MATH 585 – TOPICS IN MATHEMATICAL PHYSICS – FALL 2006
MATHEMATICS OF MEAN FIELD SPIN GLASSES AND THE REPLICA METHOD
LECTURE 2: INTRODUCTION TO SPIN GLASSES

S. STARR
MATHEMATICS DEPARTMENT, UNIVERSITY OF ROCHESTER

CONTENTS

Note: We will try to denote random variables with the sans-serif font.

1. REAL SPIN GLASSES

In the 1970's a new type of magnetic material was discovered experimentally, called spin glasses. Such a material is comprised of a crystal of one type of atom, which does not interact magnetically, with a dilute mixture of another type of atom which does interact magnetically. The dilute magnetic sites are distributed at random throughout the nonmagetic material. For example, one could imagine that the mixture was made at high temperature, and then rapidly cooled. In this case the magnetic atoms' positions will be random and not determined by the magnetic energy of the configuration. In the continuum limit, one may imagine a large sample, for example $\Lambda_N = \mathbb{T}_N^d := \mathbb{R}^d/N\mathbb{Z}^d$, and placed on this is an i.i.d sample of N^d points placed uniformly. A sample of such a process is shown in Figure 1.

There is a function $J : \mathbb{R}^d \to \mathbb{R}$, which is even, $J(r) = J(-r)$, and such that the coupling between sites at $x, y \in \mathbb{R}^d$ is given by $J(x-y)$. Therefore, the Ising-type Hamiltonian for this random configuration of points is

$$\mathsf{H}_N(\sigma) = -\sum_{1\leq i<j\leq N^d} J(\mathsf{X}_i - \mathsf{X}_j)\sigma_i\sigma_j - h\sum_{i=1}^{N^d}\sigma_i\,.$$

where $\mathsf{X}_1, \dots, \mathsf{X}_{N^d}$ are the random positions of the spins.

Date: September 6, 2006.

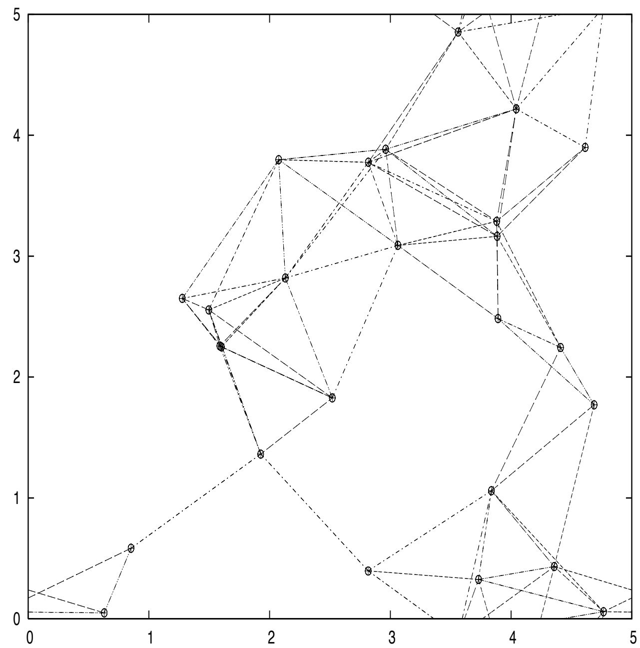

FIGURE 1. A random set of spin sites in a 5×5 square, with periodic boundary conditions, and some of the edges between them. The criterion used for including an edge in this picture is that the length is small enough.

1.1 The Edwards-Anderson Model.

This model seems very difficult at first. On the other hand, the function J supposedly has the property that it is oscillatory with an amplitude which decays slowly over a long range (it has power-law decay). If the distances between impurity atoms is also large compared to the frequency of oscillation, then, to a good approximation, the signs of the pairs $J(\mathsf{X}_i - \mathsf{X}_j)$ are equally likely to be positive or negative, and are also close to being independent. In 1975, Edwards and Anderson proposed a model that takes into account the random nature of the couplings, but leaves out the stochastic geometry induced by the Poisson point process. Namely, their model is again on a crystalline lattice[1], like the Ising model from last lecture, $\Lambda_N = \mathbb{Z}_N^d$, but now,

$$\mathsf{H}_N^{EA}(\sigma) \,=\, -\sum_{\langle x,y\rangle} \mathsf{J}_{x,y}\sigma_x\sigma_y - h\sum_{x\in\Lambda_N}\sigma_x\,,$$

with the collection of couplings $(\mathsf{J}_{x,y} \,:\, \langle x,y\rangle \subset \Lambda_N)$ being i.i.d. $N(0,1)$ random variables. I.e., they are Gaussian and have mean 0 and variance 1.

This model also seems difficult to solve. (Indeed there is a division among top mathematicians and physicists working on spin glasses, today, as to their opinions of what happens in

[1]This is not an accurate description of what Edwards and Anderson did. Actually, the nonmagnetic background is crystalline, and that may be the only reason to consider a crystalline lattice. Edwards and Anderson's treatment by molecular field theory is equally motivated if one simply considers a model where many $\mathsf{J}_{x,y}$'s are zero because there is no magnetic atom at that site. That is what they say in their paper. But in the present-day literature on the subject, the "Edwards-Anderson" model always means the spin-glass on a crystalline lattice.

this model, even qualitatively.) So Edwards and Anderson invented a type of "molecular field theory". A molecular field theory is a self-consistent ansatz, which amounts to completely changing the model one is looking at. It usually involves replacing all two-body terms, such as $\sigma_x \sigma_y$, with one-body terms, such as $\eta_x \sigma_x$, where η_x is an indeterminate, but nonrandom external field – the "molecular field". One can calculate the pressure of such a model because the partition function factorizes:

$$\sum_{\sigma \in \Lambda_N} e^{\beta \sum_{x \in \Lambda_N} (h+\eta_x)\sigma_x} = \prod_{x \in \Lambda_N} \left[\sum_{\sigma_x \in \{+1,-1\}} e^{\beta(h+\eta_x)\sigma_x} \right].$$

But then one imposes a constitutive relationship between the indeterminates $(\eta_x : x \in \Lambda_N)$ and the Boltzmann-Gibbs distributions of the $(\sigma_x : x \in \Lambda_N)$. For example, one might consider η_x to be the sum of the Boltzmann-Gibbs expectation of $\frac{1}{2}\mathsf{J}_{xy}\sigma_y$, for all nearest-neighbors y, of x. Since the distributions of the σ_x's are determined from the η_x's, this leads to nonlinear equations for the "molecular fields" η_x. Sometimes such equations can be solved, which usually leads to a wealth of fairly explict formulas for the thermodynamic functions. On the other hand, when one simply replaces one problem by a different, simpler problem, it is often not clear what direct relevance this has to the original model. (In a nutshell, this last issue is a driving force behind much of mathematical physics.)

1.2 The Sherrington-Kirkpatrick Model.

Also in 1975, in order to provide a model where the molecular field theory is more appropriate, Sherrington and Kirkpatrick considered the model on a complete graph. This is the following. For N, the lattice Λ_N is just $\{1, 2, \ldots, N\}$. So Ω_N is just $\{+1, -1\}^N$. A spin configuration is $\sigma = (\sigma_1, \ldots, \sigma_N)$, where each σ_i takes the value $+1$ or -1. Finally the Hamiltonian is

$$\mathsf{H}_N^{SK}(\sigma) = -\frac{1}{\sqrt{N}} \sum_{1 \leq i < j \leq N} \mathsf{J}_{i,j}\sigma_i\sigma_j - h\sum_{i=1}^{N} \sigma_i\,, \tag{1.1}$$

where $h \in \mathbb{R}$ is a fixed, nonrandom number, giving the strength of a magnetic field in the $\uparrow$-spin direction, and all the $\mathsf{J}_{i,j}$'s are i.i.d. $N(0,1)$ random variables. Hence, every spin σ_i interacts with every other spin σ_j, but with a random coupling, which is actually equal to $-\mathsf{J}_{ij}/\sqrt{N}$.

This model has a high amount of frustration. We define a frustrated triangle as a triple of sites $1 \leq i < j < k \leq N$, such that $\mathsf{J}_{ij}\mathsf{J}_{jk}\mathsf{J}_{ik} < 0$. An unfrustrated triangle is one where the product is positive. In an unfrustrated triangle, one can find spins $(\sigma_i, \sigma_j, \sigma_k)$ such that each of the terms, $\mathsf{J}_{ij}\sigma_i\sigma_j$, $\mathsf{J}_{jk}\sigma_j\sigma_k$, and $\mathsf{J}_{ik}\sigma_i\sigma_k$, are positive, which is good for minimizing energy. But in a frustrated triangle this is impossible, because the product of all three terms has the same sign as $\mathsf{J}_{ij}\mathsf{J}_{jk}\mathsf{J}_{ik}$. If there is only one triangle to worry about, then in a frustrated triangle, one simply chooses to frustrate that bond $\langle i, j\rangle$, $\langle j, k\rangle$, or $\langle i, k\rangle$ which corresponds to the coupling with the lowest magnitude. But when one considers that the average number of frustrated triangles is half of them, it becomes apparent that there will be many more competitions, between different triangles, and the problem of finding the best spin configuration is formidable.

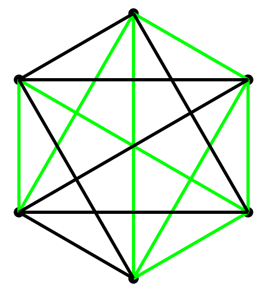

FIGURE 2. The complete graph, K_6, again: relevant for the Sherrington-Kirkpatrick model, though not for the Edwards-Anderon model. We have colored the bonds two colors at random. Let black mean $\mathsf{J}_{ij} > 0$ and green mean $\mathsf{J}_{ij} < 0$. Then there are many "frustrated triangles".

In Figure 2, we drew a complete graph with 6 vertices, and randomly chose 8 edges to make negative. One can see many frustrated triangles. (Note: The number of frustrated triangles does depend on more than just the number of negative edges.)

2. FIRST BOUNDS

We will now prove simple upper and lower bounds for the pressure of the Sherrington-Kirkpatrick model, just on the basis of Jensen's inequality. After that, we will present the "eigenvector shadowing" method of Aizenman, Lebowitz and Ruelle [1], which gives non-trivial bounds on the ground state energy.

Let us first recall what quantities interest us. The partition function, is

$$\mathsf{Z}_N(\beta) \,=\, \sum_{\sigma\in\Omega_N} e^{-\beta\,\mathsf{H}_N^{SK}(\sigma)}\,,$$

which is, of course, random because it depends on all the $(\mathsf{J}_{ij}\,:\,1 \leq i < j \leq N)$ through $\mathsf{H}_N^{SK}(\sigma)$. Then there is the pressure, which is also random,

$$\mathsf{p}_N(\beta) \,=\, \frac{1}{N}\log(\mathsf{Z}_N(\beta))\,.$$

In principle, the quantity one is most interested is the distributional limit of the pressure, if it exists. In other words, the main question is whether there is a random variable $\mathsf{p}(\beta)$ such that

$$\mathsf{p}_N(\beta) \,\xrightarrow{\mathscr{D}}\, \mathsf{p}(\beta)\,,$$

as $N \to \infty$, where $\mathscr{D}$ indicates an identity in distribution, or in this case a limit in distribution? Actually, $\mathsf{p}_N(\beta)$ "concentrates" near its mean. Therefore, it is of interest to examine that mean.

Let us introduce the notation $\mathbb{P}$ and $\mathbb{E}$ to mean the probability distribution of all the i.i.d. $N(0,1)$ random couplings J_{ij}, and the expectation with respect to that probability. Therefore, $\mathbb{E}$ is averaging over the noise in this disordered system. We define the "quenched pressure"

$$p_N^Q(\beta) \,=\, \mathbb{E}[\mathsf{p}_N(\beta)]\,.$$

This is a main quantity of interest. Actually, we are even more interested in the limit

$$p^Q(\beta) \,:=\, \lim_{N\to\infty} p_N^Q(\beta)\,,$$

if it exists. (It does exist, this is an important result of Guerra and Toninelli, which we will talk about later in the semester.)

Let us also define the ground state energy

$$\mathsf{e}_N(0) \,=\, \min_{\sigma\in\Omega_N} \frac{1}{N}\,\mathsf{H}_N(\sigma)\,.$$

This is a random quantity. Starting from the formula for the random pressure,

$$\mathsf{p}_N(\beta) \,=\, \frac{1}{N}\log(\mathsf{Z}_N(\beta)) \,=\, \frac{1}{N}\log\left(\sum_{\sigma\in\Omega_N} e^{-\beta\,\mathsf{H}_N^{SK}(\sigma)}\right)\,,$$

it is easy to see that

$$\mathsf{e}_N(0) \,=\, \lim_{\beta\to\infty} -\beta^{-1}\,\mathsf{p}_N(\beta)\,. \tag{2.1}$$

2.1 Jensen's inequality bounds.

The following are standard first bounds (for example, they are both listed early in Talagrand's book and Saint Flour notes).

Lemma 2.1 *For each N, β and h,*

$$\log(2\cosh(\beta h)) \,\le\, p_N^Q(\beta) \,\le\, \log(2\cosh(\beta h)) + \frac{N-1}{N}\cdot\frac{\beta^2}{4}\,.$$

Proof. Recall that the quenched pressure is defined as

$$p_N^Q(\beta) \,=\, \frac{1}{N}\mathbb{E}[\log(\mathsf{Z}_N(\beta)]\,.$$

There is another quantity, called the "annealed pressure", which is defined

$$p_N^A(\beta) \,=\, \frac{1}{N}\log(\mathbb{E}[\mathsf{Z}_N(\beta)])\,.$$

Note the difference: the annealed pressure has the logarithm outside the expectation. Of course the logarithm is a concave function. Therefore, the logarithm of an average is greater than or equal to the average of the logarithm. So, Jensen's inequality implies that

$$p_N^Q(\beta) \,=\, \frac{1}{N}\mathbb{E}[\log(\mathsf{Z}_N(\beta)] \,\le\, \frac{1}{N}\log(\mathbb{E}[\mathsf{Z}_N(\beta)]) \,=\, p_N^A(\beta)\,.$$

But $p_N^A(\beta)$, unlike $p_N^Q(\beta)$, is easy to calculate explicitly. Namely, because

$$\mathbb{E}[\mathsf{Z}_N(\beta)] \,=\, \mathbb{E}\left[\sum_{\sigma\in\Omega_N} e^{-\beta\,\mathsf{H}_N^{SK}(\sigma)}\right] \,=\, \sum_{\sigma\in\Omega_N}\mathbb{E}\left[e^{-\beta\,\mathsf{H}_N^{SK}(\sigma)}\right]\,.$$

Note that for each $\sigma\in\Omega_N$, the Hamiltonian

$$\mathsf{H}_N^{SK}(\sigma) \,=\, -\frac{1}{\sqrt{N}}\sum_{1\le i<j\le N}\mathsf{J}_{i,j}\sigma_i\sigma_j - h\sum_{i=1}^N\sigma_i\,,$$

is a sum of independent Gaussians, and a nonrandom shift. Therefore, it is a Gaussian. Also, recall that the fundamental property of the Gaussian X is that

$$\mathbb{E}\left[e^{\lambda \mathsf{X}}\right] = e^{\lambda \mathbb{E}[\mathsf{X}] + \frac{\lambda^2}{2}\mathrm{Var}(\mathsf{X})}.$$

For the Hamiltonian, since all the $\mathsf{J}_{i,j}$'s are centered, of unit variance, and independent,

$$\mathbb{E}[\mathsf{H}_N^{SK}(\sigma)] = -h\sum_{i=1}^N \sigma_i\,, \qquad \mathrm{Var}(\mathsf{H}_N^{SK}(\sigma)) = \frac{1}{N}\sum_{1\leq i<j\leq N}\mathrm{Var}(\mathsf{J}_{i,j})\sigma_i^2\sigma_j^2 = \frac{N-1}{2}.$$

Putting this together with the formula above for the moment generating function, gives

$$\mathbb{E}\left[e^{-\beta\,\mathsf{H}_N^{SK}(\sigma)}\right] = e^{\beta h\sum_{i=1}^N \sigma_i + \frac{(N-1)\beta^2}{4}};$$

so,

$$\mathbb{E}[\mathsf{Z}_N(\beta)] = e^{(N-1)\beta^2/4}\sum_{\sigma\in\Omega_N} e^{\beta h\sum_{i=1}^N\sigma_i}.$$

The last sum factorizes, since each σ_i is being summed over $\{+1,-1\}$, independently:

$$\mathbb{E}[\mathsf{Z}_N(\beta)] = e^{(N-1)\beta^2/4}[2\cosh(\beta h)]^N.$$

Taking the logarithm and dividing by N gives

$$p_N^Q(\beta) \leq p_N^A(\beta) = \log(2\cosh(\beta h)) + \frac{N-1}{N}\cdot\frac{\beta^2}{4},$$

which is the claimed upper bound.

The lower bound also uses Jensen's inequality. But now the probability measure isn't the average over the disorder. Instead it is the average over $\sigma\in\Omega_N$. Let us define

$$\tilde{\mu}_{\beta h}(\sigma) = \frac{e^{\beta h\sum_{i=1}^N\sigma_i}}{[2\cosh(\beta h)]^N}.$$

(We put a tilde to remind ourselves that this is not the full, random Gibbs measure, just the nonrandom Gibbs measure coming from the external magnetic field.) Then

$$\begin{aligned}\mathsf{Z}_N(\beta) &= \sum_{\sigma\in\Omega_N} e^{\frac{\beta}{\sqrt{N}}\sum_{i<j}\mathsf{J}_{ij}\sigma_i\sigma_j + \beta h\sum_i\sigma_i}\\ &= [2\cosh(\beta h)]^N\,\mathbf{E}^{\tilde{\mu}_{\beta h}}\left[e^{\frac{\beta}{\sqrt{N}}\sum_{i<j}\mathsf{J}_{ij}\sigma_i\sigma_j}\right].\end{aligned}$$

Again, the expectation here is just over $\sigma\in\Omega_N$, using the measure $\tilde{\mu}_{\beta h}$, not over the random couplings.

Since the exponential function is convex, Jensen's inequality gives

$$\begin{aligned}\mathsf{Z}_N(\beta) &\geq [2\cosh(\beta h)]^N\,\exp\left(\mathbf{E}^{\tilde{\mu}_{\beta h}}\left[\frac{\beta}{\sqrt{N}}\sum_{i<j}\mathsf{J}_{ij}\sigma_i\sigma_j\right]\right)\\ &= [2\cosh(\beta h)]^N\,\exp\left(\frac{\beta}{\sqrt{N}}\sum_{i<j}\mathsf{J}_{ij}\mathbf{E}^{\tilde{\mu}_{\beta h}}[\sigma_i\sigma_j]\right).\end{aligned}$$

The final expectation can be easily seen to be $\tanh^2(\beta h)$. We will not use that now. Rather, what we see is that, taking the logarithm and dividing by N, we have a bound

$$\mathsf{p}_N(\beta) \ \geq \ \log(2\cosh(\beta h)) + \frac{\beta}{N^{3/2}} \sum_{i<j} \mathsf{J}_{ij} \mathbf{E}^{\tilde{\mu}_{\beta h}}[\sigma_i \sigma_j] \, .$$

Taking the expectation with respect to $\mathbb{P}$, the second term vanishes, simply because all the J_{ij} are centered. So

$$p_N^Q(\beta) \ \geq \ \log(2\cosh(\beta h)) \, ,$$

as claimed. The lower bound simply expresses a well-known fact: adding an independent centered Gaussian to the exponent of an exponential only serves to increase the expectation, never lower it. □

Let us now consider why $\frac{1}{\sqrt{N}}$ is the right normalization. In the Curie-Weiss model, which is the nonrandom mean-field Ising model, the correct normalization is $\frac{1}{N}$. But that normalization is not right for the SK model. (For example, see the first $1\frac{1}{2}$ pages of Bovier's Erlangen notes.) Since the couplings are Gaussian, it is the variance that matters, and in the variance, the $\frac{1}{\sqrt{N}}$ term is squared.

The upper bound of Lemma 2.1 is a confirmation that the normalization $\frac{1}{\sqrt{N}}$ is, at least, not too big. If it were, the quenched pressure would diverge as $N \to \infty$, which it does not. Also the upper bound does use the variance, bolstering the previous argument.

On the other hand, since the lower bound is what we would obtain just from setting the random terms in the Hamiltonian to 0, we do not yet know that $\frac{1}{\sqrt{N}}$ is not too small. Among other things, the next section shows that it is just the right size to give nontrivial results.

2.2 The Eigenvector Shadowing bound.

Let us motivate the following bound by noting the failure of the previous bounds to tell us anything useful about the groundstate of H_N^{SK}. Particularly, if one takes $h = 0$ (which is often the most interesting case) then the Jensen's-inequality bounds give

$$\log(2) \ \leq \ p_N^Q(\beta) \ \leq \ \log(2) + \frac{N-1}{N} \cdot \frac{\beta^2}{4} \, .$$

But what does this tell us about $\mathsf{e}_N(0)$? Using equation (2.1) it gives us just

$$-\lim_{\beta \to \infty} \beta^{-1} \left[\log(2) + \frac{N-1}{N} \cdot \frac{\beta^2}{4} \right] \ \leq \ \mathbb{E}[\mathsf{e}_N(0)] \ \leq \ -\lim_{\beta \to \infty} \beta^{-1} \log(2) \, .$$

But all this says is that

$$-\infty \ \leq \ \mathbb{E}[\mathsf{e}_N(0)] \ \leq \ 0 \, ,$$

a fact we could have easily deduced anyway (for example because, even without minimizing over $\sigma \in \Omega_N$, we have $\mathbb{E}[\mathsf{H}_N^{SK}(\sigma)] = 0$ for any fixed σ). The following bounds, which Aizenman, Lebowitz and Ruelle call "weak bounds", are obviously much better.

Proposition 2.2 *For any $\epsilon > 0$,*

$$\lim_{N \to \infty} \mathbb{P}\Big\{ -1 - \epsilon \ \leq \ \mathsf{e}_N(0) \ \leq \ -\frac{2}{\pi} + \epsilon \Big\} \ = \ 1 \, .$$

Proof. Define a random matrix, $\mathsf{A}^{(N)}$, using the couplings of the SK Hamiltonian, and auxiliary i.i.d. $N(0,1)$ random variables, $(\mathsf{J}_i\,:\,1 \leq i \leq N)$, thus:

$$\mathsf{A}_{ij}^{(N)} \;=\; \frac{\mathsf{J}_{ij}}{2\sqrt{N}} \;=\; \mathsf{A}_{ji}^{(N)} \quad \text{for } 1 \leq i < j \leq N;$$
$$\mathsf{A}_{ii}^{(N)} \;=\; \frac{\mathsf{J}_i}{\sqrt{N}} \quad \text{for } 1 \leq i \leq N.$$

Then this is a GOE (Gaussian orthogonal ensemble) matrix. In order to check the normalization, let us observe that for $\overline{\mathrm{Tr}} := \frac{1}{N}\,\mathrm{Tr}$ being the normalized trace (so $\overline{\mathrm{Tr}}(I) = 1$),

$$\mathbb{E}\big[\overline{\mathrm{Tr}}[(\mathsf{A}^{(N)})^2]\big] \;=\; \frac{1}{N}\sum_{i,j=1}^{N} \mathbb{E}[\mathsf{A}_{ij}\mathsf{A}_{ji}] \;=\; \frac{N+1}{4N}\,.$$

By the Wigner semicircle law, one has that the normalized eigenvalue density of such a matrix converges, as $N \to \infty$, to a nonrandom function

$$\rho(\lambda) \;=\; \frac{2}{\pi}\,\mathbb{1}_{[-1,1]}(x)\,\sqrt{1-x^2}\,.$$

Refer to, for example, [3]. (One can check the normalization by observing that $\int \rho(x)\,dx = 1$, which is the limit of $\overline{\mathrm{Tr}}[I]$, and $\int x^2\sigma(x)\,dx \;=\; \frac{1}{4}$, which is the limit of the expectation of $\overline{\mathrm{Tr}}[(\mathsf{A}^{(N)})^2]$.)

Now, for any $\sigma \in \Omega_N$,

$$\langle \sigma, \mathsf{A}^{(N)}\sigma\rangle \;=\; -\mathsf{H}_N(\sigma) + \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \mathsf{J}_i\,,$$

using the fact that $\sigma_i^2 \,=\, 1$ for each i. Of course, the second term on the right hand side is just an $N(0,1)$ random variable. Therefore, modulo an order $O(1/N)$ error, we can think of $\mathsf{e}_N(0)$ as being given instead by

$$\mathsf{e}'_N(0) \;:=\; -\frac{1}{N}\max_{\sigma\in\Omega_N}\langle \sigma, \mathsf{A}^{(N)}\sigma\rangle\,.$$

On the other hand, since $\Omega_N \subset \sqrt{N}\mathbb{S}^{N-1} := \{v \in \mathbb{R}^N\,:\,\|v\| = N\}$, we have

$$\max_{\sigma\in\Omega_N}\langle \sigma, \mathsf{A}^{(N)}\sigma\rangle \;\leq\; N\max_{v\in\mathbb{S}^{N-1}} v^T\mathsf{A}^{(N)}v \;=\; N\lambda_N^{(N)}\,,$$

where $\lambda_N^{(N)}$ is the largest eigenvalue of $\mathsf{A}^{(N)}$. Because of the Wigner semicircle law, we know that the random variables $\lambda_N^{(N)}$ converges to 1, as $N \to \infty$, in probability. This gives the first bound of the proposition, which is the easy one.

The other bound requires a good choice of $\sigma \in \Omega_N$. I.e., instead of merely choosing a vector $v \in \mathbb{R}^N$, with the correct norm, we must satisfy all the combinatorial restrictions. The eigenvector shadowing method is this. Let v^N be the eigenvector for λ_N^N, and take

$$\sigma_i \;=\; \mathrm{sign}(v_i^N)\,,$$

for each i from 1 to N. In particular, then

$$\langle \sigma, v^{(N)}\rangle \;=\; \|v^{(N)}\|_1\,.$$

Certainly part of the spectral decomposition of $\mathsf{A}^{(N)}$ is $\lambda_N^N v^{(N)}\langle v^{(N)},\cdot\rangle$. The contribution to $\frac{1}{N}\langle\sigma,\mathsf{A}^{(N)}\sigma\rangle$ coming from this is λ_N^N times $\frac{1}{N}\|v^{(N)}\|_1^2$. Since we already know that λ_N^N converges to 1, it only remains to ask what $\frac{1}{N}\|v^{(N)}\|_1^2$ is? Of course, by the rotation invariance of the GOE, $v^{(N)}$ is a uniform random vector in $\mathbb{S}^{N-1}$. Therefore, we obtain

$$\lim_{N\to\infty}\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\|v^{(N)}\|_1-\sqrt{\frac{2}{\pi}}\right)^2\right] = 0\,.$$

This follows more-or-less from the usual WLLN argument, once one observes (1) that the individual components of $\sqrt{N}v^{(N)}$ each converge to $N(0,1)$ in distribution as $N\to\infty$, and moreover any fixed number of them become asymptotically independent, and (2) given an $N(0,1)$ random variable X, one has $\mathbb{E}[|\mathsf{X}|] = \sqrt{2/\pi}$ by an easy calculation. One may worry that the asymptotic pairwise independence of the components of $v^{(N)}$, for example, isn't enough to guarantee the limit above. But it is, since in the calculation of the-L^2 norm, one considers a sum of expectations, each of which involves at most two components. We leave it as an exercise to the interested reader to derive the result that the components of a properly rescaled spherical distribution converge to i.i.d. Gaussians.

It is now clear (modulo the reader's exercise) that

$$\frac{\lambda_N^{(N)}}{N}(\langle\sigma,v^{(N)}\rangle)^2 \to \frac{2}{\pi}\,,$$

in probability, as $N\to\infty$, It remains to prove that

$$\sum_{n=1}^{N-1}\frac{\lambda_n^{(N)}}{N}(\langle\sigma,v^{(n)}\rangle)^2 \to 0\,,$$

as $N\to\infty$, in probability. But it is easy to see that the components of $\sigma - v\langle v,\sigma\rangle$ are equally spread with respect to the spectral measure of $\mathsf{A}^{(N)}$. Therefore, using Wigner's semicircle law again, the limit-in-probability really is 0, since $\int x\rho(x)\,dx = 0$. □

Remark 2.3 The spherical version of the SK model is the one where the combinatorial restriction on σ is completely relaxed. It was studied by the physicists, Kosterlitz, Thouless and Jones in [2].

It seems like it would be an interesting problem to improve either the upper or lower bounds without resorting to "Parisi's ansatz".

References

[1] M. Aizenman, J. L. Lebowitz and D. Ruelle. Some rigorous results on the Sherrington-Kirkpatrick Spin Glass Model. *Commun. Math. Phys.* **112** (1987) pp. 3–20.
[2] J. M. Kosterlitz, D. J. Thouless, and Raymund C. Jones. Spherical Model of a Spin-Glass
[3] M. L. Mehta, *Random Matrices. Revised and enlarged 2nd Ed.*, Academic Press, Inc., San Diego, CA, 1991.

Mathematics Department, University of Rochester, Rochester, NY 14627
E-mail address: sstarr@math.rochester.edu