MATH 585 – TOPICS IN MATHEMATICAL PHYSICS – FALL 2006
MATHEMATICS OF MEAN FIELD SPIN GLASSES AND THE REPLICA METHOD
LECTURE 7: SOLUTION OF THE C-W MODEL BY DE FINETTI'S THEOREM

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CONTENTS

This lecture is based on the paper of Fannes, Spohn and Verbeure [2], *although specialized to the classical (non-quantum) setting. The presentation I give here owes a debt also to Eugene Kritchevski,* [4]. *To some extent, the best perspective on this approach is to first consider statistical mechanics on* $\mathbb{Z}^d$ *via the Gibbs variational principle and ergodic decompositions. But to digress into that subject now would take us too far astray. For independent study, we recommend the monographs of Israel* [3] *and Simon* [7].

1. A SECOND VERSION OF THE CURIE-WEISS MODEL

In this lecture, we will define the Curie-Weiss model slightly differently than we did before. Let $\Omega = \{+1, -1\}$. For each $N \in \mathbb{Z}_{\geq 2}$, and $t, x \in \mathbb{R}$, let us define $\varphi_N(\cdot; t, x) : \Omega^N \to \mathbb{R}$ by

$$\varphi_N(\sigma; t, x) \;=\; \frac{t}{N(N-1)} \sum_{1 \leq i < j \leq N} \sigma_i \sigma_j + \frac{x}{N} \sum_{i=1}^{N} \sigma_i \,.$$

This will represent the negative of the Hamiltonian divided by N, which represents the negative of the "energy density".

Let us be slightly pedantic and rewrite φ_N in a different, but equivalent, way. Let λ_N be the uniform measure on $[1, N]$. Let γ_N be the uniform measure on

$$\binom{[1,N]}{2} \;:=\; \big\{\{i,j\} \subset [1,N] \,:\, i \neq j\big\} \,.$$

Then we have

$$\varphi_N(\sigma; t; x) \;:=\; \frac{t}{2}\mathbf{E}^{\gamma_N}[\sigma_{\mathsf{i}}\sigma_{\mathsf{j}}] + x\mathbf{E}^{\lambda_N}[\sigma_{\mathsf{i}}] \,,$$

where $\{\mathsf{i}, \mathsf{j}\}$ is distributed by γ_N in the first expectation, and i is distributed by λ_N in the second expectation.

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Let the partition function be given by

$$Z_N(t,x) \,=\, \sum_{\sigma\in\Omega^N} e^{N\varphi_N(\sigma;t,x)}\,,$$

and the finite-volume approximation to the pressure

$$p_N(t,x) \,=\, \frac{1}{N}\log(Z_N(t,x))\,.$$

With these definitions, we can see that the Hamiltonian would be $H_N(\cdot;J,h)$ with

$$-\beta H_N(\sigma;J,h) \,=\, N\varphi_N(\sigma;\beta J,\beta h)\,.$$

Adjusting for this change, the present definition of the C-W Hamiltonian differs slightly from the definition that we gave when using the LDP. For example, now there are no diagonal terms in the spin-spin interaction. Such minor changes make no difference in the $N\to\infty$ limit. Let us define the thermodynamic pressure

$$p(t,x) \,=\, \lim_{N\to\infty} p_N(t,x)\,,$$

if the limit exists.

2. EXISTENCE OF THE THERMODYNAMIC LIMIT

Let $\alpha\in\mathscr{M}_1(\Omega)$ be the uniform measure,

$$\alpha\{+1\} \,=\, \alpha\{-1\} \,=\, \frac{1}{2}\,.$$

Let $\alpha^{\otimes N}\in\mathscr{M}_1(\Omega^N)$ be the i.i.d. product measure built from α. Note that this is also uniform on Ω^N. Let us define the "relative entropy density" for a system of size N to be the function $s_N:\mathscr{M}_1(\Omega^N)\to\mathbb{R}\cup\{-\infty\}$ defined as

$$s_N(\mu) \,=\, \frac{1}{N}S(\mu|\alpha^{\otimes N})\,.$$

This is the relative entropy density with respect to α. The measure $\alpha^{\otimes N}$ is going to play the role of *a priori* measure in the Gibbs variational principle, as usual. From discussions in previous lectures we know that s_N is strictly concave, and upper semicontinuous.

There are other important features of s_N. One of these is strong subadditivity. Let $\pi:[1,N]\to[1,N]$ be any permutation. Let us define $\pi_*:\Omega^N\to\Omega^N$ by

$$\pi_*(\sigma) \,=\, (\sigma_{\pi(1)},\dots,\sigma_{\pi(N)})\,.$$

Then, for any $\mu\in\mathscr{M}_1(\Omega^N)$,

$$s_N(\mu) \,=\, s_N(\mu\circ\pi_*^{-1})\,,$$

just because $\alpha^{\otimes N}$ is exchangeable. Given $A\subseteq[1,N]$, let $\pi:[1,|A|]\to[1,N]$ be any injection such that $\mathrm{Ran}(\pi)=A$, and define

$$s_A(\mu) \,:=\, s_{|A|}(\mu\circ\pi_*^{-1})\,.$$

Note that by the permutation invariance of $s_{|A|}$, this definition is independent of which injection π we choose.

Proposition 2.1 (Strong subadditivity (SSA) of entropy) *Let $N \in \mathbb{Z}_{>0}$ and let $\mu \in \mathscr{M}_1(\Omega^N)$. Then for any two sets $A, B \subseteq [1, N]$,*

$$|A \cup B| s_{A\cup B}(\mu) + |A \cap B| s_{A\cap B}(\mu) \leq |A| s_A(\mu) + |B| s_B(\mu) .$$

We will not prove this result. Please consult Israel's beautiful monograph, [3], Chapter II for the proof. In the case that $A \cap B = \emptyset$, one simply defines $0 \cdot s_{\emptyset}(\mu) := 0$. Also, in that case, SSA is called just subadditivity. (The proof of subadditivity is easier than SSA. This is even more true in the case of quantum states, although of course we do not venture that far afield.)

By the Gibbs variational principle, we know that

$$p_N(t, x) = \log(2) + \max_{\mu \in \mathscr{M}_1(\Omega^N)} g_N(\mu; t, x) ,$$

where $g_N(\cdot; t, x) : \mathscr{M}_1(\Omega^N) \to \mathbb{R} \cap \{-\infty\}$ is the strictly concave, upper-semicontinuous function

$$g_N(\mu; t, x) := s_N(\mu) + \mathbf{E}^{\mu}[\varphi_N(\cdot; t, x)] ,$$

and that the unique maximizer is the Boltzmann-Gibbs distribution, $\mu_{N,t,x} \in \mathscr{M}_1(\Omega^N)$,

$$\mu_{N,t,x}(\sigma) = \frac{e^{N\varphi_N(\sigma;t,x)}}{Z_N(t, x)} .$$

We will call $g(\cdot; t, x)$ the Gibbs functional.

Definition 2.2 *For $N \in \mathbb{Z}_{>0}$ and $\mu \in \mathscr{M}_1(\Omega^N)$, call μ finite exchangeable if $\mu \circ \pi_*^{-1} = \mu$ for each permutation $\pi : [1, N] \to [1, N]$. Call such a finitely exchangeable measure infinitely extendible if it is a mixture of i.i.d. product measures, and therefore is the restriction to Ω^N of an infinitely exchangeable measure on Ω^{∞}.*

This notation is co-opted from Aldous's beautiful survey of exchangeability [1], except that he calls "finite exchangeability" by "N-exchangeability" which is more descriptive but also slightly redundant. Also, for recent results along these lines, especially pertaining to the C-W model, see the paper of Liggett, Steif and Toth, [5]. Note that since $\varphi_N(\cdot; t, x) : \Omega^N \to \mathbb{R}$ is permutation-invariant, the Boltzmann-Gibbs distribution is finite-exchangeable.

With these preliminaries out of the way, it will be easy to prove the following elementary result.

Lemma 2.3 (Subadditivity of the pressure) *For $t, x \in \mathbb{R}$ and any $M, N \in \mathbb{Z}_{\geq 2}$,*

$$p_{M+N}(t, x) \leq \frac{M p_M(t, x) + N p_N(t, x)}{M + N} .$$

Proof. Suppose $\mu_{M+N} \in \mathscr{M}_1(\Omega^{M+N})$ is finite-exchangeable. Let $\pi_1 : [1, M] \to [1, M + N]$ and $\pi_2 : [1, N] \to [1, M + N]$ be $\pi_1(i) = i$ and $\pi_2(i) = M + i$. Let

$$\mu_M := \mu_{M+N} \circ (\pi_1)_*^{-1} \in \mathscr{M}_1(\Omega^M) \quad \text{and} \quad \mu_N := \mu_{M+N} \circ (\pi_2)_*^{-1} \in \mathscr{M}_1(\Omega^N) .$$

(Note that if $M = N$ then $\mu_M = \mu_N$, by exchangeability.) Then by subadditivity of entropy,

$$s_{M+N}(\mu_{M+N}) \leq \frac{M s_M(\mu_M) + N s_N(\mu_N)}{M + N} .$$

Also, by exchangeability, it is easy to see that

$$\mathbf{E}^{\mu_{M+N}}[\varphi_{M+N}(\cdot;t,x)] \;=\; \mathbf{E}^{\mu_M}[\varphi_M(\cdot;t,x)] \;=\; \mathbf{E}^{\mu_N}[\varphi_N(\cdot;t,x)]\,.$$

Therefore,

$$g_{M+N}(\mu_{M+N};t,x) \;\leq\; \frac{Mg_M(\mu_M;t,x) + Ng_N(\mu_N;t,x)}{M+N}\,. \tag{2.1}$$

Now apply (2.1) to the case that $\mu_{M+N} = \mu_{M+N,t,x}$ is the Boltzmann-Gibbs distribution. Then one obtains

$$\begin{aligned} p_{M+N}(t,x) \;&=\; \log(2) + g_{M+N}(\mu_{M+N};t,x) \\ &\leq\; \log(2) + \frac{Mg_M(\mu_M;t,x) + Ng_N(\mu_N;t,x)}{M+N} \\ &\leq\; \frac{Mp_M(t,x) + Np_N(t,x)}{M+N}\,, \end{aligned}$$

by the Gibbs variational principle. □

Establishing subadditivity is important because of the following lemma of Fekete.

Lemma 2.4 (Fekete) *If $x_n, x_{n+1}, \dots \in \mathbb{R}$ is a sequence such that*

$$x_{M+N} \;\leq\; \frac{Mx_M + Nx_N}{M+N}\,,$$

for all $M, N \in \mathbb{Z}_{\geq n}$, then

$$\lim_{N\to\infty} x_N \;=\; \inf_{N\in\mathbb{Z}_{\geq n}} x_N \;\in \mathbb{R}\cup\{-\infty\}\,.$$

We leave the proof as an exercise for the reader. In a later lecture we will prove a stronger version of this lemma. On the other hand, using this and Lemma 2.3, we easily obtain the following.

Corollary 2.5 *For each $t, x \in \mathbb{R}$, the thermodynamic pressure $p(t,x))$ exists in $\mathbb{R}$ and it is bounded below by $\log(2)$.*

Proof. It is trivial to see that $g_N(\alpha^{\otimes N}) = 0$ which leads to the lower bound of $\log(2)$ for $p_N(t,x)$ through the Gibbs variational principle. Thus, not only does $p_N(t,x))$ converge to $p(t,x)$ in the extended real line $\mathbb{R}\cup\{-\infty\}$, as guaranteed by Lemmas 2.3 and 2.4. it converges in $\mathbb{R}$. □

3. The Gibbs variational principle

Let us define, in contradistinction to certain notations in past lectures, $\alpha_m \in \mathscr{M}_1(\Omega)$ to be

$$\alpha_m \;=\; \frac{1+m}{2}\,\delta_{+1} + \frac{1-m}{2}\,\delta_{-1}\,,$$

for $m \in [-1,1]$. Then $\alpha = \alpha_0$. Also,

$$\mathbf{E}^{\alpha_m^{\otimes N}}[\varphi_N(\cdot;t,x)] \;=\; \widetilde{\varphi}(m;t,x) \;:=\; \frac{m^2 t}{2} + mx\,.$$

Similarly,

$$s_N(\alpha_m^{\otimes N}) \;=\; \widetilde{s}(m) \;:=\; -\frac{1+m}{2}\log(1+m) - \frac{1-m}{2}\log(1-m)\,.$$

Therefore, it makes sense to define

$$\widetilde{g}(m;t,x) \;=\; \widetilde{s}(m) + \widetilde{\varphi}(m;t,x)\,.$$

The main result of this lecture is the following.

Theorem 3.1 (Gibbs, de Finetti principle) *For each $t, x \in \mathbb{R}$,*

$$p(t,x) \;=\; \log(2) + \max_{m\in(-1,1)} \widetilde{g}(m;t,x)\,.$$

As usual, in analysis (and therefore probability) an identity is the conjunction of two inequalities. In this case the lower bound,

$$p(t,x) \;\geq\; \log(2) + \max_{m\in(-1,1)} \widetilde{g}(m;t,x)\,, \tag{3.1}$$

is easy. As we have just demonstrated before the statement of the theorem, we have

$$g_N(\alpha_m^{\otimes N};t,x) \;=\; \widetilde{g}(m;t,x)\,,$$

for each $N \in \mathbb{Z}_{\geq 2}$. Therefore, by the Gibbs variational principle, each $p_N(t,x)$ is bounded below by the right-hand-side of (3.1). So, taking the limit as $N \to \infty$, which is the same as the infimum over N, gives (3.1).

The corresponding upper bound is more involved. A major issue is to understand the entropy function on exchangeable measures.

Definition 3.2 *For $N \in \mathbb{Z}_{>0}$ and $M \in [1,N]$, let $\pi_M^N : [1,M] \to [1,N]$ be the injection $\pi_M^N(i) = i$. For $\mu_N \in \mathscr{M}_1(\Omega^N)$ finite exchangeable, define $\mu_N \upharpoonright \Omega_M := \mu_N \circ (\pi_M^N)_*^{-1}$.*

Lemma 3.3 (Monotonicity of entropy) *Suppose $N \in \mathbb{Z}_{>0}$, suppose $M \in [1,N]$, and suppose $\mu_N \in \mathscr{M}_1(\Omega^N)$ is finite exchangeable. Then*

$$s_M(\mu_N \upharpoonright \Omega^M) \;\geq\; s_N(\mu_N)\,.$$

Proof. It would clearly be sufficient to prove this just for $M = N-1$ if we could prove it for all $N \in \mathbb{Z}_{\geq 2}$. For example, then defining $\mu_{N-1} = \mu_N \upharpoonright \Omega^{N-1}$, we would have that this is also finite exchangeable, so the inequality could be iterated down to $M = N-2$, and so on, and so forth.

Let us suppose $N \in \mathbb{Z}_{\geq 2}$. Note that, taking $A = [1, N-1]$ and $B = [2, N]$, SSA of entropy gives

$$\begin{aligned} &N s_N(\mu_N) - (N-1) s_{N-1}(\mu_N \upharpoonright \Omega^{N-1}) \\ &\qquad\qquad \leq\; (N-1) s_{N-1}(\mu_N \upharpoonright \Omega^{N-1}) - (N-2) s_{N-2}(\mu_N \upharpoonright \Omega^{N-2})\,. \end{aligned}$$

In the case $N = 2$, this inequality again uses the formal quantity $0s_0(\mu_2 \upharpoonright \Omega^0) := 0$. Iterating this, and performing the telescopic sum, gives

$$\begin{aligned} Ns_N(\mu_N) &= \sum_{M=1}^{N} \left[Ms_M(\mu_N \upharpoonright \Omega^M) - (M-1)s_{M-1}(\mu_N \upharpoonright \Omega^{M-1})\right] \\ &\geq N\left[Ns_N(\mu_N) - (N-1)s_{N-1}(\mu_N \upharpoonright \Omega^{N-1})\right] , \\ \implies s_N(\mu_N) &\geq Ns_N(\mu_N) - (N-1)s_{N-1}(\mu_N \upharpoonright \Omega^{N-1}) , \\ \implies s_N(\mu_N) &\leq s_{N-1}(\mu_N \upharpoonright \Omega^{N-1}) . \end{aligned}$$

□

Corollary 3.4 *Let $N \in \mathbb{Z}_{\geq 2}$ and $M \in [2, N]$. If $\mu_N \in \mathscr{M}_1(\Omega^N)$ is finite exchangeable, then*

$$g_N(\mu_N; t, x) \leq g_M(\mu_N \upharpoonright \Omega^M; t, x) .$$

Proof. This follows just from the lemma and the fact that, by symmetry and exchangeability,

$$\mathbf{E}^{\mu_N \upharpoonright \Omega^M}[\varphi_M(\cdot; t, x)] = \mathbf{E}^{\mu_N}[\varphi_N(\cdot; t, x)] .$$

□

The corollary is important because it guarantees the existence of a limit. Namely, let $\mu \in \mathscr{M}_1(\Omega^\infty)$ be exchangeable. Then we can define $\mu \upharpoonright \Omega^N$ for each N. One way to do this, is just to use de Finetti's theorem directly. There must be a measure $\rho \in \mathscr{M}_1([-1, 1])$ (note that we have changed notation a little since the last lecture) such that

$$\mu(\cdot) = \int_{-1}^{1} \alpha_m^{\otimes\infty}(\cdot)\, d\rho(m) .$$

Then one also has (by definition)

$$\mu \upharpoonright \Omega_N = \int_{-1}^{1} \alpha_m^{\otimes N}\, d\rho(m) .$$

One can prove, using concavity of $g_N(\cdot; t, x)$, that

$$g_N(\mu \upharpoonright \Omega_N; t, x) \geq \int_{-1}^{1} g_N(\alpha_m^{\otimes N}; t, x)\, d\rho(m) = \int_{-1}^{1} \widetilde{g}(m; t, x)\, d\rho(m) .$$

(Here we use the fact that Ω^N is finite. If we would consider continuous spins, for example, the same could still be proved but it would have to use the correct definition of entropy, as found for example in the seminal paper of Ruelle and Robinson [6].)

By the corollary, the sequence $(g_N(\mu \upharpoonright \Omega^N; t, x) : N \in \mathbb{Z}_{\geq 2})$ is decreasing. Therefore, there is a limit, which is also the infimum

$$g_\infty(\mu; t, x) := \lim_{N\to\infty} g_N(\mu \upharpoonright \Omega^N; t, x) .$$

Note that, as this is the infimum of u.s.c. functions, it is itself a u.s.c. function. It is also the infimum of strictly concave functions, so it is concave (though not necessarily strictly so). It is also convex! This fact is not immediately obvious, but it follows because of an inequality that Simon calls "almost convexity" for the entropy: for $N \in \mathbb{Z}_{>0}$ and $\mu, \nu \in \mathscr{M}_1(\Omega^N)$,

$$s_N(\theta \cdot \mu + (1-\theta)\nu) \;\leq\; \theta s_N(\mu) + (1-\theta) s_N(\nu) - \frac{1}{N}[\theta \log(\theta) + (1-\theta)\log(1-\theta)]\,,$$

for each $\theta \in [0,1]$. Since $-\theta\log(\theta) - (1-\theta)\log(1-\theta) \leq \log(2)$, as $N \to \infty$ this implies convexity. Of course the expectation of the Hamiltonian is always linear. We will not prove "almost convexity", but consult Simon or Israel for simple proofs.

Since g is an affine, it can be proved that

$$g_\infty(\mu; t, x) \;=\; \int_{-1}^{1} \widetilde{g}(m; t, x)\, d\rho(m)\,.$$

Note, this decomposition uses upper semicontinuity. We can now prove the theorem.

of Theorem 3.1. Since the lower bound has already been proved, all that needs to be proved is the upper bound. Let $\mu_{N,t,x}$ be the Boltzmann-Gibbs distribution for each $N \in \mathbb{Z}_{\geq 2}$. By compactness and the Cantor diagonal trick, we can choose a subsequence $N_1 < N_2 < \dots$ such that $\mu_{N_k,t,x} \restriction M$ converges for each $M \in \mathbb{Z}_{>0}$. Therefore, by the last fact stated in the last lecture, we know that there is a $\mu \in \mathscr{M}_1(\Omega^\infty)$ such that $\mu \restriction \Omega^M$ is the limit for each M. Since each $\mu_{N,t,x}$ is exchangeable, so is μ.

Also, by Corollary 3.4, we know that

$$p_N(t,x) - \log(2) \;=\; g_N(\mu_{N,t,x}; t, x) \;\leq\; g_M(\mu_{N,t,x} \restriction \Omega^M; t, x)\,,$$

for each $M \in [2, N]$. Therefore, since an upper semicontinuous function can only "jump up",

$$p(t,x) - \log(2) \;\leq\; g_M(\mu \restriction \Omega^M; t, x)\,,$$

for each $M \in \mathbb{Z}_{\geq 2}$. In particular,

$$p(t,x) - \log(2) \;\leq\; g_\infty(\mu; t, x) \;=\; \int_{-1}^{1} \widetilde{g}(m; t, x)\, d\rho(m)\,.$$

But this is clearly bounded by $\max_{m \in [-1,1]} \widetilde{g}(m; t, x)$. The restriction $-1 < m < 1$ is obtained by checking the derivatives at -1 and 1, as usual. □

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