MATH 585 – TOPICS IN MATHEMATICAL PHYSICS – FALL 2006
MATHEMATICS OF MEAN FIELD SPIN GLASSES AND THE REPLICA METHOD
LECTURE 13: PARISI'S ANSATZ 2 AND GUERRA'S BOUNDS

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CONTENTS

1. GUERRA'S REPLICA SYMMETRY "UNBROKEN" BOUNDS

We are going to present this lecture out of all chronological order, mainly because it seems the presentation is easier if one starts by applying Guerra's bounds to the replica symmetric solution of Sherrington and Kirkpatrick.

Last lecture we did calculus type manipulations to arrive at the function of $q_0 \in [0,1]$,

$$f(\beta, x; q_0) \,=\, f^{(1)}(\beta, x; q_0) - f^{(2)}(\beta; q_0)\,,$$

where

$$f^{(1)}(\beta, x; q_0) \,=\, \mathbb{E}^{(0)}\left[\log\left(\mathbb{E}^{(1)}\left[2\cosh\left(x + \beta\sqrt{q_0}\,\mathsf{X}^{(0)} + \beta\sqrt{1-q_0}\,\mathsf{X}^{(1)}\right)\right]\right)\right]\,,$$

and

$$f^{(2)}(\beta; q_0) \,=\, \mathbb{E}^{(0)}\left[\log\left(\mathbb{E}^{(1)}\left[\exp\left(\beta\sqrt{\frac{q_0^2}{2}}\,\mathsf{X}^{(0)} + \beta\sqrt{\frac{1-q_0^2}{2}}\,\mathsf{X}^{(1)}\right)\right]\right)\right]\,.$$

This was extrapolated from the $n = 1, 2, 3, \ldots$ moments, where there was a rigorous connection between maximizing the function $F_n(\beta, x; \boldsymbol{q})$ and calculating $\mathcal{P}(\beta, x; n)$. Taking, formally, $\lim_{n\to 0} \frac{1}{n} F_n(\beta, x; \boldsymbol{q})$, is supposed to give information on $p(\beta, x)$. But as we alluded to last time, one solves a minimization problem not a maximization problem. This fact is easiest to see by introducing Guerra's completely rigorous theorem, in the context of Sherrington and Kirkpatrick's ansatz. Later on we will state Parisi's ansatz and state how Guerra's theorem generalizes to it.

Theorem 1.1 (Guerra 2001 (specialization))

$$p(\beta, x) \,\leq\, \min_{0\leq q_0\leq 1}\left[f^{(1)}(\beta, x; q_0) - f^{(2)}(\beta; q_0)\right]\,.$$

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More precisely, for every $N > 0$,

$$p_N(\beta,x) \le \min_{0\le q_0\le 1} \left[f^{(1)}(\beta,x;q_0) - f^{(2)}(\beta;q_0)\right] .$$

This is specialization of Guerra's main replica symmetry breaking bounds. But even this simpler result involves a lot of notation. Let us define a 1-parameter family of Gaussian vectors, which are all coupled in the simplest way

$$\mathsf{G}_N(\sigma;t) = \sqrt{t}\,\mathsf{H}_N(\sigma) + \sqrt{t}\,\mathsf{K}_N + \sqrt{1-t}\,\mathsf{L}_N(\sigma)\,,$$

where $\mathsf{H}_N(s)$ is the regular Sherrington-Kirkpatrick Hamiltonian, while

$$\mathsf{K}_N = \sqrt{\frac{q_0^2}{2}}\,\mathsf{K}_N^{(0)} + \sqrt{\frac{1-q_0^2}{2}}\,\mathsf{K}_N^{(1)}\,,$$

and

$$\mathsf{L}_N(\sigma) = \sqrt{q_0}\,\mathsf{L}_N^{(0)}(\sigma) + \sqrt{1-q_0}\mathsf{L}_N^{(1)}\,,$$

and all of the following Gaussian random variables and/or vectors are independent of one another: $(\mathsf{H}_N(\sigma) : \sigma \in \Omega_N)$, $(\mathsf{L}_N^{(0)}(\sigma) : \sigma \in \Omega)$, $(\mathsf{L}_N^{(1)}(\sigma) : \sigma \in \Omega)$, $\mathsf{K}_N^{(0)}$, and $\mathsf{K}_N^{(1)}$. The covariances are as follows,

$$\mathbb{E}[\mathsf{H}_N(\sigma)\mathsf{H}_N(\sigma')] = \frac{N}{2}[q_N(\sigma,\sigma')]^2\,,$$

where $q_N(\sigma,\sigma') = \frac{1}{N}\sum_{i=1}^N \sigma_i\sigma_i'$ is the usual spin-spin overlap. Also,

$$\mathbb{E}[\mathsf{L}_N^{(i)}(\sigma)\mathsf{L}_N^{(i)}(\sigma')] = N q_N(\sigma,\sigma') \quad \text{for} \quad i=1,2\,.$$

Finally,

$$\mathbb{E}[(\mathsf{K}_N^{(i)})^2] = \frac{N}{2} \quad \text{for} \quad i=1,2\,.$$

We are going to define an interpolated pressure function. Let us recall that

$$w_N(\sigma;x) = e^{x\sum_{i=1}^N \sigma_i}\,,$$

which is the factor in the Gibbs weight coming from a pure external magnetic field. Now we define

$$\mathsf{W}_N(\sigma;\beta,x;t) = w_N(\sigma;x)e^{-\beta \mathsf{G}_N(\sigma;t)}\,,$$

which is the full random Gibbs weight. And define the interpolated pressure

$$p_N(\beta,x;t) = \frac{1}{N}\mathbb{E}\mathbb{E}^{(0)}\left[\log\left(\mathbb{E}^{(1)}\left[\sum_{\sigma\in\Omega^N}\mathsf{W}_N(\sigma;\beta,x;t)\right]\right)\right]\,,$$

where $\mathbb{E}^{(1)}$ is the expectation over $(\mathsf{L}_N^{(1)}(\sigma) : \sigma \in \Omega)$ and $\mathsf{K}_N^{(1)}$, $\mathbb{E}^{(0)}$ is the expectation over $(\mathsf{L}_N^{(0)}(\sigma) : \sigma \in \Omega)$ and $\mathsf{K}_N^{(0)}$, and $\mathbb{E}$ is the expectation over $(\mathsf{H}_N(\sigma) : \sigma \in \Omega_N)$. We will abbreviate $\mathsf{W}_N(\sigma;\beta,x;t)$ by $\mathsf{W}_N(\sigma;t)$ since it causes no confusion. We will begin by showing that $p_N(\beta,x;0) \ge p_N(\beta,x;1)$.

By interchanging the derivative and expectations, by the usual claim that all can be justified from DCT, we have

$$\frac{d}{dt}p_N(\beta,x;t) \;=\; \frac{1}{N}\mathbb{E}\mathbb{E}^{(0)}\left[\left(\widetilde{\mathbb{E}}^{(1)}\left[\sum_{\widetilde{\sigma}\in\Omega^N}\widetilde{\mathsf{W}}_N(\widetilde{\sigma};t)\right]\right)^{-1}\mathbb{E}^{(1)}\left[\sum_{\sigma\in\Omega^N}\frac{d}{dt}\mathsf{W}_N(\sigma;t)\right]\right].$$

We define $\widetilde{G}_N(\beta;t)$ to be defined just as $G_N(\beta;t)$, but replacing $(\mathsf{L}_N^{(1)}(\sigma)\,:\,\sigma\in\Omega)$ and $\mathsf{K}_N^{(1)}$ by independent replicas of those random variables, $(\widetilde{\mathsf{L}}_N^{(1)}(\sigma)\,:\,\sigma\in\Omega)$ and $\widetilde{\mathsf{K}}_N^{(1)}$. The tilde expectation takes the expectation over these new random variables. We define $\tilde{\mathsf{W}}_N(\sigma;t)$ as $\mathsf{W}_N(\sigma;t)$ was defined, but relative to $\widetilde{G}_N(\beta;t)$ instead of $G_N(\beta;t)$. Of course, $\widetilde{\sigma}$ is just another spin configuration than σ. We need both expectations by the definition of the derivative and the chain rule. We introduce the new random variables because momentarily we will want to put the two random variables together under the same expectation.

Note that

$$\frac{d}{dt}\mathsf{W}_N(\sigma;t) \;=\; -\beta\mathsf{W}_N(\sigma;t)\,\frac{d}{dt}\mathsf{G}_N(\sigma;t)\,.$$

We could simplify $\frac{d}{dt}\mathsf{G}_N(\sigma;t)$, but we choose not to. The only important thing is that it is a Gaussian and therefore Wick's rule applies. Remember Wick's rule, also called Gaussian integration by parts, says

$$\mathbb{E}[\mathsf{X}_0\,\varphi(\mathsf{X}_1,\ldots,\mathsf{X}_n)] \;=\; \sum_{i=1}^{n}\mathbb{E}\left[\frac{\partial\varphi}{\partial x_i}(\mathsf{X}_1,\ldots,\mathsf{X}_n)\right]\,\mathbb{E}[\mathsf{X}_0\mathsf{X}_i]\,,$$

when $(\mathsf{X}_0,\mathsf{X}_1,\ldots,\mathsf{X}_n)$ are jointly Gaussian random variables and φ is smooth enough (and not growing too fast at ∞; e.g., polynomial growth is fine). (Also note that it is perfectly acceptable for X_0 to be a dependent function of some of the $\mathsf{X}_1,\ldots,\mathsf{X}_n$. It is not required for all these components of the Gaussian vector to be independent.)

Therefore,

$$\begin{aligned}\frac{d}{dt}p_N(\beta,x;t) &= \frac{1}{N}\mathbb{E}\mathbb{E}^{(0)}\mathbb{E}^{(1)}\left[\sum_{\sigma\in\Omega^N}\left(-\beta\frac{d}{dt}\mathsf{G}_N(\sigma;t)\right)\mathsf{W}_N(\sigma;t)\left(\widetilde{\mathbb{E}}^{(1)}\left[\sum_{\widetilde{\sigma}\in\Omega^N}\widetilde{\mathsf{W}}_N(\widetilde{\sigma};t)\right]\right)^{-1}\right]\\ &= \frac{1}{N}\sum_{\sigma\in\Omega^N}\mathbb{E}\mathbb{E}^{(0)}\mathbb{E}^{(1)}\left[\left(-\beta\frac{d}{dt}\mathsf{G}_N(\sigma;t)\right)\mathsf{W}_N(\sigma;t)\left(\widetilde{\mathbb{E}}^{(1)}\left[\sum_{\widetilde{\sigma}\in\Omega^N}\widetilde{\mathsf{W}}_N(\widetilde{\sigma};t)\right]\right)^{-1}\right]\\ &= \frac{1}{N}\sum_{\sigma\in\Omega^N}\mathbb{E}\mathbb{E}^{(0)}\mathbb{E}^{(1)}\left[\mathsf{A}(\sigma;t)+\mathsf{B}(\sigma;t)\right].\end{aligned}$$

The last step is merely anticipating what we will get from Wick's rule, knowing that there are two terms and that they are fairly messy. The first term involves the derivative of $\mathsf{W}_N(\sigma;t)$. Since

$$\mathsf{W}_N(\sigma;t) \;=\; w_N(\sigma;x)e^{-\beta\mathsf{G}_N(\sigma;t)} \quad\Rightarrow\quad \frac{\partial\mathsf{W}_N(\sigma;t)}{\partial\mathsf{G}_N(\sigma;t)} \;=\; -\beta\mathsf{W}_N(\sigma;t)\,,$$

we obtain

$$\mathsf{A}(\sigma;t) \,=\, \beta^2\, \mathbb{E}\mathbb{E}^{(0)}\mathbb{E}^{(1)}\left[\mathsf{G}_N(\sigma;t)\frac{d}{dt}\mathsf{G}_N(\sigma;t)\right]\, \mathsf{W}_N(\sigma;t)\, \left(\widetilde{\mathbb{E}}^{(1)}\left[\sum_{\widetilde{\sigma}\in\Omega^N}\widetilde{\mathsf{W}}_N(\widetilde{\sigma};t)\right]\right)^{-1}.$$

Note that the first expectation is the covariance term coming from Wick's rule. But on the other hand,

$$\mathbb{E}\mathbb{E}^{(0)}\mathbb{E}^{(1)}\left[\mathsf{G}_N(\sigma;t)\frac{d}{dt}\mathsf{G}_N(\sigma;t)\right] \,=\, \frac{1}{2}\cdot\frac{d}{dt}\mathbb{E}\mathbb{E}^{(0)}\mathbb{E}^{(1)}\left[(\mathsf{G}_N(\sigma;t))^2\right]\,,$$

and

$$\mathbb{E}\mathbb{E}^{(0)}\mathbb{E}^{(1)}\left[(\mathsf{G}_N(\sigma;t))^2\right] \,=\, \frac{N}{2}\big(t[q_N(\sigma,\sigma)]^2+t+2(1-t)q_N(\sigma,\sigma)\big) \,=\, N\,,$$

independent of t, because $q_N(\sigma,\sigma)=1$. Therefore the covariance term alone makes $\mathsf{A}(\sigma;t)=0$. The second term is more complicated.

By introducing double-tilde random variables in complete analogy to the tilde random variables, we have

$$\mathsf{B}(\sigma;t) \,=\, -\left(\widetilde{\widetilde{\mathbb{E}}}^{(1)}\left[\sum_{\widetilde{\widetilde{\sigma}}\in\Omega^N}\widetilde{\widetilde{\mathsf{W}}}_N(\widetilde{\widetilde{\sigma}};t)\right]\right)^{-2}\widetilde{\mathbb{E}}^{(1)}\left[\sum_{\widetilde{\sigma}\in\Omega^N}\mathsf{C}(\sigma,\widetilde{\sigma};t)\right]\,\mathsf{W}_N(\sigma;t)\,,$$

where

$$\mathsf{C}(\sigma,\widetilde{\sigma};t) \,=\, \beta^2\,\widetilde{\mathsf{W}}_N(\widetilde{\sigma};t)\,\mathbb{E}\mathbb{E}^{(0)}\mathbb{E}^{(1)}\widetilde{\mathbb{E}}^{(1)}\left[\widetilde{\mathsf{G}}_N(\widetilde{\sigma};t)\frac{d}{dt}\mathsf{G}_N(\sigma;t)\right].$$

Again, the expectation term here comes from the covariance in Wick's rule. But now we have, by a straightforward calculation involving the covariances we wrote before

$$\mathbb{E}\mathbb{E}^{(0)}\mathbb{E}^{(1)}\widetilde{\mathbb{E}}^{(1)}\left[\widetilde{\mathsf{G}}_N(\widetilde{\sigma};t)\frac{d}{dt}\mathsf{G}_N(\sigma;t)\right] \,=\, \frac{N}{4}\left([q_N(\sigma,\widetilde{\sigma})]^2+q_0^2-2q_0q_N(\sigma,\widetilde{\sigma})\right).$$

Exercise: *This calculation is critically important to understanding Guerra's theorem. Do it carefully.*

Putting this all together, and again noting that $\mathsf{A}(\sigma;t)$ is identically zero, we have

$$\frac{d}{dt}p_N(\beta,x;t) \,=\, -\frac{1}{N}\mathbb{E}\mathbb{E}^{(0)}\mathbb{E}^{(1)}\widetilde{\mathbb{E}}^{(1)}\left[\sum_{\sigma,\widetilde{\sigma}\in\Omega^N}\frac{\mathsf{W}_N(\sigma;t)\widetilde{\mathsf{W}}_N(\widetilde{\sigma};t)}{\left(\widetilde{\widetilde{\mathbb{E}}}^{(1)}\left[\sum_{\widetilde{\widetilde{\sigma}}\in\Omega^N}\widetilde{\widetilde{\mathsf{W}}}_N(\widetilde{\widetilde{\sigma}};t)\right]\right)^2}\left(\frac{\beta^2N}{4}[q_N(\sigma,\widetilde{\sigma})-q_0]^2\right)\right].$$

Let us agree to represent by $\mathbb{E}$ the expectation over all the random variables. Also, let us define a new random variable, which is the "partially annealed" Gibbs weight, normalized,

$$\mathsf{U}_N^{(0)}(\sigma;t) \,=\, \frac{\mathsf{W}_N(\sigma;t)}{\mathbb{E}^{(1)}\left[\sum_{\sigma\in\Omega^N}\mathsf{W}_N(\sigma;\beta,x;t)\right]}\,.$$

We no longer bother to write the denominator with a double tilde because we are not going to apply Wick's rule anymore. Then we can rewrite

$$\frac{d}{dt}p_N(\beta,x;t) \;=\; -\frac{\beta^2}{4}\mathbb{E}\left[\sum_{\sigma,\widetilde{\sigma}\in\Omega^N} \mathsf{U}_N^{(0)}(\sigma;t)\,\widetilde{\mathsf{U}}_N^{(0)}(\widetilde{\sigma};t)\,[q_N(\sigma,\widetilde{\sigma})-q_0]^2\right].$$

Note that $\mathsf{U}_N^{(0)}(\sigma;t)$ and $\widetilde{\mathsf{U}}_N^{(0)}(\widetilde{s};t)$ are densities. I.e., summing over σ and taking the expectation of this random variable gives 1, and the random variable is always nonnegative. Francesco Guerra writes a new symbol for the expectation, thinking of this as a function on $\Omega^N\times\Omega^N$. For any function $f(\sigma,\widetilde{\sigma})$, define

$$\langle f(\sigma,\widetilde{\sigma})\rangle_t^{(0)} \;=\; \mathbb{E}\left[\sum_{\sigma,\widetilde{\sigma}\in\Omega^N} \mathsf{U}_N^{(0)}(\sigma;t)\,\widetilde{\mathsf{U}}_N^{(0)}(\widetilde{\sigma};t)\,f(\sigma,\widetilde{\sigma})\right].$$

Then one gets the even more concise notation

$$\frac{d}{dt}p_N(\beta,x;t) \;=\; -\frac{\beta^2}{4}\,\left\langle [q_N(\sigma,\widetilde{\sigma})-q_0]^2\right\rangle_t^{(0)}.$$

In particular,

$$\frac{d}{dt}p_N(\beta,x;t) \;\leq\; 0\,.$$

It remains to finish the proof, by evaluating $p_N(\beta,x;0)$ and $p_N(\beta,x;1)$. Since we haven't formally started the proof before now, let us do that.

Proof. At $t=0$, we have just

$$\mathsf{G}_N(\sigma;t) \;=\; \mathsf{L}_N(\sigma)\,.$$

But notice that one way to produce the random variable $\mathsf{L}_N(\sigma)$ is to start with i.i.d. $N(0,1)$ random variables $\mathsf{X}_1^{(0)},\dots,\mathsf{X}_N^{(0)}$ and $\mathsf{X}_1^{(1)},\dots,\mathsf{X}_N^{(1)}$, and take

$$\mathsf{L}_N(\sigma) \;=\; -\sum_{i=1}^N\left(\sqrt{q_0}\,\mathsf{X}_i^{(0)}+\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\,\sigma_i\,.$$

Therefore,

$$\begin{aligned}
\mathsf{W}_N(\sigma;\beta,x;0) \;&=\; w_N(\sigma;x)\,e^{-\beta\mathsf{G}_N(\sigma;0)}\\
&=\; w_N(\sigma;x)\,\exp\left[\beta\sum_{i=1}^N\left(\sqrt{q_0}\,\mathsf{X}_i^{(0)}+\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\,\sigma_i\right]\\
&=\; \exp\left[\sum_{i=1}^N\left(x+\beta\sqrt{q_0}\,\mathsf{X}_i^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\,\sigma_i\right]\\
&=\; \prod_{i=1}^N\exp\left[\left(x+\beta\sqrt{q_0}\,\mathsf{X}_i^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\,\sigma_i\right].
\end{aligned}$$

Therefore, by calculations which are now very familiar to us, we have

$$\begin{aligned} p_N(\beta,x;0) &= \frac{1}{N}\mathbb{E}\mathbb{E}^{(0)}\left[\log\left(\mathbb{E}^{(1)}\left[\sum_{\sigma\in\Omega^N}\mathsf{W}_N(\sigma;\beta,x;t)\right]\right)\right] \\ &= \frac{1}{N}\mathbb{E}\mathbb{E}^{(0)}\left[\log\left(\mathbb{E}^{(1)}\left[\sum_{\sigma\in\Omega^N}\prod_{i=1}^N \exp\left[\left(x+\beta\sqrt{q_0}\,\mathsf{X}_i^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right)\sigma_i\right]\right]\right)\right] \\ &= \frac{1}{N}\mathbb{E}\mathbb{E}^{(0)}\left[\log\left(\mathbb{E}^{(1)}\left[\prod_{i=1}^N\left(2\cosh\left[x+\beta\sqrt{q_0}\,\mathsf{X}_i^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}_i^{(1)}\right]\right)\right]\right)\right]. \end{aligned}$$

Note that this does not depend at all on the random variables whose expectation is $\mathbb{E}$. So we can take $\mathbb{E}$ away. Also, by using the independence of the $\mathsf{X}_i^{(0)}$ and $\mathsf{X}_i^{(1)}$ for various i, we can take the product outside the expectation, then with the logarithm it becomes a sum, which we can also take outside the expectation. This gives us, at first, a power of N, but after the logarithm a factor of N. This cancels the $\frac{1}{N}$. Therefore,

$$p_N(\beta,x;0) = \mathbb{E}^{(0)}\left[\log\left(\mathbb{E}^{(1)}\left[2\cosh\left(x+\beta\sqrt{q_0}\,\mathsf{X}^{(0)}+\beta\sqrt{1-q_0}\,\mathsf{X}^{(1)}\right)\right]\right)\right].$$

Or, in other words,

$$p_N(\beta,x;0) = f^{(1)}(\beta,x;q_0).$$

The calculation of $p_N(\beta,x;1)$ is similar, so let us leave it as an exercise.

Exercise: *Prove that* $p_N(\beta,x;1) = p_N(\beta,x)+f^{(2)}(\beta;q_0)$.

One may be slightly surprised by the fact that there are two terms for $p_N(\beta,x;1)$ instead of just one. First of all, this is exactly what we need, as we will see momentarily. Second of all, one can understand this without great difficulty. Namely,

$$\mathsf{G}_N(\sigma;1) = \mathsf{H}_N(\sigma)+\mathsf{K}_N,$$

and K_N does not depend on σ at all. On the other hand $\mathsf{H}_N(\sigma))$ is completely independent of the random variables going into $\mathbb{E}^{(0)}$ and $\mathbb{E}^{(1)}$. Therefore, essentially one can decompose as follows

$$\begin{aligned} \mathbb{E}\mathbb{E}^{(0)}\left[\log\left(\mathbb{E}^{(1)}\left[\sum_{\sigma\in\Omega^N} w_N(\sigma;x)e^{-\beta\mathsf{G}_N(\sigma;1)}\right]\right)\right] &= \mathbb{E}\left[\log\left(\sum_{\sigma\in\Omega^N} w_N(\sigma;x)e^{-\beta\mathsf{H}_N(\sigma)}\right)\right] \\ &\quad + \mathbb{E}^{(0)}\left[\log\left(\mathbb{E}^{(1)}\left[e^{-\beta\mathsf{K}_N}\right]\right)\right]. \end{aligned}$$

We leave it to the reader to fill in the details, and find a good way to produce the random variable K_N.

Because we know that $p_N(\beta,x;0)\geq p_N(\beta,x;1)$, due to the fact that $\frac{d}{dt}p_N(\beta,x;t)\leq 0$, we have

$$f^{(1)}(\beta,x;q_0) \geq p_N(\beta,x)+f^{(2)}(\beta;q_0) \quad\Rightarrow\quad p_N(\beta,x) \leq f^{(1)}(\beta,x;q_0)-f^{(2)}(\beta;q_0).$$

Since this is true for every $q_0\in[0,1]$, we see that

$$p_N(\beta,x) \leq \min_{q_0\in[0,1]}\left[f^{(1)}(\beta,x;q_0)-f^{(2)}(\beta;q_0)\right],$$

which is what we wanted to prove. (We do need that $0 \leq q_0 \leq 1$ in order to take all the square-roots such as $\sqrt{q_0}$ and $\sqrt{1-q_0}$.) □

Guerra also pointed out an important identity by exactly characterizing the remainder in the inequality. This is often called a "sum-rule" in mathematical physics, although one usually hears that term used in connection with subjects such as spectral theory for Schrödinger operators. In the present case, this takes the following form, whose proof is contained in the proof of the previous theorem.

Corollary 1.2 *For any* $q_0 \in [0,1]$,

$$f^{(1)}(\beta, x; q_0) - f^{(2)}(\beta; q_0) - p_N(\beta, x) \;=\; \frac{\beta^2}{4} \int_0^1 \left\langle [q_N(\sigma, \widetilde{\sigma}) - q_0]^2 \right\rangle_t^{(0)} \, dt\, .$$

Therefore, if one wanted to prove that Sherrington and Kirkpatrick's ansatz gave the correct value, then one could proceed as follows. Identify the conjectured optimal q_0. Then prove that

$$\lim_{N \to \infty} \int_0^1 \left\langle [q_N(\sigma, \widetilde{\sigma}) - q_0]^2 \right\rangle_t^{(0)} \, dt \;=\; 0\, .$$

In fact, this works. There is a region of β and x where Sherrington and Kirkpatrick's "solution" is correct. The boundary of this region is called the Almeida-Thouless line. In their first paper on the subject, Guerra and Toninelli proved that Sherrington and Kirkpatrick's solution is correct, although they could not push their argument all the way to the Almeida-Thouless line. In later work Guerra improved this. See the papers [3, 2].

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