A Thinning Analogue of de Finetti's Theorem

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`http://front.math.ucdavis.edu/math.PR/0406364`

All measure spaces are compact, metric spaces. $\mathcal{M}_{+,1}$ always refers to the set of Borel probability measures on any such space. With the vague topology this is, in turn, compact and metrizable.

Ω compact metric space.
For $n \in \mathbb{N}$, let $\mu_n \in \mathcal{M}_{+,1}(\Omega^n)$.
Let $(X_{n,1}, \dots, X_{n,n})$ be r.v.s with joint distr. μ_n.
If $n > 0$, define $\theta(\mu_n) \in \mathcal{M}_{+,1}(\Omega^{n-1})$ as follows:
random combination $1 \leq k(1) < \cdots < k(n-1) \leq n$,
independent of $(X_{n,1}, \dots, X_{n,n})$;
$\theta(\mu_n)$ is distribution of $(X_{n,k(1)}, \dots, X_{n,k(n-1)})$.
Call θ "thinning".
Sequence $(\mu_n : n \in \mathbb{N})$ is θ-invariant if $\forall n$, $\mu_n = \theta(\mu_{n+1})$.

Examples

1. The i.i.d. case. If $\mu \in \mathcal{M}_{+,1}(\Omega)$ define $\mu_n = \mu^{\otimes n}$, all n.
2. The exchangeable case. If $\rho \in \mathcal{M}_{+,1}(\mathcal{M}_{+,1}(\Omega))$, define $\mu_n(E) = \int_{\mathcal{M}_{+,1}(\Omega)} \mu^{\otimes n}(E)\, \rho(d\mu)$, all n.
3. If $\Omega = [0,1]$ and $\mu \in \mathcal{M}_{+,1}([0,1])$, let $(X_{n,1}, \dots, X_{n,n})$ be order statistics for n i.i.d. μ-distr. r.v.s, and μ_n their distr.

Examples continued

4. Let $\mu : [0,1] \to \mathcal{M}_{+,1}(\Omega)$ be a Borel measurable mapping. Let $\mathfrak{t} = (t_{n,1}, \dots, t_{n,n})$ be the order statistics for n i.i.d. $U([0,1])$.

Let μ_n be the measure such that, for any Borel $A \subset \Omega^n$,

$$\mu_n(A) \;=\; \mathbb{E}^{\mathfrak{t}}[\mu(t_{n,1}) \otimes \cdots \otimes \mu(t_{n,n})(A)]\,.$$

It turns out that this is the most general extreme point of the thinning invariant simplex.

Let us define the entire set of all Borel maps $\mu : [0,1] \to \mathcal{M}_{+,1}(\Omega)$ by $\mathcal{K}$.

Technicality – Topology: Given $\mu : [0,1] \to \mathcal{M}_{+,1}(\Omega)$, and Borel $E \subset [0,1]$, let $\mu_E \in \mathcal{M}_{+}(\Omega)$ by

$$\mu_E(A) = \int_E \mu(t)(A)\, dt$$

all Borel $A \subset \Omega$.

For a sequence of $\mu^{(n)} : [0,1] \to \mathcal{M}_{+,1}(\Omega)$, say that it converges iff, for all Borel $E \subset [0,1]$, it is true $\mu_E^{(n)} \to \mu_E$.

This is the topology dual to the natural action of $L^1([0,1], dt; \mathcal{C}(\Omega))$ on $\mathcal{K}$.

Since latter is separable Banach space, $\mathcal{K}$ is compact and metrizable (as in Banach-Alaoglu).

Theorem. *Let* $(\mu_n : n \in \mathbb{N})$ *be a thinning invariant sequence. Then there exists a unique* $\rho \in \mathcal{M}_{+,1}(\mathcal{K})$*, such that, for every* $n \in \mathbb{N}$ *and Borel* $A \subset \Omega^n$*,*

$$\mu_n(A) \, = \, \int_{\mathcal{K}} \mathbb{E}^{\mathrm{t}}[\mu(t_{n,1}) \otimes \cdots \otimes \mu(t_{n,n})(A)] \, \rho(d\mu) \, .$$

Example. $\Omega = \{0, 1\}$.
Then $\mathcal{M}_{+,1}(\Omega) \cong [0, 1]$ by $p = \mu(\{1\})$.
$\mathcal{K} \cong$ the set of all Borel mappings $p : [0, 1] \to [0, 1]$.
Topology is the weak topology with respect to $L^1([0, 1])$.
The same as vague topology restricted to the subset $\{p(t)dt \in \mathcal{M}_{+,1}([0, 1]) \,|\, p : [0, 1] \to [0, 1], \text{Borel}\}$.
Since each p is bounded by 1, the set is compact in $\mathcal{M}_{+,1}([0, 1])$.
Then $\mathcal{M}_{+,1}(\mathcal{K}) \cong$ the subset of all Borel measures on $\mathcal{M}_{+,1}([0, 1])$ supported on this small set of measures.

Application 1. "Asymmetric mean-field" statistical mechanics models.
Let $r \in \mathbb{N}$ and $f : \Omega^r \to \mathbb{R} \cup \{+\infty\}$.
Do not assume that f is symmetric in r variables.
Assume that f is bounded below.
Let $\alpha \in \mathcal{M}_{+,1}(\Omega)$ be the a priori measure.
Assume $\alpha^{\otimes r}(f) < \infty$.

For each N, define $H_N : \Omega^N \to \mathbb{R} \cap \{+\infty\}$ by

$$H_N(x) = \binom{N}{r}^{-1} \sum_{1 \le i(1) < \cdots < i(r) \le N} f(x_{i(1)}, \ldots, x_{i(r)}) \,.$$

The finite, volume$= N$ approximation to the pressure is

$$p_N(\beta) = N^{-1} \log \left(\alpha^{\otimes N}(e^{-\beta H_N}) \right) \,.$$

If it exists, the pressure is $p(\beta) = \lim_{N \to \infty} p_N(\beta)$.
The finite-volume Gibbs measures are $\rho_{\beta,N} \in \mathcal{M}_{+,1}(\Omega^N)$ such that

$$\frac{d\rho_{\beta,N}}{d\alpha^{\otimes N}}(x) = e^{-\beta H_N(x) - N p_N(\beta)} \,.$$

Motivation : Scaling There is a famous scaling for spin systems: the Lebowitz-Penrose limit.
Means, (1) first take thermodynamic limit, (2) then take coupled limit where range of interaction $\to \infty$, amplitude of interaction $\to 0$ so that L^1 is preserved.
Necessarily recovers exchangeability.
This largely motivates mean-field models (e.g., van der Waals or Curie-Weiss)
as well, it is an initial step for some perturbative results in real short-range models (e.g., Lebowitz, Mazel, Presutti: liquid-vapor transition in continuum model).

But consider another limit, where volume and range $\to \infty$ together, with amplitude $\to 0$ so as to preserve L^1.
This is equivalent to the asymmetric mean-field limit (using Stone-Weierstrass).

This is connected to thinning-invariant arrays why?

Little-known result : Fannes, Spohn and Verbeure
One of the most elegant solution of mean-field models was done by Fannes, Spohn and Verbeure.
They actually did it for mean-field quantum models (e.g. Dicke maser model following Hepp and Lieb) using Stormer's generalization of de Finetti's theorem to symmetric states on C^*-algebra, but let's stick to classical case.

Mean-field: Model is as before, but f is assumed to be symmetric on Ω^r.
Interested in $p(\beta)$ and some constraints on the set of limit points of the Gibbs states $(\rho_{\beta,N} : N \geq r)$.

The finite approx. to pressure is given by the Gibbs variational formula

$$p_N(\beta) = \sup_{\rho \in \mathcal{M}_{+,1}(\Omega^N)} G_N(\beta; \rho)$$
$$G_N(\beta; \rho) = N^{-1}[S(\rho, \alpha^{\otimes N}) - \beta\rho(H_N)],$$

where S is relative entropy.
The functional G_N is concave and upper semicontinuous.
The maximum is attained, and the unique argmax is the Gibbs state (for N).

The relative entropy is

$$S(\rho, \alpha^{\otimes N}) = \int_{\Omega^N} g\left(\frac{d\rho}{d\alpha^{\otimes N}}(x)\right) \alpha^{\otimes N}(dx)\,,$$

where $g(t) = -t\log(t)$ for $t > 0$ and 0 at $t = 0$.
It satisfies:
(1) It is expressed as infimum of continuous functions ranging over Borel partitions-of-unity (Ruelle and Robinson);
(2) It is upper semicontinuous on $\mathcal{M}_{+,1}(\Omega)$;
(3) It is concave and "almost convex" i.e.,
$S(t\rho_1 + (1-t)\rho_2) \leq tS(\rho_1) + (1-t)S(\rho_2) + g(t) + g(1-t)$ independent of N;
(4) It is strongly subadditive.

Strong subadditivity has the following consequence.
Let $\rho \in \mathcal{M}_{+,1}(\Omega^N)$ and let $n \leq N$.
Let $\rho^{(n)}$ denote the n-particle reduced measure in $\mathcal{M}_{+,1}(\Omega^n)$.
Then $n^{-1}S(\rho^{(n)}, \alpha^{\otimes n}) \leq N^{-1}S(\rho, \alpha^{\otimes N})$.
Mean density is decreasing.
Given any infinite exchangeable measure, the limit of the mean entropies of the N-particle reduced states exists, using subadditivity or monotonicity.

But monotonicity implies (happy to explain privately) that

$$p(\beta) = \sup_{\rho \in \mathcal{M}_{+,1}(\mathcal{M}_{+,1}(\Omega))} G(\beta; \rho)$$

$$G(\beta; \rho) = s(\rho) - \beta \rho(\mu^{\otimes r}(f)),$$

where $s(\rho)$ is the mean entropy associated to infinite exchangeable measure uniquely associated to ρ.
Note s is affine in the limit so G is affine and upper semicontinuous.
Therefore, G is optimized on extreme points.
More generally the optimizers of G form a union of faces.
(Nonuniqueness can occur: phase transition.)
Using the property of Ruelle and Robinson one can even do more by proving that

$$s(\rho) = \int_{\mathcal{M}_{+,1}(\Omega)} S(\mu, \alpha)\, \rho(d\mu),$$

from which it is obvious that the optimizers are a face.
Therefore, suffices to calculate optimizer of the "gap equation"

$$\gamma(\mu) = S(\mu, \alpha) - \beta \mu^{\otimes r}(f).$$

Every vague limit point of Gibbs measure is in the face spanned by optimizers of γ.

In the weakly asymmetric case, we have a similar set-up, but do not require that f be symmetric.

Then we can take weak of Gibbs measures:

ρ_{β,N_i} converges iff $N_i \to \infty$ and for all $n \in \mathbb{N}$, the sequences $\theta^{N_i-n}(\rho_{\beta,N_i})$ converges in $\mathcal{M}_{+,1}(\Omega^n)$.

Define limit as μ_n. Then $(\mu_n \,:\, n \in \mathbb{N})$ is θ-invariant by construction.

Also, if $\mu_N \in \mathcal{M}_{+,1}(\Omega^N)$ then $\theta^{N-n}(\mu)(H_n/n) = \mu(H_N/N)$.

Strong subadditivity still implies monotonicity of mean entropy upon thinning.

Therefore, using the argument of Fannes, Spohn and Verbeure, mutatis mutandis, we draw a similar conclusion.

One difference. Now $\mu : [0,1] \to \mathcal{M}_{+,1}(\Omega)$ and

$$\Gamma(\mu) = \int_0^1 S(\mu(t), \alpha)\, dt - \beta \mathbb{E}^{\mathrm{t}}[\mu(t_1) \otimes \cdots \otimes \mu(t_r)(f)]\,.$$

The simplest example turns out to be rather interesting:
$\Omega = \mathbb{R}$ and $f : \Omega^2 \to \mathbb{R}$ by $f(x, y) = \chi_{x>y}$.
(Everything extends to Borel subsets of compact, metric spaces – just let α have smaller support.)
Example is related to "Mallows model":
Select permutations, not according to uniform distribution, but with log of weights proportional to number of crossings in diagram (modulo constant shift).
Action on appropriate vector in $(\mathbb{C}^2)^{\otimes N}$ gives kink groundstates of XXZ model, invariant measures for constant coefficient nn ASEP on $[1, N]$: scaling is weakly anisotropic.

Euler-Lagrange for Γ leads to PDE for $u(x, t)$ if
$\mu(t)(dx) = u(x, t)dx$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial t}\log(u(x, t)) \;=\; 2\beta u(x, t)\,.$$

Need solution for $0 < t < 1$ and $-\infty < x < \infty$.
Assume travelling wave solution $u(x, t) = U(x - ct)$.
Solution is $U(z) = (\lambda/2)\,\mathrm{sech}^2(\lambda z)$ where $\lambda = \beta/2c$.
Take $c = \beta/2$.
Get $u(x, t) = \frac{1}{2}\,\mathrm{sech}^2(x - \frac{\beta}{2}t)$ for $0 < t < 1$.
It is a solution but requires a special choice of a priori measure $\alpha(dx) = \phi(x)\,dx$ with

$$\phi(x) \propto \mathrm{sech}(x)\,\mathrm{sech}(x - \beta/2)\,.$$

Application 2. "Asymmetric mean-field" Simple Exclusion Process (and generalizations???)

Consider the simple exclusion process on $[1, N]$ whose generator is

$$\Omega f(\eta) = \sum_{x,y\in[1,N]} \chi[\eta(x) = 1, \eta(y) = 0] \, (f(\eta_{xy}) - f(\eta)) \, p(x, y) \, ,$$

where

$$\eta_{xy}(z) = \begin{cases} \eta(y) & z = x, \\ \eta(x) & z = y, \\ \eta(z) & z \in \{x, y\}^c, \end{cases}$$

and

$$p(x, y) = \begin{cases} q & x < y, \\ 1 - q & y < x. \end{cases}$$

For each pair of particles, (x, y) there are two Poisson clocks with rates q and $1 - q$.
One transports particles from x to y at rings, the other transports particles in the opposite direction.
Note $|x - y|$ is irrelevant, but the order does matter.

Question: Can one identify all the (weak) limit points of invariant measures (with a limiting density of particles) in the $N \to \infty$ limit?
Answer: I don't know.

But one can determine all the extremal thinning-invariant states which have the potential of being limit points.
Note now $\Omega = \{0,1\}$.
Let $p(t) = \mu(t)(\{1\})$ for $t \in [0,1]$.
Then a necessary condition for $\mu : [0,1] \to \mathcal{M}_{+,1}(\Omega)$ to be a limit point is that

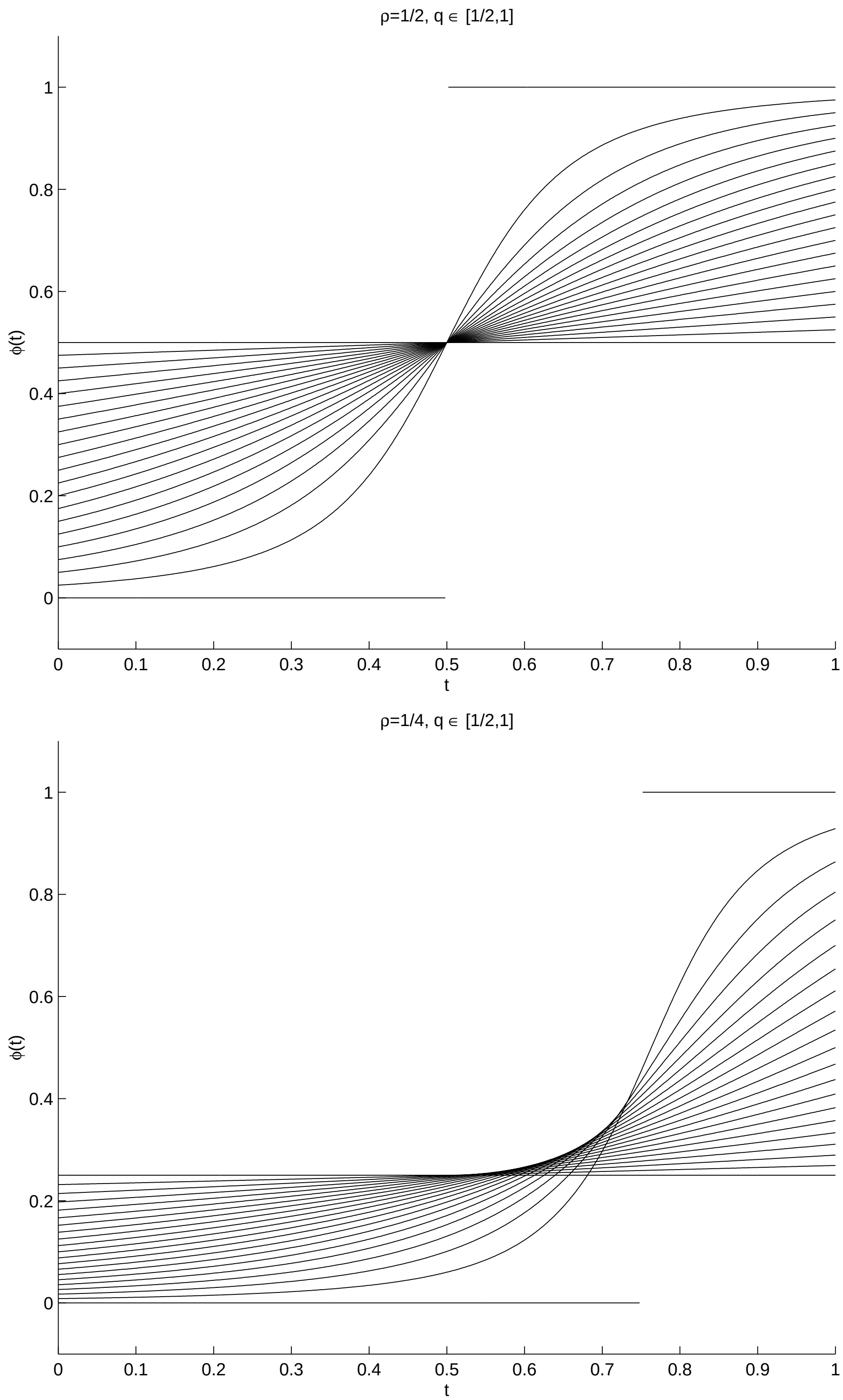
$$0 = \int_0^t ((1-q)p(t)[1-p(s)] + qp(s)[1-p(t)])ds + \int_t^1 (qp(t)[1-p(s)] + (1-q)p(s)[1-p(t)])ds$$

for all $t \in [0,1]$.
Fix the density $\rho = \int_0^1 p(t)dt$.
The solution is presented in two steps.
Let $u(t) = \frac{1}{2}(\int_0^t p(s)ds - \int_t^1 p(s)ds)$.
Then $u'(t) = -p(t)$ and we determine

$$2p(t) = \frac{\frac{1}{2}\rho + (2q-1)u(t)}{\frac{1}{2}[(1-q)t + q(1-t)] + (2q-1)u(t)}\,.$$

This is solved by

$$2u(t) = -\left(\frac{q}{2q-1} - t\right) + \left[\left(\frac{q}{2q-1} - t\right)^2 + \rho\left(\rho + \frac{2t-2q}{2q-1}\right)\right]^{1/2}.$$


ρ=1/2, q ∈ [1/2,1]
φ(t)
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
ρ=1/4, q ∈ [1/2,1]
φ(t)
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t