MATH 585 – TOPICS IN MATHEMATICAL PHYSICS – FALL 2006
MATHEMATICS OF MEAN FIELD SPIN GLASSES AND THE REPLICA METHOD
LECTURE 4: LARGE DEVIATIONS AND THE CURIE-WEISS MODEL

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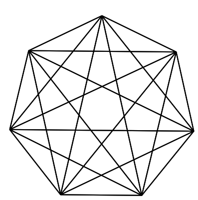
1. DEFINITION OF THE CURIE-WEISS MODEL

In the last lecture we introduced the molecular field theory for the Ising model. It is not exact in finite dimensions. But it does become exact if one considers the Ising model on a complete graph, in the thermodynamic limit. The complete graph on N vertices is the graph with every possible edge present. For example, the complete graph with 7 vertices looks like this:



For each N, $\Lambda_N = \{1, 2, \dots, N\}$, and $\Omega_N = \{+1, -1\}^N$. So a general spin configuration is $\sigma = (\sigma_1, \dots, \sigma_N)$, with each $\sigma_j \in \{+1, -1\}$. The Curie-Weiss Hamiltonian is

$$H_N(\sigma) := -\frac{J}{2N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_j \sigma_k - h \sum_{j=1}^{N} \sigma_j .$$

For convenience, we have included diagonal self-interaction terms. The thermodynamic limit of the pressure is unaffected by this choice. Also, the finite-volume Gibbs states are unaffected by this choice (although they do depend on the normalizing prefactor being $J/2N$

Date: October 6, 2006.

instead of $J/2(N-1)$). Recalling the magnetization function

$$m_N(\sigma) := \frac{1}{N}\sum_{j=1}^{N}\sigma_j ,$$

the Curie-Weiss Hamiltonian can be rewritten as

$$H_N(\sigma) = -N\left[\frac{J}{2}m_N(\sigma)^2 + hm_N(\sigma)\right] .$$

1.1 Thermodynamic Functions.

Let us break with convention and define the thermodynamic functions not in terms of β but in terms of the two parameters $t = \beta J$ and $x = \beta h$. At least this way, nothing is lost; it just means that if we ever want to take ∂_β, instead we take $J\partial_t + h\partial_x$. Then let $\mu_{N,t,x}$ be the probability measure on Ω_N, defined as

$$\mu_{N,t,x}(\sigma) := \frac{e^{N[(t/2)m_N(\sigma)^2 + hm_N(\sigma)]}}{Z_N(t,x)} .$$

The normalization the partition function,

$$Z_N(t,x) := \sum_{\sigma\in\Omega_N} e^{N[(t/2)m_N(\sigma)^2 + hm_N(\sigma)]} .$$

The pressure is

$$p_N(t,x) := \frac{1}{N}\log(Z_N(t,x)) .$$

We call the limit the "thermodynamic pressure" if it exists:

$$p(t,x) := \lim_{N\to\infty} p_N(t,x)) .$$

An interesting parameter is the average magnetization, which should be defined as

$$\bar{m}(t,x) := \partial_x p(t,x) ,$$

assuming the thermodynamic pressure exists and is differentiable. It is easy to see that, whenever the pressure does exist, it is convex. Hence it has left and right derivatives everywhere. We define

$$\bar{m}_\pm(t,x) := \lim_{y\to x^\pm}\frac{p(t,y)-p(t,x)}{y-x} .$$

By soft arguments regarding limits of convex functions,

$$\bar{m}_-(t,x) \le \liminf_{N\to\infty}\mathbf{E}^{\mu_{N,t,x}}[m_N(\sigma)] \le \limsup_{N\to\infty}\mathbf{E}^{\mu_{N,t,x}}[m_N(\sigma)] \le \bar{m}_+(t,x) .$$

In particular, whenever $\bar{m}(t,x)$ exists, it equals the thermodynamic limit of the average magnetization.

2. Main Theorem

The model on the complete graph is called "solvable". This does not mean that it is explicitly solvable in terms of elementary functions. But it can be expressed as the solution of a 1-dimensional variational problem. This represents significant progress over the "many-body" problem we started with.

Theorem 2.1 *For each value of x and t in $\mathbb{R}$, the pressure $p(t,x)$ does exist, and it equals*

$$p(t,x) \;=\; \max_{m\in(-1,1)} \left[\frac{t}{2}m^2 + mx - \frac{1+m}{2}\log\left(\frac{1+m}{2}\right) - \frac{1-m}{2}\log\left(\frac{1-m}{2}\right)\right],$$

which is the same as

$$p(t,x) \;=\; \max_{x^*\in\mathbb{R}} \left[\log(2\cosh(x^*)) - (x^*-x)\tanh(x^*) + \frac{t}{2}\tanh^2(x^*)\right],$$

the molecular field equation for all Ising models.

Remark 2.2 Going from the first to the second equation in the theorem is trivial, simply substitute $m = \tanh(x^*)$ and do the necessary calculations.

Remark 2.3 Recall from Lecture 3′ (section 3.2) that this semi-implicit function is explicit enough to allow one to obtain asymptotic formulas for several thermodynamic quantities of interest near the critical point, $(t,x) = (1,0)$.

Remark 2.4 When $t < 0$, the function being optimized is strictly concave. Therefore, there is a *unique* maximizer. Physically, this corresponds to the fact that the mean-field *antiferromagnet* never has a phase transition. In the present context, a phase transition means there are multiple maximizers.

We will prove this simple theorem by using *large deviations theory*. We take the opportunity to review some elementary aspects of that topic. Our two main references are Varadhan's short monograph [4] and Dembo and Zeitouni's textbook [1]. We only scratch the surface covering just the first sections of these excellent books.

2.1 Large Deviations Theorems: Statements.

Let $\mathscr{X}$ be a complete, separable metric space, and let $\mathscr{F}$ be the Borel σ-field. For each $N \in \mathbb{Z}_{>0}$, let P_N be a probability measure on the probability space $(\mathscr{X},\mathscr{F})$. Then Varadhan says that the sequence $(P_1, P_2, \dots)$ satisfies the *large deviation principle* with rate function $I(\cdot) : \mathscr{X} \to [0,\infty]$ if and only if

- $I(\cdot)$ is lower semicontinuous, meaning that for each $t \in [0,\infty]$ the set $\{x : I(x) \le t\}$ is closed in $\mathscr{X}$;
- for each closed set $F \subseteq \mathscr{X}$,

$$\limsup_{N\to\infty} \frac{1}{N}\log(P_N(F)) \;\le\; -\inf_{x\in F} I(x)\,;$$

- for each open set $U \subseteq \mathscr{X}$,

$$\liminf_{N\to\infty} \frac{1}{N}\log(P_N(U)) \;\ge\; -\inf_{x\in U} I(x)\,.$$

This is not the most general formulation of the large deviation principle. (See the introduction to [1].) But Varadhan immediately proves the following useful theorem, which has come to be known as "Varadhan's lemma":

Theorem 2.5 *Let $(P_1, P_2, \dots)$ satisfy the LDP with rate function $I(\cdot)$. Then for any bounded continuous function $\Phi : \mathscr{X} \to \mathbb{R}$,*

$$\lim_{N\to\infty} \frac{1}{N} \log \left(\mathbf{E}^{P_N}\left[e^{N\Phi(x)}\right]\right) = \sup_{x\in\mathscr{X}}[\Phi(x) - I(x)]\,.$$

For us $\mathscr{X} = [0,1]$ which satisfies the topological requirements. (It is even compact.) The measures we start with are the counting measures on Ω_N (not the Boltzmann-Gibbs measures). These are not normalized, but after normalizing they yield the uniform measures on Ω_N:

$$\mu_{N,0,0}(\sigma) = |\Omega_N|^{-1} = 2^{-N}\,.$$

Then the measure P_N is the induced measure on $[0,1]$ obtained from the map $m_N : \Omega_N \to [0,1]$. Hence, for any measurable subset $E \subset [0,1]$,

$$P_N(E) = \mu_{N,0,0}\{\sigma : m_N(\sigma) \in E\}\,.$$

An equivalent way to get the same thing is to consider $\mathsf{X}_1, \mathsf{X}_2, \dots$ to be i.i.d. random variables with values in $\{+1,-1\}$ with equal probabilities for both, so the distributions are $\mu_{1,0,0}$. Then P_N is the probability distribution for the random variable

$$\frac{\mathsf{X}_1 + \dots + \mathsf{X}_N}{N}\,.$$

Therefore, the following result is applicable.

Theorem 2.6 (Cramér's theorem for finite subsets of $\mathbb{R}$) *Let $\mathsf{X}_1, \mathsf{X}_2, \dots$ be i.i.d. random variables, with distribution P_1 such that $\mathrm{supp}(P_1)$ is a finite subset of $\mathbb{R}$. For each $N \in \mathbb{Z}_{>0}$, let P_N be the probability distribution of*

$$\frac{\mathsf{X}_1 + \dots + \mathsf{X}_N}{N}\,.$$

This sequence of measures satisfies the LDP with rate function

$$I(x) = \sup_{\lambda\in\mathbb{R}}[\lambda x - \Lambda(\lambda)]\,,$$

where $\Lambda : \mathbb{R} \to \mathbb{R}$ is the logarithmic moment generating function of P_1,

$$\Lambda(\lambda) := \log\left(\mathbf{E}^{P_1}\left[e^{\lambda x}\right]\right)\,.$$

Remark 2.7 The support of each measure is a subset of $\mathscr{X} = \mathrm{cch}(\mathrm{supp}(P_1))$, where cch denotes "closed, convex hull". For every $x \in \mathscr{X}^{\complement}$, $I(x) = +\infty$.

We will follow Dembo and Zeitouni in proving this as a consequence of Sanov's theorem. But we will delay this until lecture 5, which we devote to the topic of proving Cramér's theorem. However, with the theorems as stated, we can prove Theorem 2.1.

2.2 Conditional Proof of Main Theorem.

We give the proof of the main theorem now. But it still must be considered a conditional proof, because it relies on Theorem 2.5, which we will prove in the next section, and Theorem 2.6 which we will prove in the next lecture.

As stated above, we can think of $P_1, P_2, \dots$ as being defined on $\mathscr{X} = \mathrm{cch}(\mathrm{supp}(P_1))$. Since $\mathrm{supp}(P_1) = \{+1, -1\}$, this means $\mathscr{X} = [-1, 1]$ as stated before. It is trivial to calculate

$$\Lambda(\lambda) = \log\left(\mathbf{E}^{P_1}\left[e^{\lambda m}\right]\right) = \log(\cosh(\lambda)),$$

since $P_1 = \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$. We are writing the variable of integration as m, not x, because that is what it is in our case. Therefore, Cramér's theorem gives

$$I(m) = \sup_{\lambda \in \mathbb{R}}[\lambda m - \log(\cosh(\lambda))],$$

for each $m \in [-1, 1]$. But this optimum is attained when $\tanh(\lambda) = m$; i.e.,

$$\lambda = \frac{1}{2}\log(1+m) - \frac{1}{2}\log(1-m).$$

(We allow for $\lambda = \pm\infty$, which occurs at $m = \pm 1$, because those are the limits that would be obtained via a "supremizing" sequence.) Therefore by calculations from before,

$$I(m) = \frac{1+m}{2}\log(1+m) + \frac{1-m}{2}\log(1-m).$$

As noted before, P_N is the image of the *uniform probability measure* on Ω_N under the measurable mapping $m_N : \Omega_N \to \mathbb{R}$. But the pressure is not defined with respect to the uniform probability measure, it is defined with respect to the counting measure. Therefore,

$$p_N(t, x) = \log(2) + \frac{1}{N}\log\left(\mathbf{E}^{P_N}\left[e^{N[(t/2)m^2 + mx]}\right]\right),$$

where P_N is a measure on m (not x). By Varadhan's lemma, the thermodynamic limit of the pressure exists, and

$$\begin{aligned} p(t, x) &= \log(2) + \max_{m \in \mathscr{X}}\left[\frac{t}{2}m^2 + mx - I(m)\right] \\ &= \max_{m \in \mathscr{X}}\left[\frac{t}{2}m^2 + mx - \frac{1+m}{2}\log\left(\frac{1+m}{2}\right) - \frac{1-m}{2}\log\left(\frac{1-m}{2}\right)\right]. \end{aligned}$$

We have written the $\max$ instead of $\sup$, which is justified whenever $\mathscr{X}$ is compact, because a lower semicontinuous function always attains its maximum on compact sets. Actually, in the present context, it is easy to see that the maximum is never attained at $m = 1$ or $m = -1$ because the right-derivative of this function equals $+\infty$ at $m = -1$ and the left-derivative equals $-\infty$ at $m = 1$. That is why in the theorem we restricted to $m \in (-1, 1)$.

3. Varadhan's Proof

Here we will simply restate Varadhan's proof of his lemma, paraphrased from [4]. As is common in analysis, an identity is actually proved as the conjunction of two inequalities.

The upper bound relies on a principle which is an essential element of large deviations theory. Given a sequence of nonnegative numbers $\boldsymbol{a} = (a_1, a_2, \dots)$, define

$$L(\boldsymbol{a}) := \limsup_{N \to \infty} \frac{1}{N}\log(a_N).$$

Then, given n different sequences $\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(n)}$, one has

$$L\left(\boldsymbol{a}^{(1)} + \cdots + \boldsymbol{a}^{(n)}\right) \;=\; \max\left\{L\left(\boldsymbol{a}^{(1)}\right), \dots, L\left(\boldsymbol{a}^{(n)}\right)\right\}\,.$$

This is easy to check and we encourage the reader to do it. So the function L turns the binary operation of $+$ into the binary operation of $\vee$.

With this in mind, given any finite number of sets $F_1, \dots, F_n \in \mathscr{F}$, whose union covers $\mathscr{X}$, it is obvious that

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N} \log\left(\mathbf{E}^{P_N}\left[e^{N\Phi(x)}\right]\right) \;&=\; \limsup_{N\to\infty} \frac{1}{N}\left(\int_{\mathscr{X}} e^{N\Phi(x)}\,dP_N(x)\right)\\
&\leq\; \limsup_{N\to\infty} \frac{1}{N}\left(\sum_{k=1}^{n} \int_{F_k} e^{N\Phi(x)}\,dP_N(x)\right)\\
&=\; \max_{1\leq k\leq n} \limsup_{N\to\infty} \frac{1}{N} \log\left(\int_{F_k} e^{N\Phi(x)}\,dP_N(x)\right)\,.
\end{aligned}$$

On the other hand, clearly

$$\log\left(\int_{F_k} e^{N\Phi(x)}\,dP_N(x)\right) \;\leq\; \sup_{x\in F_k} \Phi(x) + \log(P_N(F_k))\,,$$

so that, if F_k is closed, the LDP gives

$$\limsup_{N\to\infty} \frac{1}{N} \log\left(\int_{F_k} e^{N\Phi(x)}\,dP_N(x)\right) \;\leq\; \sup_{x\in F_k} \Phi(x) - \inf_{x\in F_k} I(x)\,.$$

If we also suppose that Φ has oscillation at most ϵ on F_k, then

$$\sup_{x\in F_k} \Phi(x) - \inf_{x\in F_k} I(x) \;\leq\; \epsilon + \sup_{x\in F_k}[\Phi(x) - I(x)]\,.$$

Therefore, with all the suppositions we have made, one concludes

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N} \log\left(\mathbf{E}^{P_N}\left[e^{N\Phi(x)}\right]\right) \;&\leq\; \epsilon + \max_{1\leq k\leq n} \sup_{x\in F_n}[\Phi(x) - I(x)]\\
&=\; \epsilon + \sup_{x\in\mathscr{X}}[\Phi(x) - I(x)]\,.
\end{aligned}$$

So, proving

$$\limsup_{N\to\infty} \frac{1}{N} \log\left(\mathbf{E}^{P_N}\left[e^{N\Phi(x)}\right]\right) \;\leq\; \sup_{x\in\mathscr{X}}[\Phi(x) - I(x)]\,,$$

the upper bound for Varadhan's lemma, is reduced to checking that, for each $\epsilon > 0$, there is an $n \in \mathbb{Z}_{>0}$ and closed sets $F_1, \dots, F_n$, which cover $\mathscr{X}$, and such that Φ has oscillation at most ϵ on each one. But, since $\Phi : \mathscr{X} \to \mathbb{R}$ is bounded and continuous, such sets are easy to find. Namely, just cover the range of Φ by a finite number, n, of closed intervals each of length ϵ. The preimage of each such set under Φ is still closed, their union covers $\mathscr{X}$, and by construction Φ has oscillation at most ϵ on each one.

To conclude Varadhan's proof, we just have to establish the lower bound. He proves this by an "$\epsilon/2$" argument. First, given an arbitrary $\epsilon > 0$, choose $x_0 \in \mathscr{X}$ such that

$$\Phi(x_0) - I(x_0) \;\geq\; -\frac{\epsilon}{2} + \sup_{x\in\mathscr{X}}[\Phi(x) - I(x)]\,.$$

Let U be the open set $\{y \in \mathscr{X} \, : \, |\Phi(y) - \Phi(x)| < \epsilon/2\}$. Then

$$\frac{1}{N} \log \left(\int_U e^{N\Phi(x)} \, dP_N(x) \right) \, \geq \, \Phi(x_0) - \frac{\epsilon}{2} + \frac{1}{N} \log(P_N(U)) \, .$$

By the LDP,

$$\liminf_{N\to\infty} \frac{1}{N} \log(P_N(U)) \geq \, - \inf_{x\in U} I(x) \, \geq \, -I(x_0) \, ,$$

since x_0 is obviously in U. Therefore,

$$\begin{aligned} \liminf_{N\to\infty} \frac{1}{N} \log \left(\mathbf{E}^{P_N} \left[e^{N\Phi(x)} \right] \right) \, &\geq \, \liminf_{N\to\infty} \frac{1}{N} \log \left(\int_U e^{N\Phi(x)} \, dP_N(x) \right) \\ &\geq \, -\frac{\epsilon}{2} + \Phi(x_0) - I(x_0) \\ &\geq \, -\epsilon + \inf_{x\in\mathscr{X}} [\Phi(x) - I(x)] \, . \end{aligned}$$

That is the lower bound. □

Remark 3.1 We state again, just to be clear, that this is the proof given in Varadhan's book, except that we have inserted many more words than he did, spoiling the elegance characteristic of his writing.

4. Comment: the Minimax Theorem

One can observe that by combining Cramér's theorem and Varadhan's lemma, before solving, one obtains

$$\lim_{N\to\infty} \frac{1}{N} \log \left(\mathbf{E}^{P_N} \left[e^{N\Phi(x)} \right] \right) \, = \, \sup_{x\in\mathscr{X}} \inf_{\lambda\in\mathbb{R}} [\Phi(x) - \lambda x + \Lambda(\lambda)] \, .$$

In principle, one can ask whether it is allowed to exchange the order of first taking the min, then taking the max. The answer is "yes" under certain conditions. We quote the following theorem.

Theorem 4.1 (Kneser-Fan Theorem) *Let $\mathscr{X}$ be a compact, convex subset of a Banach space, and let $\mathscr{Y}$ be any convex subset of a vector space. Suppose that $\mathcal{L}$ is a function on $\mathscr{X} \times \mathscr{Y}$ that is concave with respect to $x \in \mathscr{X}$ and convex with respect to $y \in \mathscr{Y}$. Then, if $\mathcal{L}$ is also upper semicontinuous on $\mathscr{X}$, for each fixed $y \in \mathscr{Y}$, one concludes*

$$\sup_{x\in\mathscr{X}} \inf_{y\in\mathscr{Y}} \mathcal{L}(x,y) \, = \, \inf_{y\in\mathscr{Y}} \sup_{x\in\mathscr{X}} \mathcal{L}(x,y) \, .$$

This purely topogical result is proved, for example, in [3]. If $\mathscr{X}$ is compact, as in the hypotheses of the theorem, and if Φ is concave, then we are allowed to switch the order of optimization problems arising from Varadhan's lemma and Cramér's theorem. In our case, where we are calculating the pressure of the Curie-Weiss model, Φ is concave only for the antiferromagnet. It may seems a rather minor point to be allowed to exchange the order of optimization problems, since both are solved more-or-less explicitly by finding critical points through the derivative. In the present context it is a minor point. But, for the antiferromagnetic Curie-Weiss model, the exchanged problem can also be derived by a different technique, called the extended variational principle. (See, for example, [2].) This technique is useful when it comes to spin glasses.

Remark 4.2 The Kneser-Fan theorem, is a nice generalization of a famous result called "von Neumann's minimax theorem" which is the specialization to the case that $\mathcal{L}$ is bilinear and the convex sets are both finite-dimensional simplexes, such as $\Delta_{n-1} = \{(x_1, \dots, x_n) : x_1, \dots, x_n \geq 0$ and $x_1 + \cdots + x_n = 1\}$. If $\mathscr{X} = \Delta_{n-1}$ and $\mathscr{Y} = \Delta_{m-1}$, then $\mathcal{L}$ is derived from an $n \times m$ matrix. In that case, the two optimization problems correspond to finding the best strategy for two different players in a "zero-sum, matrix game". The equality corresponds to the existence of a saddle-point or equilibrium. This problem is apparently useful in economics, which is presumably why von Neumann's minimax theorem is famous.

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