A Thinning Analogue of de Finetti’s Theorem

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All measure spaces are compact, metric spaces. \( \mathcal{M}_{+,1} \) always refers to the set of Borel probability measures on any such space. With the vague topology this is, in turn, compact and metrizable.

\( \Omega \) compact metric space.

For \( n \in \mathbb{N} \), let \( \mu_n \in \mathcal{M}_{+,1}(\Omega^n) \).

Let \((X_{n,1}, \ldots, X_{n,n})\) be r.v.s with joint distr. \( \mu_n \).

If \( n > 0 \), define \( \theta(\mu_n) \in \mathcal{M}_{+,1}(\Omega^{n-1}) \) as follows:

random combination \( 1 \leq k(1) < \cdots < k(n-1) \leq n \),

independent of \((X_{n,1}, \ldots, X_{n,n})\);

\( \theta(\mu_n) \) is distribution of \((X_{n,k(1)}, \ldots, X_{n,k(n-1)})\).

Call \( \theta \) “thinning”.

Sequence \((\mu_n : n \in \mathbb{N})\) is \( \theta \)-invariant if \( \forall n, \mu_n = \theta(\mu_{n+1}) \).

Examples

1. The i.i.d. case. If \( \mu \in \mathcal{M}_{+,1}(\Omega) \) define \( \mu_n = \mu^{\otimes n} \), all \( n \).
2. The exchangeable case. If \( \rho \in \mathcal{M}_{+,1}(\mathcal{M}_{+,1}(\Omega)) \), define \( \mu_n(E) = \int_{\mathcal{M}_{+,1}(\Omega)} \mu^{\otimes n}(E) \rho(d\mu) \), all \( n \).
3. If \( \Omega = [0,1] \) and \( \mu \in \mathcal{M}_{+,1}([0,1]) \), let \((X_{n,1}, \ldots, X_{n,n})\) be order statistics for \( n \) i.i.d. \( \mu \)-distr. r.v.s, and \( \mu_n \) their distr.
Examples continued

4. Let $\mu : [0, 1] \rightarrow \mathcal{M}_{+1}(\Omega)$ be a Borel measurable mapping. Let $t = (t_{n,1}, \ldots, t_{n,n})$ be the order statistics for $n$ i.i.d. $U([0, 1])$.

Let $\mu_n$ be the measure such that, for any Borel $A \subset \Omega^n$,

$$\mu_n(A) = \mathbb{E}^t[\mu(t_{n,1}) \otimes \cdots \otimes \mu(t_{n,n})(A)].$$

It turns out that this is the most general extreme point of the thinning invariant simplex.

Let us define the entire set of all Borel maps $\mu : [0, 1] \rightarrow \mathcal{M}_{+1}(\Omega)$ by $\mathcal{K}$.

**Technicality – Topology:** Given $\mu : [0, 1] \rightarrow \mathcal{M}_{+1}(\Omega)$, and Borel $E \subset [0, 1]$, let $\mu_E \in \mathcal{M}_+(\Omega)$ by

$$\mu_E(A) = \int_E \mu(t)(A) \, dt$$

all Borel $A \subset \Omega$.

For a sequence of $\mu^{(n)} : [0, 1] \rightarrow \mathcal{M}_{+1}(\Omega)$, say that it converges iff, for all Borel $E \subset [0, 1]$, it is true $\mu^{(n)}_E \rightarrow \mu_E$.

This is the topology dual to the natural action of $L^1([0, 1], dt; C(\Omega))$ on $\mathcal{K}$.

Since latter is separable Banach space, $\mathcal{K}$ is compact and metrizable (as in Banach-Alaoglu).
**Theorem.** Let \((\mu_n : n \in \mathbb{N})\) be a thinning invariant sequence. Then there exists a unique \(\rho \in \mathcal{M}_{+,1}(\mathcal{K})\), such that, for every \(n \in \mathbb{N}\) and Borel \(A \subset \Omega^n\),

\[
\mu_n(A) = \int_{\mathcal{K}} \mathbb{E}^t[\mu(t_{n,1}) \otimes \cdots \otimes \mu(t_{n,n})(A)] \rho(d\mu).
\]

**Example.** \(\Omega = \{0, 1\}\).
Then \(\mathcal{M}_{+,1}(\Omega) \cong [0, 1]\) by \(p = \mu(\{1\})\).
\(\mathcal{K} \cong\) the set of all Borel mappings \(p : [0, 1] \to [0, 1]\).
Topology is the weak topology with respect to \(L^1([0,1])\).
The same as vague topology restricted to the subset
\(\{p(t)dt \in \mathcal{M}_{+,1}([0,1]) \mid p : [0, 1] \to [0, 1], \text{Borel}\}\).
Since each \(p\) is bounded by 1, the set is compact in \(\mathcal{M}_{+,1}([0,1])\).
Then \(\mathcal{M}_{+,1}(\mathcal{K}) \cong\) the subset of all Borel measures on \(\mathcal{M}_{+,1}([0,1])\) supported on this small set of measures.
**Application 1.** “Asymmetric mean-field” statistical mechanics models.

Let \( r \in \mathbb{N} \) and \( f : \Omega^r \to \mathbb{R} \cup \{+\infty\} \).

Do not assume that \( f \) is symmetric in \( r \) variables.

Assume that \( f \) is bounded below.

Let \( \alpha \in \mathcal{M}_{+,1}(\Omega) \) be the a priori measure.

Assume \( \alpha^{\otimes r}(f) < \infty \).

For each \( N \), define \( H_N : \Omega^N \to \mathbb{R} \cap \{+\infty\} \) by

\[
H_N(x) = \binom{N}{r}^{-1} \sum_{1 \leq i(1) < \cdots < i(r) \leq N} f(x_{i(1)}, \ldots, x_{i(r)}).
\]

The finite, volume= \( N \) approximation to the pressure is

\[
p_N(\beta) = N^{-1} \log \left( \alpha^{\otimes N}(e^{-\beta H_N}) \right).
\]

If it exists, the pressure is \( p(\beta) = \lim_{N \to \infty} p_N(\beta) \).

The finite-volume Gibbs measures are \( \rho_{\beta,N} \in \mathcal{M}_{+,1}(\Omega^N) \) such that

\[
\frac{d\rho_{\beta,N}}{d\alpha^{\otimes N}}(x) = e^{-\beta H_N(x) - Np_N(\beta)}.
\]
**Motivation: Scaling** There is a famous scaling for spin systems: the Lebowitz-Penrose limit. Means, (1) first take thermodynamic limit, (2) then take coupled limit where range of interaction $\to \infty$, amplitude of interaction $\to 0$ so that $L^1$ is preserved. Necessarily recovers exchangeability. This largely motivates mean-field models (e.g., van der Waals or Curie-Weiss) as well, it is an initial step for some perturbative results in real short-range models (e.g., Lebowitz, Mazel, Presutti: liquid-vapor transition in continuum model).

But consider another limit, where volume and range $\to \infty$ together, with amplitude $\to 0$ so as to preserve $L^1$. This is equivalent to the asymmetric mean-field limit (using Stone-Weierstrass).

This is connected to thinning-invariant arrays why?
Little-known result: Fannes, Spohn and Verbeure

One of the most elegant solution of mean-field models was done by Fannes, Spohn and Verbeure. They actually did it for mean-field quantum models (e.g. Dicke maser model following Hepp and Lieb) using Stormer’s generalization of de Finetti’s theorem to symmetric states on $C^*$-algebra, but let’s stick to classical case.

Mean-field: Model is as before, but $f$ is assumed to be symmetric on $\Omega^r$. Interested in $p(\beta)$ and some constraints on the set of limit points of the Gibbs states ($\rho_{\beta,N} : N \geq r$).

The finite approx. to pressure is given by the Gibbs variational formula

$$p_N(\beta) = \sup_{\rho \in \mathcal{M}_{+,1}(\Omega^N)} G_N(\beta; \rho)$$

$$G_N(\beta; \rho) = N^{-1}[S(\rho, \alpha^\otimes N) - \beta \rho(H_N)],$$

where $S$ is relative entropy.

The functional $G_N$ is concave and upper semicontinuous. The maximum is attained, and the unique argmax is the Gibbs state (for $N$).
The relative entropy is

$$S(\rho, \alpha^\otimes N) = \int_{\Omega^N} g \left( \frac{d\rho}{d\alpha^\otimes N}(x) \right) \alpha^\otimes N(dx),$$

where $g(t) = -t \log(t)$ for $t > 0$ and $0$ at $t = 0$. It satisfies:

(1) It is expressed as infimum of continuous functions ranging over Borel partitions-of-unity (Ruelle and Robinson);

(2) It is upper semicontinuous on $M_{+,1}(\Omega)$;

(3) It is concave and “almost convex” i.e.,

$$S(t\rho_1 + (1-t)\rho_2) \leq tS(\rho_1) + (1-t)S(\rho_2) + g(t) + g(1-t)$$

independent of $N$;

(4) It is strongly subadditive.

Strong subadditivity has the following consequence.

Let $\rho \in M_{+,1}(\Omega^N)$ and let $n \leq N$.

Let $\rho^{(n)}$ denote the $n$-particle reduced measure in $M_{+,1}(\Omega^n)$.

Then $n^{-1}S(\rho^{(n)}, \alpha^\otimes n) \leq N^{-1}S(\rho, \alpha^\otimes N)$.

Mean density is decreasing.

Given any infinite exchangeable measure, the limit of the mean entropies of the $N$-particle reduced states exists, using subadditivity or monotonicity.
But monotonicity implies (happy to explain privately) that

\[ p(\beta) = \sup_{\rho \in \mathcal{M}_{+,1}(\mathcal{M}_{+,1}(\Omega))} G(\beta; \rho) \]

and

\[ G(\beta; \rho) = s(\rho) - \beta \rho(\mu^{\otimes r}(f)), \]

where \( s(\rho) \) is the mean entropy associated to infinite exchangeable measure uniquely associated to \( \rho \).

Note \( s \) is affine in the limit so \( G \) is affine and upper semicontinuous.

Therefore, \( G \) is optimized on extreme points.

More generally the optimizers of \( G \) form a union of faces.

(Nonuniqueness can occur: phase transition.)

Using the property of Ruelle and Robinson one can even do more by proving that

\[ s(\rho) = \int_{\mathcal{M}_{+,1}(\Omega)} S(\mu, \alpha) \rho(d\mu), \]

from which it is obvious that the optimizers are a face.

Therefore, suffices to calculate optimizer of the “gap equation”

\[ \gamma(\mu) = S(\mu, \alpha) - \beta \mu^{\otimes r}(f). \]

Every vague limit point of Gibbs measure is in the face spanned by optimizers of \( \gamma \).
In the weakly asymmetric case, we have a similar set-up, but do not require that $f$ be symmetric. Then we can take weak of Gibbs measures:

$$\rho_{\beta,N_i} \text{ converges iff } N_i \to \infty \text{ and for all } n \in \mathbb{N}, \text{ the sequences } \theta^{N_i-n}(\rho_{\beta,N_i}) \text{ converges in } M_{+,1}(\Omega^n).$$

Define limit as $\mu_n$. Then $(\mu_n : n \in \mathbb{N})$ is $\theta$-invariant by construction.

Also, if $\mu_N \in M_{+,1}(\Omega^N)$ then $\theta^{N-n}(\mu)(H_n/n) = \mu(H_N/N)$. Strong subadditivity still implies monotonicity of mean entropy upon thinning.

Therefore, using the argument of Fannes, Spohn and Verbeure, mutatis mutandis, we draw a similar conclusion. One difference. Now $\mu : [0, 1] \to M_{+,1}(\Omega)$ and

$$\Gamma(\mu) = \int_0^1 S(\mu(t), \alpha) \, dt - \beta \mathbb{E}[\mu(t_1) \otimes \cdots \otimes \mu(t_r)(f)].$$
The simplest example turns out to be rather interesting:
\( \Omega = \mathbb{R} \) and \( f : \Omega^2 \to \mathbb{R} \) by \( f(x, y) = \chi_{x>y} \).
(Everything extends to Borel subsets of compact, metric spaces - just let \( \alpha \) have smaller support.)
Example is related to “Mallows model”:
Select permutations, not according to uniform distribution, but with log of weights proportional to number of crossings in diagram (modulo constant shift).
Action on appropriate vector in \( (\mathbb{C}^2)^\otimes N \) gives kink groundstates of XXZ model, invariant measures for constant coefficient nn ASEP on \([1, N]\): scaling is weakly anisotropic.
Euler-Lagrange for \( \Gamma \) leads to PDE for \( u(x, t) \) if \( \mu(t)(dx) = u(x, t)dx \)
\[
\frac{\partial}{\partial x} \frac{\partial}{\partial t} \log(u(x, t)) = 2\beta u(x, t).
\]
Need solution for \( 0 < t < 1 \) and \(-\infty < x < \infty \).
Assume travelling wave solution \( u(x, t) = U(x - ct) \).
Solution is \( U(z) = (\lambda/2) \text{sech}^2(\lambda z) \) where \( \lambda = \beta/2c \).
Take \( c = \beta/2 \).
Get \( u(x, t) = \frac{1}{2} \text{sech}^2(x - \frac{\beta}{2} t) \) for \( 0 < t < 1 \).
It is a solution but requires a special choice of a priori measure \( \alpha(dx) = \phi(x)\, dx \) with
\[
\phi(x) \propto \text{sech}(x) \text{sech}(x - \beta/2) .
\]
**Application 2.** “Asymmetric mean-field” Simple Exclusion Process (and generalizations???)

Consider the simple exclusion process on $[1, N]$ whose generator is

$$
\Omega f(\eta) = \sum_{x, y \in [1, N]} \chi[\eta(x) = 1, \eta(y) = 0] (f(\eta_{xy}) - f(\eta)) p(x, y),
$$

where

$$
\eta_{xy}(z) = \begin{cases}
\eta(y) & z = x, \\
\eta(x) & z = y, \\
\eta(z) & z \in \{x, y\}^c,
\end{cases}
$$

and

$$
p(x, y) = \begin{cases}
q & x < y, \\
1 - q & y < x.
\end{cases}
$$

For each pair of particles, $(x, y)$ there are two Poisson clocks with rates $q$ and $1 - q$.

One transports particles from $x$ to $y$ at rings, the other transports particles in the opposite direction.

Note $|x - y|$ is irrelevant, but the order does matter.

**Question:** Can one identify all the (weak) limit points of invariant measures (with a limiting density of particles) in the $N \to \infty$ limit?

**Answer:** I don’t know.
But one can determine all the extremal thinning-invariant states which have the potential of being limit points.

Note now $\Omega = \{0, 1\}$.

Let $p(t) = \mu(t)(\{1\})$ for $t \in [0, 1]$.

Then a necessary condition for $\mu : [0, 1] \to \mathcal{M}_{+, 1}(\Omega)$ to be a limit point is that

$$0 = \int_0^t ((1 - q)p(t)[1 - p(s)] + qp(s)[1 - p(t)]) ds$$

$$+ \int_t^1 (qp(t)[1 - p(s)] + (1 - q)p(s)[1 - p(t)]) ds$$

for all $t \in [0, 1]$.

Fix the density $\rho = \int_0^1 p(t) dt$.

The solution is presented in two steps.

Let $u(t) = \frac{1}{2} (\int_0^t p(s) ds - \int_t^1 p(s) ds)$.

Then $u'(t) = -p(t)$ and we determine

$$2p(t) = \frac{1}{2} \rho + (2q - 1)u(t)$$

$$\frac{1}{2}[(1 - q)t + q(1 - t)] + (2q - 1)u(t).$$

This is solved by

$$2u(t) = -\left(\frac{q}{2q - 1} - t\right)$$

$$+ \left[\left(\frac{q}{2q - 1} - t\right)^2 + \rho \left(\rho + \frac{2t - 2q}{2q - 1}\right)\right]^{1/2}.$$
$\rho = 1/2, \ q \in [1/2,1]$  

$\rho = 1/4, \ q \in [1/2,1]$