0 \in [0, 1]$. This is a major restriction, when all one knows *a priori* is that $\boldsymbol{q} \in \mathrm{Ov}(n)$, but it has the advantage that it is then easier to calculate $F_n(\boldsymbol{q})$. Let us define new matrices $\boldsymbol{Q}^{(0)}$ and $\boldsymbol{Q}^{(1)}$ so that

$$Q_{ij}^{(0)} \;=\; 1 \;\text{ for all } i, j \in \{1, \dots, n\}, \quad \text{and} \quad Q_{ij}^{(1)} \;=\; \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Then $\boldsymbol{q} = q_0 \boldsymbol{Q}^{(0)} + (1 - q_0) \boldsymbol{Q}^{(1)}$. Recall the definition of the inner product of two matrices that we defined before,

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle \;=\; \sum_{i,j=1}^n A_{ij} B_{ij} \,.$$

Because of the form of these two matrices,

$$\|\boldsymbol{Q}^{(0)}\|^2 \;=\; n^2 \quad \text{while} \quad \|\boldsymbol{Q}^{(1)}\|^2 \;=\; \langle \boldsymbol{Q}^{(0)}, \boldsymbol{Q}^{(1)} \rangle \;=\; n \,.$$

Therefore

$$\begin{aligned} \|\boldsymbol{q}\|^2 \;&=\; q_0^2 \|\boldsymbol{Q}^{(0)}\|^2 + (1 - q_0)^2 \|\boldsymbol{Q}^{(1)}\|^2 + 2q_0(1 - q_0) \langle \boldsymbol{Q}^{(0)}, \boldsymbol{Q}^{(1)} \rangle \\ &=\; n^2 q_0^2 + n[(1 - q_0)^2 + 2q_0(1 - q_0)] \\ &=\; n^2 q_0^2 + n(1 - q_0^2) \,. \end{aligned}$$

Hence,

$$F_n^{(2)}(\beta; \boldsymbol{q}) \;=\; \frac{\beta^2}{4} \|\boldsymbol{q}\|^2 \;=\; \frac{\beta^2}{4} [q_0^2 n^2 + (1 - q_0^2) n] \,.$$

We will use the same decomposition, $\boldsymbol{q} = q_0 \boldsymbol{Q}^{(0)} + (1 - q_0) \boldsymbol{Q}^{(1)}$, for calculating $F_n^{(1)}(\beta, x; \boldsymbol{q})$. We see that

$$\sum_{a,b=1}^n q_{a,b} \sigma_a \sigma_b \;=\; q_0 \left(\sum_{a=1}^n \sigma_a \right)^2 + (1 - q_0) \sum_{a=1}^n \sigma_a^2 \,.$$

Of course, the second sum is trivially n. But instead of writing it as n, let us leave it the way it is. To some extent, this keeps the second term looking as much like the first term as possible. Then we have

$$F_n^{(1)}(\boldsymbol{q}) \;=\; \log\left(\sum_{\sigma\in\Omega^n}\exp\left(x\sum_{i=1}^n\sigma_i+\frac{\beta^2 q_0}{2}\left(\sum_{i=1}^n\sigma_i\right)^2+\frac{\beta^2(1-q_0)}{2}\sum_{i=1}^n(\sigma_i)^2\right)\right)\,.$$

The following calculation is involved enough to warrant setting it apart as a lemma.

Lemma 1.1 *Let $\mathsf{X}^{(0)}$ and $\mathsf{X}^{(1)}$ be i.i.d., $N(0,1)$ random variables, and let $\mathbb{E}^{(0)}$ and $\mathbb{E}^{(1)}$ denote the expectation with respect to these two random variables. Then*

$$F_n^{(1)}(\beta,x;\boldsymbol{q}) \;=\; \log\left(\mathbb{E}^{(0)}\left[\left(\mathbb{E}^{(1)}\left[2\cosh\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}^{(1)}\right)\right]\right)^n\right]\right)\,.$$

Proof. Note the simple, but important, identity

$$e^{x^2/2} \;=\; \mathbb{E}[e^{x\mathsf{X}}]\,,$$

valid for every real x, where X is a standard normal random variable. We introduce $n+1$ i.i.d. standard normal random variables $\mathsf{X}^{(0)}$ and $\mathsf{X}_1^{(1)},\dots,\mathsf{X}_n^{(1)}$, and write $\mathbb{E}^{(0)},\mathbb{E}_1^{(1)},\dots,\mathbb{E}_n^{(1)}$ to denote the average over these random variables. Then

$$\begin{aligned}
&\exp\left(x\sum_{i=1}^n\sigma_i+\frac{\beta^2 q_0}{2}\left(\sum_{i=1}^n\sigma_i\right)^2+\frac{\beta^2(1-q_0)}{2}\sum_{i=1}^n(\sigma_i)^2\right)\\
&\quad=\;\exp\left(x\sum_{i=1}^n\sigma_i\right)\,\mathbb{E}^{(0)}\left[\exp\left(\beta\sqrt{q_0}\,\mathsf{X}^{(0)}\sum_{i=1}^n\sigma_i\right)\right]\prod_{i=1}^n\mathbb{E}^{(0)}\left[\exp\left(\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\sigma_i\right)\right]\\
&\quad=\;\mathbb{E}^{(0)}\left[\prod_{i=1}^n\mathbb{E}_i^{(1)}\left[\exp\left(\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\sigma_i\right)\right]\right]\,.
\end{aligned}$$

If we now consider the sum over all σ, and interchange the sum with $\mathbb{E}^{(0)}$, first, and then with all the $\mathbb{E}_i^{(1)}$, next, we obtain (we are writing every small step of this calculation)

$$\begin{aligned}
&\sum_{\sigma\in\Omega^n}\mathbb{E}^{(0)}\left[\prod_{i=1}^n\mathbb{E}_i^{(1)}\left[\exp\left(\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\sigma_i\right)\right]\right]\\
&\quad=\;\mathbb{E}^{(0)}\left[\sum_{\sigma\in\Omega^n}\prod_{i=1}^n\mathbb{E}_i^{(1)}\left[\exp\left(\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\sigma_i\right)\right]\right]\\
&\quad=\;\mathbb{E}^{(0)}\left[\prod_{i=1}^n\left(\sum_{\sigma_i\in\{+1,-1\}}\mathbb{E}_i^{(1)}\left[\exp\left(\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\sigma_i\right)\right]\right)\right]\\
&\quad=\;\mathbb{E}^{(0)}\left[\prod_{i=1}^n\mathbb{E}_i^{(1)}\left[\sum_{\sigma_i\in\{+1,-1\}}\exp\left(\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\sigma_i\right)\right]\right]\\
&\quad=\;\mathbb{E}^{(0)}\left[\prod_{i=1}^n\mathbb{E}_i^{(1)}\left[\cosh\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\right]\right]\,.
\end{aligned}$$

Now, introducing a new, independent $N(0,1)$ random variable $\mathsf{X}^{(1)}$, and letting $\mathbb{E}^{(1)}$ denote its expectation, we have

$$\mathbb{E}_i^{(1)}\left[\cosh\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\right] \,=\, \mathbb{E}^{(1)}\left[\cosh\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}^{(1)}\right)\right]\,,$$

for all $i=1,\dots,n$. Therefore, we have

$$\begin{aligned}\mathbb{E}^{(0)}&\left[\prod_{i=1}^{n}\mathbb{E}_i^{(1)}\left[\cosh\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\right]\right]\\ &\qquad=\,\mathbb{E}^{(0)}\left[\left(\mathbb{E}^{(1)}\left[\cosh\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}^{(1)}\right)\right]\right)^n\right]\,.\end{aligned}$$

Putting this together with the previous formulas, and inserting in the definition of $F_n^{(1)}$ gives the desired formula. □

Although we already have a simpler formula for $F_n^{(2)}(\beta;\boldsymbol{q})$, we can also write it in a similar form

$$F_n^{(2)}(\beta;\boldsymbol{q})\,=\,\log\left(\mathbb{E}^{(0)}\left[\left(\mathbb{E}^{(1)}\left[\exp\left(\beta\sqrt{\frac{q_0^2}{2}}\,\mathsf{X}^{(0)}+\beta\sqrt{\frac{1-q_0^2}{2}}\,\mathsf{X}^{(1)}\right)\right]\right)^n\right]\right)\,.$$

This will be important later on.

Exercise: *Prove that formula.*

Remark 1.2 If time had permitted we would have talked about this approach, called the Hubbard-Stratonovich transformation, to give yet another solution of the Curie-Weiss model. Indeed, the method was the beginning step of an important investigation by Ellis and Newman [1] that we should have discussed.

2. THE "REPLICA TRICK"

We will present the version of the replica trick as in the paper of van Hemmen and Palmer.

We begin, for finite N, by generalizing the definition of the nth moment for $n=0,1,2,\dots$. Indeed, for arbitrary $r\in\mathbb{R}$, define

$$\mathcal{P}_N(\beta,x;r)\,=\,\frac{1}{N}\log\left(\mathbb{E}\left[\left(\mathsf{Z}_N(\beta,x)\right)^r\right]\right)\,.$$

Actually, even for $\zeta\in\mathbb{C}$, one can define this function, simply by taking

$$\mathcal{P}_N(\beta,x;\zeta)\,=\,\frac{1}{N}\log\left(\mathbb{E}\left[e^{\zeta\log\left(\mathsf{Z}_N(\beta,x)\right)}\right]\right)\,,$$

which is completely unambiguous because $\mathsf{Z}_N(\beta,x)>0$. Note that this function is analytic in ζ. (For this, one only needs to make elementary use of the dominated convergence theorem to deal with the technicalities involved in interchanging the derivative and the expectation.)

Moreover,

$$\begin{aligned}
\frac{d}{d\zeta}\mathcal{P}_N(\beta,x;\zeta)\bigg|_{\zeta=0} \;&=\; \frac{1}{N}\cdot\frac{\mathbb{E}\left[e^{\zeta\log\left(\mathsf{Z}_N(\beta,x)\right)}\log\left(\mathsf{Z}_N(\beta,x)\right)\right]\bigg|_{\zeta=0}}{\mathbb{E}\left[e^{\zeta\log\left(\mathsf{Z}_N(\beta,x)\right)}\right]\bigg|_{\zeta=0}} \\
&=\; \frac{1}{N}\mathbb{E}\left[\log\left(\mathsf{Z}_N(\beta,x)\right)\right] \\
&=\; p_N(\beta,x)\,.
\end{aligned}$$

So this gives one method of calculating the quenched pressure. Also note that, by L'Hospital's rule,

$$\frac{d}{d\zeta}\mathcal{P}_N(\beta,x;\zeta)\bigg|_{\zeta=0} \;=\; \lim_{r\to 0}\frac{1}{r}\mathcal{P}_N(\beta,x;r)\,,$$

where the limit is taken along real r.

We should note two points. Firstly, we cannot calculate $\mathcal{P}_N(\beta,x;r)$ exactly for any real r, other than 0 and 1. We can calculate $\mathcal{P}(\beta,x;n)=\lim_{N\to\infty}\mathcal{P}_N(\beta,x;n)$ for $n=1,2,3,\dots$. Secondly, we are interested in $p(\beta,x)=\lim_{N\to\infty}p_N(\beta,x)$. Therefore, the following lemmas are of some interest.

Lemma 2.1 *For each real r,*

$$\lim_{N\to\infty}\mathcal{P}_N(\beta,x;r) \;=:\; \mathcal{P}(\beta,x;r)$$

exists.

Lemma 2.2 *If the two-sided limit*

$$\lim_{r\to 0}\frac{1}{r}\mathcal{P}(\beta,x;r)$$

exists, then it equals $p(\beta,x)$. More precisely,

$$\lim_{r\to 0^-}\frac{1}{r}\mathcal{P}(\beta,x;r) \;\le\; p(\beta,x) \;\le\; \lim_{r\to 0^+}\frac{1}{r}\mathcal{P}(\beta,x;r)\,,$$

and the one-sided limits are guaranteed to exist.

We will not prove Lemma 2.1. It is a straightforward generalization of Guerra and Toninelli's theorem proving the existence of the thermodynamic limit $p(\beta,x)=\lim_{N\to\infty}p_N(\beta,x)$. More precisely, for $r\ge 1$ or $r\le 0$, one has

$$(N_1+N_2)\mathcal{P}_{N_1+N_2}(\beta,x;r) \;\le\; N_1\mathcal{P}_{N_1}(\beta,x;r)+N_2\mathcal{P}_{N_2}(\beta,x;r)\,,$$

while for $0\le r\le 1$, one has

$$(N_1+N_2)\mathcal{P}_{N_1+N_2}(\beta,x;r) \;\ge\; N_1\mathcal{P}_{N_1}(\beta,x;r)+N_2\mathcal{P}_{N_2}(\beta,x;r)\,.$$

Note that this means that for $r=0,1$ one must have constancy in N. But that is easy to check directly, anyway. The reason for super-additivity for some r and sub-additivity for other r is because of the convexity/concavity properties of the function $t\mapsto t^r$. The function is convex or concave depending on whether $r(r-1)=2\binom{r}{2}$ is nonnegative or nonpositive.

This convexity or concavity enters into Guerra and Toninelli's theorem because it determines the signs of the off-diagonal entries of Hessian matrix for the relevant function.

Let us prove Lemma 2.2.

Proof. We have mentioned before the fact, which is easy itself to prove, that if f_n is a sequence of differentiable, convex functions on $\mathbb{R}$, converging pointwise to f, then

$$D^- f(x) \,\leq\, \liminf_{n\to\infty} f_n'(x) \,\leq\, \limsup_{n\to\infty} f_n'(x) \,\leq\, D^+ f(x)\,,$$

where

$$D^{\pm} f(x) \,=\, \lim_{h\to 0^{\pm}} \frac{f(x+h)-f(x)}{h}\,,$$

are the one-sided derivatives, which are guaranteed to exist because f is convex (being the pointwise limit of convex functions, and convexity being a pointwise property). This is easy to see, for example, in the sequence of functions

$$f_n(x) \,=\, \frac{1}{n}\log(\cosh(nx))\,,$$

which converges pointwise to $f(x) = |x|$. Note that

$$D^- f(0) \,=\, -1 \quad \text{and} \quad D^+ f(0) \,=\, +1\,,$$

but $f_n'(0) \,=\, 0$ for all n. So this example has strict inequality. This fact can be applied to the sequence $\mathcal{P}_N(\beta,x;r)$ and its pointwise limit $\mathcal{P}(\beta,x;r)$ (guaranteed to exist by our unproven Lemma 2.1) if we check that each $\mathcal{P}_N(\beta,x;r)$ is convex in r. But that is true just because $\mathcal{P}_N(\beta,x;r)$ is a logarithmic moment generating function. One can use either Hölder's inequality or Jensen's inequality to prove that. Let us use Cauchy-Schwarz to prove a specific case of convecity. Namely, suppose that $r, s \in \mathbb{R}$, then

$$\begin{aligned}
\mathcal{P}_N\left(\beta,x;\frac{1}{2}r+\frac{1}{2}s\right) \,&=\, \frac{1}{N}\log\left(\mathbb{E}\left[e^{(\frac{1}{2}r+\frac{1}{2}s)\log\left(\mathsf{Z}_N(\beta,x)\right)}\right]\right)\\
&=\, \frac{1}{N}\log\left(\mathbb{E}\left[e^{\frac{1}{2}r\log\left(\mathsf{Z}_N(\beta,x)\right)}e^{\frac{1}{2}s\log\left(\mathsf{Z}_N(\beta,x)\right)}\right]\right)\\
&\leq\, \frac{1}{N}\log\left(\mathbb{E}\left[e^{r\log\left(\mathsf{Z}_N(\beta,x)\right)}\right]^{1/2}\,\mathbb{E}\left[e^{s\log\left(\mathsf{Z}_N(\beta,x)\right)}\right]^{1/2}\right)\\
&=\, \frac{1}{2N}\log\left(\mathbb{E}\left[e^{r\log\left(\mathsf{Z}_N(\beta,x)\right)}\right]\right)+\frac{1}{2N}\log\left(\mathbb{E}\left[e^{s\log\left(\mathsf{Z}_N(\beta,x)\right)}\right]\right)\\
&=\, \frac{1}{2}\mathcal{P}_N\left(\beta,x;r\right)+\frac{1}{2}\mathcal{P}_N\left(\beta,x;s\right)\,.
\end{aligned}$$

$\square$

Because of the lemma, if one could prove that the two-sided limit exists, and calculate it somehow, then one would have the pressure for the Sherrington-Kirkpatrick model. That is what Sherrington and Kirkpatrick purported to do, somehow.

3. Sherrington and Kirkpatrick's "Solution"

Everything which has come previously was rigorous. Everything which follows is not.

There is some Euler-Lagrange or critical point equation which must be satisfied by q_0 to give the optimal value for $F_n(\beta, x; \boldsymbol{q})$. This is assuming that Sherrington and Kirkpatrick's ansatz is correct. (But come to class for the explanation of why to take it.) Let us suppose that somehow this leads to a determination of the optimal q_0. Now consider the formal limit $n \to \infty$, obtained by taking the new formulas for $F_n^{(1)}(\beta, x; \boldsymbol{q})$ and $F_n^{(2)}(\beta; \boldsymbol{q})$, derived in Section 1, and then treating n as a parameter. By L'Hospital's rule again, and a derivative calculation, one sees that

$$\begin{aligned} f^{(1)}(\beta, x; q_0) &:= \frac{1}{r} \log \left(\mathbb{E}^{(0)} \left[\left(\mathbb{E}^{(1)} \left[2 \cosh \left(x + \beta \sqrt{q_0}\, \mathsf{X}^{(0)} + \beta \sqrt{1 - q_0}\, \mathsf{X}^{(1)} \right) \right] \right)^r \right] \right) \\ &= \mathbb{E}^{(0)} \left[\log \left(\mathbb{E}^{(1)} \left[2 \cosh \left(x + \beta \sqrt{q_0}\, \mathsf{X}^{(0)} + \beta \sqrt{1 - q_0}\, \mathsf{X}^{(1)} \right) \right] \right) \right] . \end{aligned}$$

Similarly,

$$\begin{aligned} f^{(2)}(\beta; q_0) &:= \frac{1}{r} \log \left(\mathbb{E}^{(0)} \left[\left(\mathbb{E}^{(1)} \left[\exp \left(\beta \sqrt{\frac{q_0^2}{2}}\, \mathsf{X}^{(0)} + \beta \sqrt{\frac{1 - q_0^2}{2}}\, \mathsf{X}^{(1)} \right) \right] \right)^r \right] \right) \\ &= \mathbb{E}^{(0)} \left[\log \left(\mathbb{E}^{(1)} \left[\exp \left(\beta \sqrt{\frac{q_0^2}{2}}\, \mathsf{X}^{(0)} + \beta \sqrt{\frac{1 - q_0^2}{2}}\, \mathsf{X}^{(1)} \right) \right] \right) \right] . \end{aligned}$$

Note that we no longer write $\boldsymbol{q}$ because, since we are now replacing n by r, which ultimately approaches 0, there is no matrix anymore. At best our interpretation would have to be that the matrix was fractional-dimensional, which is clearly not sensible. (On the other hand, for an array, if one probes the array through only statistical information, it is possible to arrive at something sensible for non-integer parameters. Indeed, this is closely related to the hypergeometric distribution, which is well-known from elementary probability. An investigation into that direction would ultimately lead us to a particular type of point process called the random probability cascade of Ruelle, or the Bolthausen-Sznitman coalescent. But we do not have time for that, now.)

One should evaluate $f^{(1)}(\beta, x; q_0) - f^{(2)}(\beta; q_0)$ at the q_0 given by the Euler-Lagrange equation. But now we come to an interesting question. Will this give a local max or a local min? The answer, as we will see next time, is a local min. The reason this is so surprising is that for $n = 1, 2, 3, \dots$ the Euler-Lagrange equations arise from taking maxima not minima. Parisi was the first to propose solving the opposite optimization problem when $0 < r < 1$. Here is a justification analogous to his. The function $t \mapsto t^r$ is convex when $r \in (1, \infty)$ and concave when $r \in (0, 1)$. We solved the maximization problem when that function was convex. So we should solve the minimization problem when that function is concave.

It may seem that whether $r > 1$ or $0 < r < 1$, we should solve the opposite problem because, after all, concave functions are more closely associated with maximum problems and convex functions are more closely associated with minimum problems. There are two things to note. First of all, the function which is being maximized or minimized is a difference of two functions $f^{(1)}(\beta, x; q_0) - f^{(2)}(\beta; q_0)$. Therefore, it would not be convex or concave even if both $f^{(1)}(\beta, x; \cdot)$ and $f^{(2)}(\beta; \cdot)$ were. Second of all, recall that from Slepian's lemma there is a counterintuitive dependence on the sign of the second-derivative matrix.

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