MATH 585 – TOPICS IN MATHEMATICAL PHYSICS – FALL 2006

MATHEMATICS OF MEAN FIELD SPIN GLASSES AND THE REPLICA METHOD

LECTURE 3: THE CURIE-WEISS MODEL, PART I

S. STARR
MATHEMATICS DEPARTMENT, UNIVERSITY OF ROCHESTER

CONTENTS

1. MOLECULAR FIELD THEORY

In this section, we will start by explaining one of the physicists' approaches, called "molecular field theory", or "mean-field theory". We previously used the term "mean-field model" to mean a statistical mechanical model on a complete graph. Therefore, we will stick with the term "molecular field theory" for the idea we are about to present. Let us begin by reconsidering the Ising model, on the d-dimensional graph $\Lambda_N = \mathbb{Z}_N^d$, for some fixed d. Let us redefine the Hamiltonian, to take into account the explicit dependence on dimension:

$$H_N(\sigma) \;=\; -\frac{J}{d}\sum_{\langle x,y\rangle}\sigma_x\sigma_y - h\sum_x \sigma_x\,,$$

where $\sigma = (\sigma_x \,:\, x \in \Lambda)$, with each σ_x in $\{+1, -1\}$, as before. The only difference, relative to what we have done previously, is that we divided J by d. The reason we did this is because if we count the ration of the number of edges $\langle x, y\rangle$ to the number of vertices x, it is exactly d. (There are $2d$ edges incident to each vertex, but each edge is comprised of two vertices.) Also, let us agree henceforth in this lecture to let $J = 1$, because for this model we only need two of the three parameters, (β, J, h), to be independent.

Date: September 12, 2006.

Let us recall the thermodynamic quantities of interest to us. There is the partition function,

$$Z_N(\beta) := \sum_{\sigma \in \Omega_N} e^{-\beta H_N(\sigma)} .$$

Of course, here $\Omega_N = \{+1, -1\}^{\Lambda_N}$ means the set of all spin configurations. The "pressure" is

$$p_N(\beta) := \frac{1}{|\Lambda_N|} \log(Z_N(\beta)) .$$

(Actually $p_N(\beta)$ is really a function of the two independent parameters β and h. We will write $p_N(\beta, h)$ to emphasize this whenever necessary.) The Gibbs measure on Ω_N is given by the measure

$$\mu_{\beta,N}(\sigma) := \frac{e^{-\beta H_N(\sigma)}}{Z_N(\beta)} .$$

Also, in this lecture, it will be important to consider the magnetization

$$m_N(\sigma) = \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \sigma_x ,$$

and the average magnetization

$$\overline{m}_N(\beta, h) = \mathbf{E}^{\mu_{\beta,N}}[m_N(\sigma)] .$$

Recall the simple calculation (let us write ∂_h for $\partial/\partial h$ when convenient)

$$\begin{aligned} \beta^{-1} \partial_h \, p_N(\beta, h) &= (\beta N)^{-1} \, \partial_h \, \log(Z_N(\beta, h)) \\ &= \frac{1}{\beta N} \cdot \frac{\partial_h Z_N(\beta, h)}{Z_N(\beta, h)} \\ &= \frac{1}{N} \cdot \frac{\sum_{\sigma \in \Omega_N} e^{-\beta H_N(\sigma)} \, \partial_h(-H_N(\sigma))}{Z_N(\beta)} \\ &= \sum_{\sigma \in \Omega_N} \frac{e^{-\beta H_N(\sigma)}}{Z_N(\beta)} \, m_N(\sigma) \\ &= \overline{m}_N(\beta, h) . \end{aligned}$$

This gives a useful way to calculate $\overline{m}_N(\beta, h)$, but only if one can calculate $p_N(\beta, h)$ first, which is usually pretty difficult.

The molecular field theory replaces the entire formalism presented above with a single nonlinear equation:

$$\eta_* = \tanh(\beta[h + \eta_*]) .$$

This is an equation which should be solved to give $\eta_* = \eta_*(\beta, h)$.

1.1 Heuristic derivation.

In its essence, the molecular field theory applied to the Ising model is just this equation, without necessarily having any relationship to the original model. (Indeed, it is not particularly relevant to the original model near the critical points at all.) But let us give a completely heuristic and nonrigorous explanation for how one might motivate this equation starting from the Ising model.

We begin by trying to write the Ising Hamiltonian to look like the Hamiltonian for a pure external magnetic field. Thus, we introduce random variables $\eta_x(\sigma)$ for each $x \in \Lambda_N$ such that

$$H_N(\sigma) \;=\; -\sum_{x\in\Lambda_N} (h+\eta_x(\sigma))\,\sigma_x\,.$$

It is clear how we should define $\eta_x(\sigma)$ to make this a true equation:

$$\eta_x(\sigma) \;=\; \frac{1}{2d}\sum_{\substack{y:\\ \langle x,y\rangle}} \sigma_y\,.$$

Here, the sum means to sum over all those $y \in \Lambda_N$ such that $\langle x, y\rangle$ is an actual edge in the graph. (Again, there is a factor of 2 multiplying d, because each edge is incident to two vertices.)

So far there has been no error, and everything is rigorous at the level of analyzing the Ising model. But let us now intentionally make an error. We will suppose that $\mu_{\beta,N}$ is such that $\eta_x(\sigma)$ is "concentrated near" a nonrandom fixed value η_*, the same for all $x \in \Lambda$, and in such a strong way that in the definition of the pressure one can simply replace every $\eta_x(\sigma)$ by η_*. This is completely incorrect! But let us proceed, anyway.

We will put tildes to remind ourselves that nothing that follows is correct for the original model. Thus, we have

$$\tilde{H}_N(\sigma) \;:=\; -(h+\eta_*)\sum_{x\in\Lambda_N}\sigma_x\,.$$

Also, by performing the same type of calculation we have done before,

$$\begin{aligned}
\tilde{Z}_N(\beta) \;&:=\; \sum_{\sigma\in\Omega_N} e^{-\beta\tilde{H}_N(\sigma)} \\
&=\; \sum_{\sigma_x\in\{+1,-1\}\text{ for each }x} e^{\beta[h+\eta_*]\sum_x\sigma_x} \\
&=\; \prod_{x\in\Lambda_N}\left[\sum_{\sigma_x\in\{+1,-1\}} e^{\beta[h+\eta_*]\sigma_x}\right] \\
&=\; [2\cosh(\beta[h+\eta_*])]^{|\Lambda_x|}\;.
\end{aligned}$$

Therefore, the pressure has become

$$\tilde{p}_N(\beta) \;:=\; \frac{1}{|\Lambda_N|}\log(\tilde{Z}_N(\beta)) \;=\; \log(2\cosh(\beta[h+\eta_*]))\,.$$

Taking the derivative gives the ostensible value of the mean magnetization,

$$\tilde{\overline{m}}_N(\beta,h) \;=\; \beta^{-1}\partial_h\tilde{p}_N(\beta,h) \;=\; \tanh(\beta[h+\eta_*])\,.$$

We now have to derive the "constitutive law" relating this back to η_*.

We are pretending to believe that each $\eta_x(\sigma)$ concentrates near η_*. By averaging $\eta_x(\sigma)$ over all x, we obtain

$$\begin{aligned}
\frac{1}{|\Lambda_N|}\sum_{x\in\Lambda_N}\eta_x(\sigma) &= \frac{1}{2d|\Lambda_N|}\sum_{x\in\Lambda_N}\sum_{y\,:\,\langle x,y\rangle}\sigma_y \\
&= \frac{1}{2d|\Lambda_N|}\sum_{\langle x,y\rangle}(\sigma_x+\sigma_y) \\
&= \frac{1}{2d|\Lambda_N|}\sum_{x\in\Lambda_N}\sigma_x\,\#\{y\,:\,\langle x,y\rangle\} \\
&= \frac{1}{|\Lambda_N|}\sum_{x\in\Lambda_N}\sigma_x \\
&= m_N(\sigma)\,.
\end{aligned}$$

Therefore, if we really believe that each $\eta_x(\sigma)$ concentrates near η_* (in the Boltzmann-Gibbs measure) then we should also accept that $m_N(\sigma)$ is concentrated near this same value. Of course, then the expectation of $m_N(\sigma)$ must be this same value. So,

$$\tilde{\overline{m}}_N(\beta,h) \;=\; \eta_*\,.$$

This is the consitutive law for η_*. Combining it with the previous heuristic equation, we obtain

$$\eta_* \;=\; \tanh(\beta[h+\eta_*])\,.$$

Thus we have heuristically derived the molecular field equation. On the other hand, we know we have committed an error in our logic to get here.

1.2 Graphical analysis.

Let us now turn to the issue of analyzing this equation. Since $\tanh$ is the more complicated function, let us make up a name for its argument x. Then

$$\eta_* \;=\; \frac{x}{\beta}-h\,. \tag{1.1}$$

So we can rewrite the equation as

$$\tanh(x) \;=\; \frac{x}{\beta}-h\,.$$

This is hard to solve explicitly analytically, but easy to solve graphically.

One simply graphs $\tanh(x)$ and then graphs the line $y=\beta^{-1}x-h$ following the formula for η_* in (1.1). Then the points of intersection give the solution to the molecular field equation. If one thinks that molecular field theory gives the right answer, then one still comes up against an obstacle: for certain values of β and h there are multiple solutions. For example, while in Figure 1 there are three values of β and h with unique solutions, in Figure 2 there is one value $(\beta,h)=(2,0)$ that has multiple solutions. At this point one needs an external decision of which solution is the best. Any such explanation will inevitably require more explanation of the molecular field theory than we have given previously. Therefore, let us stop considering things in a specious manner, and resume doing rigorous mathematics.

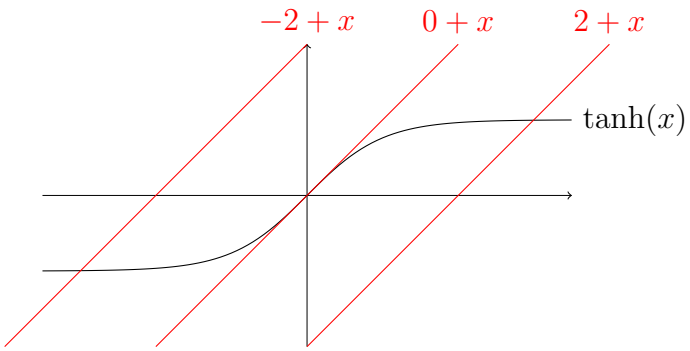


FIGURE 1. Plot of $\tanh(x)$ and three lines. This corresponds to $\beta = 1$ and $h = -2, 0, 2$. One would look for the x-coordinate of the intersection point. One can then obtain η_* from equation (1.1).

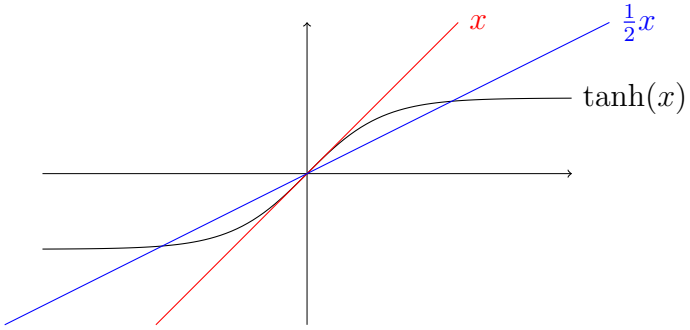


FIGURE 2. Plot of $\tanh(x)$ for $h = 0$ and two values of β: 1 and 2. At $\beta = 1$ the unique intersection point is $(0, 0)$, but at $\beta = 2$, there are three points of intersections. One then needs an external criterion for deciding which of these solutions are correct.

2. THE CURIE-WEISS MODEL: SOLUTION #1

The molecular field theory was stated for the Ising model in d-dimensions, on the graph $\Lambda_N = \mathbb{Z}_N^d$. As a methodology it can be applied to such problems. But then we know it gives incorrect results! It is true that the random variable $m_N(\sigma)$, with the Boltzmann-Gibbs distribution for σ, does "concentrate" near some value which we could call η_*, in the $N \to \infty$, as long as there is no phase transition in h at that point. One justification of why this is true will come in the next lecture, on the basis of large deviations. Another will come in the two lectures after that, on the basis of "tail triviality".

On the other hand, the same is not true of the individual "fields": $\eta_x(\sigma) = \frac{1}{d}\sum_{y\,:\,\langle x,y\rangle} \sigma_y$. The reason it cannot be true is that, if it were, there would be no nontrivial spatial correlations, $\mathbf{E}^{\mu_{\beta,N}}[(\sigma_x - \eta_*)(\sigma_y - \eta_*)]$, among the spin variables, themselves. (Essentially this is because the vector of "fields", $\eta = (\eta_x(\sigma) \,:\, x \in \mathbb{Z}_N^d)$, is the discrete Laplace matrix times the vector of spins, $\sigma = (\sigma_x \,:\, x \in \mathbb{Z}_N^d)$.) Experience tells us that this is unreasonable: the underlying geometry of the lattice $\Lambda_N = \mathbb{Z}_N^d$ *must* show up in the behavior of the spins σ, if only in some "universal" way that depends on symmetry and/or dimensionality.

On the other hand, if one considers the "mean-field" model, on the complete graph, then one can prove that certain conclusions of the molecular field theory are correct in the thermodynamic limit. This is what we will do next. We are going to see several other solutions of this model. But in this lecture, we will consider an approach to the solution which uses PDE.

The "lattice" for the Curie-Weiss model is $\Lambda_N = \{1, 2, \dots, N\}$, and the sample space is $\Omega_N = \{+1, -1\}^N$. This is the set of all spin configurations, $\sigma = (\sigma_1, \dots, \sigma_N)$. In this lecture we will define the Curie-Weiss model to include diagonal terms:

$$H_N(\sigma) \;=\; -\frac{J}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i\,.$$

Note that when $i = j$ the contribution of $\sigma_i\sigma_j = \sigma_1^2$ is 1 independent of the choice of σ. That is why the diagonal terms are usually left out. (In a later lecture we will revert to that convention.) But it is sometimes convenient to include them, which is why we do.

Of course, the partition and pressure are as usual,

$$Z_N(\beta, J, h) \;=\; \sum_{\sigma\in\Omega_N} e^{-\beta H_N(\sigma)}\,,$$

and

$$p_N(\beta, J, h) \;=\; \frac{1}{N}\log(Z_N(\beta, J, h))\,.$$

Let us also define the "average magnetization",

$$\overline{m}_N(\beta, J, h) \;=\; \beta^{-1}\partial_h p_N(\beta, h, J)\,.$$

Recall that, fairly generally,

$$\overline{m}_N(\beta, J, h) \;=\; \mathbf{E}^{\mu_{\beta,N}}[m_N(\sigma)]\,,$$

where $\mu_{\beta,N}$ is the Boltzmann-Gibbs measure on Ω_N (which we should actually write as $\mu_{\beta,J,h,N}$, but we will not) and

$$m_N(\sigma) \;=\; \frac{1}{N}\sum_{i=1}^{N}\sigma_i\,.$$

Thus, knowing $\overline{m}_N(\beta, J, h)$ tells us the average magnetization in the Boltzmann-Gibbs measure. One may also note that

$$-\beta H_N(\sigma) \;=\; N\left(\frac{1}{2}\beta J m_N(\sigma)^2 + \beta h m_N(\sigma)\right)\,.$$

The parameters β, J and h enter into the formula only through the combinations βJ and βh, written above. Therefore, we introduce variables $t = \beta J$ and $x = \beta h$. We define a function $u_N(t, x)$ by

$$u_N(t, x) \;=\; \overline{m}_N(1, t, x)\,,$$

so that $\overline{m}_N(\beta, J, h) = u_N(\beta J, \beta h)$. In order to restrict to the ferromagnetic region, as opposed to the antiferromagnetic region, we consider $(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$.

2.1 The viscous Burgers equation.

It is obvious from the formula that $p_N(\beta, J, h)$ is a smooth function for finite N (in fact real analytic). Therefore, so is $m_N(\beta, J, h) = \beta^{-1}\partial_h p_N(\beta, J, h)$, as long as $\beta > 0$. So $u_N(t, x)$ is smooth. We have the following characterization.

Lemma 2.1 *For each $N \in \mathbb{Z}_{>0}$, the function $u_N : \mathbb{R}_{\geq 0} \times \mathbb{R}$ solves the following well-posed initial-value-problem type PDE,*

$$\begin{cases} \partial_t u_N - u_N\, \partial_x u_N \;=\; \frac{1}{2N}\partial_x^2 u_N & \text{for } (t,x) \text{ in } \mathbb{R}_{>0} \times \mathbb{R}; \\ u_N(0,x) \;=\; \tanh(x) & \text{for } x \text{ in } \mathbb{R}. \end{cases}$$

Proof. We have the formula

$$Z_N(1,t,x) \;=\; \sum_{\sigma \in \Omega_N} \exp\left(N \left[\frac{t}{2} m_N(\sigma)^2 + x m_N(\sigma) \right] \right) .$$

Therefore,

$$\partial_t Z_N \;=\; \sum_{\sigma \in \Omega_N} \exp\left(N \left[\frac{t}{2} m_N(\sigma)^2 + x m_N(\sigma) \right] \right) \frac{N}{2} m_N(\sigma)^2 ,$$

while

$$\partial_x Z_N \;=\; \sum_{\sigma \in \Omega_N} \exp\left(N \left[\frac{t}{2} m_N(\sigma)^2 + x m_N(\sigma) \right] \right) N m_N(\sigma) .$$

On the other hand, we also have

$$\partial_x^2 Z_N \;=\; \sum_{\sigma \in \Omega_N} \exp\left(N \left[\frac{t}{2} m_N(\sigma)^2 + x m_N(\sigma) \right] \right) N^2 m_N(\sigma)^2 .$$

Therefore,

$$\partial_x^2 Z_N \;=\; 2N \partial_t Z_N .$$

This means that

$$\partial_x^2 e^{N p_N} \;=\; 2N \partial_t e^{N p_N} .$$

Carrying out the chain-rule leads to

$$e^{N p_N} N^2 (\partial_x p_N)^2 + e^{N p_N} N \partial_x^2 p_N \;=\; 2N^2 e^{N p_N} \partial_t p_N .$$

Dividing through by $N^2 e^{N p_N}$, we get

$$\partial_t p_N - \frac{1}{2}\big(\partial_x p_N\big)^2 \;=\; \frac{1}{2N}\, \partial_x^2 p_N .$$

Finally, considering the formula $u_N(t,x) \;=\; \partial_x [p_N(1,t,x)]$ and taking the derivative of the previous formula gives

$$\partial_t u_N - u_N\, \partial_x u_N \;=\; \frac{1}{2N}\, \partial_x^2 u_N .$$

In order to get the initial value condition we start from the formula we have calculated before:

$$p_N(\beta, 0, h) \;=\; \log(2\cosh(\beta h)) .$$

This means that $p_N(1,0,x) = \log(2) + \log(\cosh(x))$. Taking the derivative gives $u_N(0,x) = \tanh(x)$, as claimed. □

This equation is known as the viscous Burgers equation. It is a famous PDE. It is integrable by the nonlinear Hopf-Cole transformation. It arises when considering shallow water waves, or gas dynamics. A standard reference is Chapter 4 of Whitham's book [6]. Other good references are Part I of LeVeque's book [3] and various parts of Evans's textbook [1]. (For Evans, you might start with Sections 4.4.1 and 4.5.2 and then move to Chapter 3.)

2.2 The Hopf-Cole transform.

Actually, the Hopf-Cole transform will undo the procedure that we introduced to begin with. I.e., it results in needing to solve the PDE for $Z_N(1,t,x)$, which is the heat equation (as we saw before). Ordinarily, we would have started by deriving the formula for $u_N(t,x)$, directly, by brute force. Then we would have arrived at the nonlinear PDE. Generally nonlinear PDE's are difficult, if not impossible, to solve directly because there is no superposition principle, so important and useful for linear equations. On the other hand, we would have miraculously observed the following transformation: assume that there exists a function $Z_N(t,x)$ such that

$$u_N(t,x) \;=\; \frac{1}{N}\cdot\frac{\partial_x Z_N(t,x)}{Z_N(t,x)}\,,$$

and then realize that this means that the equation for $Z_N(t,x)$ is solvable. Namely, it is just

$$\partial_t Z_N(t,x) \;=\; \frac{1}{2N}\partial_x^2 Z_N(t,x)\,.$$

This precisely undoes the steps we did to begin with. But if we hadn't done these steps to begin with, we would have been surprised.

The point is that the heat equation is considered to be an integrable problem. In particular, as probabilists, we know that the solution is obtained using Brownian motion: Let B_t be a standard Brownian motion, normalized so that

$$\mathbb{E}[\mathsf{B}_s\mathsf{B}_t] \;=\; \min(s,t)\,,$$

for $s,t\geq 0$. The entire Wiener measure for Brownian motion is a measure on continuous paths in $\mathbb{R}$. But for solving the heat equation, we only need the the single-time marginals. (I.e., we consider it just as a set of Gaussian processes indexed by $t>0$, but never consider the joint distributions for any different times.) This is given by the probability density function on $\mathbb{R}$,

$$f_t(x) \;=\; \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\,.$$

By a direct calculation, one easily sees that for any sufficiently smooth function $\phi:\mathbb{R}\to\mathbb{R}$,

$$\frac{d}{dt}\mathbb{E}[\phi(\mathsf{B}_t)] \;=\; \frac{1}{2}\mathbb{E}[\phi''(\mathsf{B}_t)]\,,$$

for all $t>0$. Therefore, in particular, if we *define*, for $t>0$,

$$Z_N(t,x) \;:=\; \mathbb{E}\Big[Z_N\Big(0,x+\frac{1}{\sqrt{N}}\mathsf{B}_t\Big)\Big]\,,$$

we will see that

$$\partial_t Z_N(t,x) \;=\; \frac{\partial}{\partial t}\mathbb{E}\Big[Z_N\Big(0,x+\frac{1}{\sqrt{N}}\mathsf{B}_t\Big)\Big] \;=\; \frac{1}{2N}\mathbb{E}\Big[\partial_x^2 Z_N\Big(0,x+\frac{1}{\sqrt{N}}\mathsf{B}_t\Big)\Big] \;=\; \frac{1}{2N}\partial_x^2 Z_N(t,x)\,.$$

Since we do know $Z_N(0,x)$ – it is $e^{N\log(2\cosh(x))}$ – this solves the PDE for $Z_N(t,x)$, for all later times t, in terms of an expectation with respect to Brownian motion. (Actually, since we are just considering single-time marginals, it would be even more appropriate to say that it gives the solution just in terms of a family of expectations with respect to Gaussian random variables.)

We could now undo the transformation, to determine that

$$u_N(t,x) \;=\; \frac{1}{N}\,\frac{\partial_x \mathbb{E}[e^{N\log(2\cosh(x+\mathsf{B}_t))}]}{\mathbb{E}[e^{N\log(2\cosh(x+N^{-1/2}\mathsf{B}_t))}]}\,,$$

which can also be written more simply as

$$u_N(t,x) \;=\; \frac{1}{N}\,\partial_x \log(\mathbb{E}[e^{N\log(2\cosh(x+N^{-1/2}\mathsf{B}_t))}])\,.$$

In fact, the reason is that we actually have the formula

$$p_N(t,x) = \frac{1}{N}\log(\mathbb{E}[e^{N\log(2\cosh(x+\mathsf{B}_t))}])\,.$$

In fact, the Hopf-Cole transform is even easier to motivate for this problem. The PDE for $p_N(t,x)$ is, as we determined before,

$$\partial_t p_N - \frac{1}{2}\,(\partial_x p_N)^2 = \frac{1}{2N}\partial^2 p_N\,.$$

This is called the "integrated Burgers equation". It arises, for example, in relation to the stochastic linear-quadratic regulator problem of optimal control theory. But then it usually runs in reverse and the terminal condition (then) is related to the terminal cost function. We will actually run across the backwards-in-time integrated Burgers equation in relation to the spin glass. Therefore, let us briefly digress to explicate the Hopf-Cole transform for it.

The usual Hopf-Cole transform works by simply exponentiating the undetermined function. So we define a new function

$$\psi(t,x) \;=\; e^{\alpha p_N(t,x)}\,.$$

Then (as before) we see that $\partial_t\psi \;=\; \alpha e^{\alpha p_N}\partial_t p_N$, $\partial_x\psi \;=\; \alpha e^{\alpha p_N}\partial_x p_N$, and

$$\partial_x^2\psi \;=\; e^{\alpha p_N}\left[\alpha^2(\partial_x p_N)^2 + \alpha\partial_x^2 p_N\right]\,.$$

If we choose α correctly, this last term has the correct proportions to get rid of both the diffusion term and nonlinear first-order term, together. From the integrated Burgers equation, above, it is clear that what we need is that $\alpha/\alpha^2 = 1/N$. Or in other words, $\alpha = N$, as we had it before.

We might mention, in passing, that the same trick of exponentiating works for a very important ODE, the Ricatti equation. Let us mention only the simplest version:

$$a'(t) \;=\; 2a(t)^2\,.$$

This is what the integrated Burgers equation reduces to if the initial condition is pure quadratic, $u_N(0,x) = a_0x^2 + c_0$, and if we make the ansatz that u_N remains quadratic in x, thereafter: $u_N(t,x) = a(t)x^2 + c(t)$. Plugging this in, leads to $c'(t) = \frac{1}{N}a(t)$ and $a'(t) = 2a(t)^2$. (This is actually relevant because a model equation in control theory puts the terminal cost equal to a pure quadratic. There are also statistical mechanical models where it seems to

apply, such as the Curie-Weiss model where each spin is *a priori* an independent Gaussian random variable instead of an independent Bernoulli. The problem is that then the Gibbs weight itself can dominate the *a priori* Gaussian measure, so that one obtains a direction in spin space which has infinite variance. When that happens, the pressure is undefined, even for finite N.)

3. The inviscid limit

We are interested in the $N \to \infty$ limit of $u_N(t,x)$, which we have from before,

$$u_N(t,x) \;=\; \frac{1}{N}\,\frac{\partial_x \mathbb{E}[e^{N\log(2\cosh(x+\mathsf{B}_t))}]}{\mathbb{E}[e^{N\log(2\cosh(x+N^{-1/2}\mathsf{B}_t))}]}\,,$$

Evidently, this is a slightly singular type of asymptotic limit, and we have to be careful in taking it. The method that works is Laplace's method, which is closely related (or identical) to: the stationary phase method, the steepest descents method, and the saddle-point method. (See, [4], for example, if you are interested in learning more for such asymptotic techniques than what we will touch on this semester.) We are not going to do the analysis, here. The reason is that we will do a similar step, in the context of large deviation theory, in the next lecture (or two) using Varadhan's lemma. We will save that analysis until then. The asymptotic analysis is performed in references [1] and [6].

What we will do is to consider the limiting PDE directly, and use the method of characteristics, supplemented with the appropriate jump conditions in case of a shock. The reason the PDE,

$$\partial_t u_N - u_N\,\partial_x u_N \;=\; \frac{1}{2N}\partial_x^2 u_N\,,$$

is called the *viscous* Burgers equation is that the second derivative term represents viscosity. In the $N \to \infty$ limit, this term vanishes and we are left with the usual inviscid Burgers equation

$$\partial_t u - u\,\partial_x u \;=\; 0\,,$$

but this must be supplemented by stating how one deals with shocks in the weak solution. A weak solution to this PDE is a function $u(t,x)$, which is in an appropriate Sobolev space, so that for any smooth function $\phi(t,x)$ with compact support in space at all times,

$$\int_0^\infty \int_{-\infty}^\infty \left[(\partial_t\phi)\cdot u - \frac{1}{2}(\partial_x\phi)\cdot u^2\right]\,dx\,dt \;=\; -\int_{-\infty}^\infty \phi(0,x)\,u(0,x)\,dx\,. \tag{3.1}$$

This is what one would obtain if one assumed that $u(t,x)$, itself, was sufficiently smooth, and integrated

$$\int_0^\infty \int_{-\infty}^\infty \phi\cdot[\partial_t u - u\partial_x u]\,dx\,dt\,,$$

by parts. On the other hand, equation (3.1) may possess solutions which are not smooth enough to qualify as strong solutions to Burgers' equation, directly. In fact, this happens and the issue becomes to distinguish among the infinitely many possible weak solutions to find the right one.

3.1 The method of characteristics.

It turns out to be very easy to find implicit solutions to the Burgers equation for small times. Suppose one takes a curve $x(t)$ for $t \geq 0$ such that $x(0) = x_0$. Then one finds that

$$\frac{d}{dt}u(t, x(t)) \;=\; [\partial_t u](t, x(t)) + [\partial_x u](t, x(t)) \cdot x'(t)\,.$$

If we take $x'(t) = -u(t, x(t))$, then because of Burgers' equation, this means that $u(t, x(t))$ is constant in time. Therefore, in particular, $u(t, x(t)) = u(0, x_0)$. So $x(t) = x_0 - u(0, x_0)$. This leads to the proposition that one point on the curve at time t is $x = x_0 - u(0, x_0)$ and $u = u(0, x_0)$. Incidentally, this leads to a very easy method for drawing such curves at times $t > 0$: given the x-set $(x_1, \ldots, x_K)$ and the set for u, $(U_1, \ldots, U_K)$, then at time t, simply leave the u set alone, and replace the x-set by $(x_1 - t * U_1, \ldots, x_K - t * U_K)$. I have done this for several time steps in Figure 3.

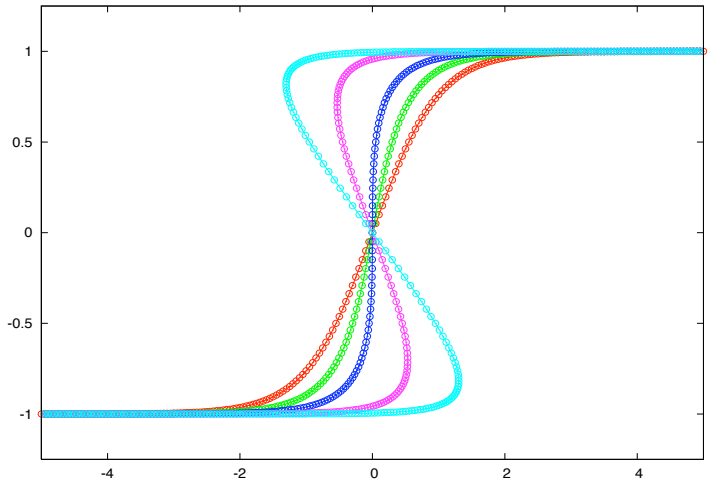
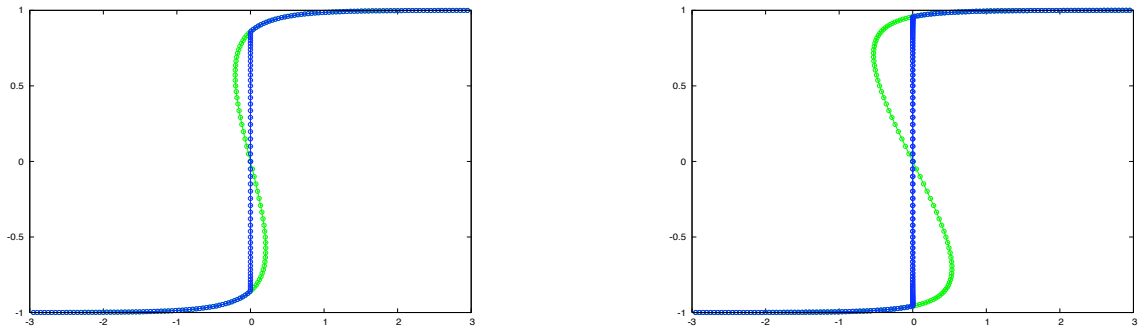


FIGURE 3. The method of characteristics allows one to easily obtain plots of the supposed waveform at times $t > 0$. Here we have plotted: red, $t = 0$; green, $t = 0.5$; blue, $t = 1$ (which is the critical time when a shock first occurs); maroon, $t = 2$; and cyan, $t = 3$. The problem is that after $t = 1$, the waveforms obtained this way are multi-valued with overhangs.

Looking at Figure 1, one can immediately see the problem with this approach. For small times, namely for $t < 1$, the solution one obtains this way is fine. In fact, it is evidently smooth. Therefore, it must be the (unique) strong solution of Burgers' equation. At time $t = 1$ it develops an infinite slope at $x = 0$, though it still remains single-valued. For time $t > 1$, it becomes multivalued. The resolution of this problem is to make a vertical cut at $x = 0$ to lop-off the overhangs. The fact that the vertical cut must be at $x = 0$ is evident through spin flip symmetry, which means that $u(t, x)$ should equal $-u(t, -x)$. More generally, the vertical cut is always made so as to have equal areas chopped off to the left and right of the cut. This is known as the equal-areas rule in the context of hyperbolic PDE's. It is known as Maxwell's construction in statistical physics.



(a) (b)

FIGURE 4. Here we demonstrate the cut-off procedure that allows for shocks with the appropriate entropy condition. The green curve is the one obtained by the method of characteristics and is multivalued. The blue curve is the correct curve. It is single-values except at $x = 0$ where there is a jump discontinuity, signifying a phase transition. (a) Time equals $t = 1.5$. (b) Time equals $t = 2$.

The equal-areas rule can be motivated by appealing to a conservation law for "$\int_{\mathbb{R}} u(t,x)\,dx$". (See, for example, Section 2.8 of [6].) This requires a fair amount of interpretation and explanation in the present context because that integral is not absolutely convergent. On the other hand, the equal areas formula will be verified in the next lecture(s) using the result of Varadhan's lemma and large deviations. Those techniques, shed new light on the present problem. We are starting to make a dictionary, albeit in a slightly disorganized fashion, between statistical mechanics, large deviations theory, and hyperbolic PDE's. Therefore, let us mention that what Varadhan's lemma will give us is a variational principle for the pressure, which turns out to be the same as the correct version of the infinite-volume analogue of the Gibbs variational principle for mean-field models. In the context of Burgers' equation, it would be called the Lax-Oleinik formula.

In the context of limits of short-ranged classical spin systems, the Maxwell construction was first explained rigorously by Kac, for a very specific model. He obtained the mean-field limit by considering long-ranged, but not "infinite-ranged", Ising models on $\mathbb{Z}^d$, and letting the range approach infinity, while the strength of the interaction approaches 0, so that the total L^1-norm remains constant. This is now known as the Kac limit. The ultimate theorem about the Kac limit is the Lebowitz-Penrose theorem, which was originally proved for the van-der-Waal's molecular field theory of gases. The ultimate reference for the Lebowitz-Penrose theorem, specialized to classical spin systems, is Appendix C of Colin Thompson's nice textbook on mathematical statistical mechanics [5]. One can find further references, such as to quantum spin system, there. (We may return to this issue later in the semester, because there is a nice paper, within the last five years, by Silvio Franz and Fabio Toninelli on the Kac-limit of the short-ranged spin glass model that is informative.)

As a slight digression, let us mention that one reason that the Lebowitz-Penrose theorem is so important, besides verifying the Maxwell construction, (which can be verified in other ways) is that it actually says something about the short-ranged models. Indeed, if one is willing to take the range to be large enough, *but still finite*, then one can use the mean-field

result as a zeroth-order approximation to perturb off of. In fact, Lebowitz, Mazel and Presutti used just this method, along with other high-powered techniques of statistical mechanics, to prove a gas-liquid phase transition in a continuous model with short-ranged interaction [2]. Their result is the first such result (and as far as I know may be the only one) for a very important problem in mathematical physics. There is another way that the mean-field model is supposed to arise, in addition to Kac-type limits. It is supposed to describe the limit of short-ranged models on $\mathbb{Z}^d$, in the limit that $d \to \infty$. Unfortunately, it does not seem to be as easy to prove this limit, except in some special cases (at criticality) using highly sophisticated methods (such as the renormalization group).

3.2 Conclusion for PDE approach to the Curie-Weiss model.

For now, let us simply observe that, modulo proving that the equal areas construction is the correct one, we can see evidence for a phase transition. For $t > 1$ there is evidently a jump discontinuity in $u(t, x)$ with respect to x at $x = 0$. This means that in zero-field, $h = 0$, there is a first-order phase transition in h for $\beta > 1/J$. There is a more interesting phase transition at the critical point, $h = 0$ and $\beta = 1/J$. We will discuss the critical points more after we develop the large-deviations approach.

We can also now come to a better understanding of the molecular field equation via the graphical method. Suppose we use the method of characteristics. We may want to know $u(t, x)$ for a given t and x. Since we know that x must be given by $x_0 - u(0, x_0) \cdot t$ to be on a characteristic, we seek that value x_0 such that $u(0, x_0) = [x_0 - x]/t$. Let us set $J = 1$, as we did in the first section, and let us identify $x_0- = \beta[h + \eta_*]$ and $x = \beta h$. Then $t = \beta$ since $J = 1$. So the equation reads

$$\eta_* \; = \; \tanh(\beta[h + \eta_*]) \, .$$

Since the constant value of u on the characteristic is $u(0, x_0)$, we see that η_* is the value of $u(t, x) = \overline{m}(\beta, 0, h)$. This is exactly as predicted by molecular field theory. On the other hand, the method of characteristics leads to better pictures!

REFERENCES

[1] L. C. Evans, *Partial Differential Equations.* Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

[2] J. L. Lebowitz, A. Mazel, and E. Presutti. Liquid-vapor phase transitions for systems with finite-range interactions. *J. Statist. Phys.* **94** (1999), no. 5-6, 955–1025. 82B26 (82B21 82D05 82D15)

[3] R. J. LeVeque, *Numerical Methods for Conservation Laws. 2nd Ed.*, Birkhauser Verlag, Basel, Germany, 1992.

[4] L. Sirovich, *Techniques of Asymptotic Analysis.* Applied Mathematical Sciences, 2. Springer Verlag, Heidelberg, Germany, 1971.

[5] C. J. Thompson, *Mathematical Statistical Mechanics.* The Macmillan Company, New York, 1972.

[6] G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley and Sons, New York, 1974.

MATHEMATICS DEPARTMENT, UNIVERSITY OF ROCHESTER, ROCHESTER, NY 14627
E-mail address: sstarr@math.rochester.edu