Short Course on the Sherrington Kirkpatrick Model Lecture 2: The Quadratic Interpolation Method and Existence of the Pressure

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1 Set-up Again

We recall the basic model, taking the opportunity to change the notation slightly. We will define a pure Gaussian component of the Hamiltonian, which we will also normalize to be order $O(1)$, $K_N(\cdot, J) : \mathbb{Z}_2^N \to \mathbb{R}$ wherein, for $\sigma = (\sigma_1, \dots, \sigma_N) \in \mathbb{Z}_2^N$, we have

$$K_N(\sigma) \, = \, - \sum_{i=1}^N \sum_{j=1}^N \frac{J_{ij}}{N} \sigma_i \sigma_j \, , \tag{1}$$

where all coupling constants J_{ij} are i.i.d. $\mathcal{N}(0,1)$ random variables. The real Hamiltonian of interest is the random function

$$H_N(\sigma, h) \, = \, \sqrt{N/2}\, K_N(\sigma) \, - \, h \sum_{i=1}^N \sigma_i \, . \tag{2}$$

These are actually random variables. To denote this we should consider a probability measure space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}$-measurable functions $J_{ij}(\omega)$ such that the P-induced joint distribution of the $J_{ij}(\omega)$ is i.i.d., $\mathcal{N}(0,1)$. Then $K_N(\sigma) = K_N(\sigma, \omega)$ and $H_N(\sigma, h) = H_N(\sigma, h, \omega)$. We will suppress the ω, unless we want to emphasize that something is a random variable. Our choice of normalization for $K_N(\sigma)$ is so that $(K_N(\sigma) \, : \, \sigma \in \mathbb{Z}_2^N)$ is a centered, Gaussian process, with order-1 covariance

$$\mathbb{E}\{K_N(\sigma) K_N(\sigma')\} \, = \, \mathcal{R}_N(\sigma, \sigma')^2 \, , \tag{3}$$

where $\mathcal{R}_N(\sigma, \sigma')$ is the spin-spin overlap

$$\mathcal{R}_N(\sigma, \sigma') \, = \, \frac{1}{N} \sum_{i=1}^N \sigma_i \sigma'_i \, . \tag{4}$$

This will be useful for us at a later point.

The thermodynamic quantities are:
the random partition function

$$Z_N(\beta, h, \omega) = \sum_{\mathbb{Z}_2^N} 2^{-N} \exp\left(-\beta H_N(\sigma, h, \omega)\right) ; \tag{5}$$

the finite-volume approximation to the pressure

$$P_N(\beta, h, \omega) = N^{-1} \log\left(Z_N(\beta, h, \omega)\right) ; \tag{6}$$

the finite-volume "quenched pressure"

$$p_N(\beta, h) = \mathbb{E}\{P_N(\beta, h, \omega)\} ; \tag{7}$$

the limiting "quenched pressure", if it exists,

$$p(\beta, h) = \lim_{N \to \infty} p_N(\beta, h) ; \tag{8}$$

and the almost-sure limiting pressure, if it exists,

$$P(\beta, h) \quad \text{defined so that} \quad \lim_{\varepsilon \to 0} \lim_{N \to \infty} \mathbb{P}(|P_N(\beta, h, \omega) - P(\beta, h)| > \varepsilon) = 0 . \tag{9}$$

(For notational reasons we use $\mathbb{P}$ to indicate probability since P is taken for pressure.)

One interesting point in the history of the Sherrington-Kirkpatrick model is that it was first proved that the random finite approximations to the pressure satisfy an L^2 estimate which proves that they are asymptotically identical to the finite approximations to the quenched pressure, before it was known that the finite approximations to the quenched pressure exists. Specifically, in [5], Pastur and Shcherbina had proved the following theorem

Theorem 1.1 (Pastur and Shcherbina 1991) *There exists an absolute constant $C < \infty$ such that for any β and h,*

$$\mathbb{E}\{[P_N(\beta, h, \omega) - p_N(\beta, h)]^2\} \le C\beta^2 N^{-1} . \tag{10}$$

Because of this theorem, one knows that if the limiting quenched pressure exists, then there is an almost sure limit of the random pressures, and $P(\beta, h) = p(\beta, h)$. The plan for this lecture is to present Guerra and Toninelli's proof that $p(\beta, h)$ exists. We will make one minor deviation from Guerra and Toninelli's proof [2] by employing the lemma of Slepian, as Talagrand also does in his book [6].

After presenting Guerra and Toninelli's proof, we will give Talagrand's concentration estimate, which is proved quite similarly to Slepian's lemma. Talagrand's concentration estimate gives an even stronger result than Pastur and Schcerbina's. In particular, using the existence of $p(\beta, h)$, the concentration estimate proves that $P(\beta, h)$ exists and equals $p(\beta, h)$.

In the last section we will prove a simple, but important result, which was first proved by Aizenman, Lebowitz and Ruelle in [1]. However, the proof we will give will follow Talagrand's presentation from Section 2.1 of his book, [6]. Particularly, it uses only the concentration estimate, and some very easy calculations.

2 Guerra and Toninelli's Result

The result which Guerra and Toninelli proved is actually something more powerful than that the limiting quenched pressure exists. They proved that the finite-volume approximations to the quenched pressure, times the volume, form a superadditive sequence. Specifically, in [2], they proved:

Theorem 2.1 (Guerra and Toninelli 2001) *For any $N_1, N_2 \in \mathbb{N}_+$, it is the case that*

$$(N_1 + N_2) p_{N+1+N_2}(\beta, h) \ \geq\ N_1 p_{N_1}(\beta, h) + N_2 p_{N_2}(\beta, h)\,. \tag{11}$$

We will prove this theorem in the next section. But first, we observe the result that the limiting pressure exists. In fact, we will prove two results, one of which is the existence of the pressure, and one of which is different. The second result will aid us in eventually obtaining an "extended variational principle" for the quenched pressure. But it is also natural to consider both parts together, because their proofs are connected.

Recall that a real sequence $(X(N)\,:\,N \in \mathbb{N}_+)$ has a limit in the extended real line $\mathbb{R} \cup \{+\infty, -\infty\}$ if $X(N) \to x \in \mathbb{R}$ in the usual sense, or if $\liminf_{N\to\infty} X(N) = +\infty$, in which case the limit is $+\infty$, or if $\limsup_{N\to\infty} X(N) = -\infty$, in which case the limit is $-\infty$. A necessary and sufficient condition for the existence of the limit in the extended sense is that $\liminf_{N\to\infty} X(N) \geq \limsup_{N\to\infty} X(N)$.

Lemma 2.2 *Let $(X(N)\,:\,N \in \mathbb{N}_+)$ be a sequence in $\mathbb{R}$, such that for every $N_1, N_2 \in \mathbb{N}_+$,*

$$X(N_1 + N_2) \ \geq\ X(N_1) + X(N_2)\,. \tag{12}$$

Then the following two limits exist, and are equal

$$\lim_{N\to\infty} \frac{X(N)}{N} \ =\ \lim_{N\to\infty} \liminf_{M\to\infty} \frac{X(M+N) - X(M)}{N}\,. \tag{13}$$

The limits may possibly (both) equal $+\infty$, but not $-\infty$.

Proof. Let n and r be integers. Then

$$\begin{aligned} \frac{X(nN+r) - X(r)}{nN} \ &=\ \frac{1}{n} \sum_{k=0}^{n-1} \frac{X(N+kN+r) - X(kN+r)}{N} \\ &\geq\ \inf_{\{m\,:\,m\geq r\}} \frac{X(m+N) - X(m)}{N}\,. \end{aligned} \tag{14}$$

Given $k, N \in \mathbb{N}$, we define nonnegative-integer-valued functions $n(M)$ and $r(M)$, defined for all integers $M \geq k$, such that $M = n(M)N + r(M)$ and $r(M) \in [k, k+N-1]$. Then, since the set $\{X(r(M))\,:\,M \geq k\}$ is bounded, as is $\{r(M)\,:\,M \geq k\}$, equation (14) implies

$$\begin{aligned} \liminf_{M\to\infty} \frac{X(M)}{M} \ &=\ \liminf_{M\to\infty} \frac{X(n(M)N + r(M)) - X(r(M))}{n(M)N} \\ &\geq\ \inf_{\{m\,:\,m\geq k\}} \frac{X(m+N) - X(m)}{N}\,. \end{aligned} \tag{15}$$

Taking the limit $k \to \infty$ gives

$$\liminf_{M\to\infty} \frac{X(M)}{M} \geq \liminf_{m\to\infty} \frac{X(m+N)-X(m)}{N}\,. \tag{16}$$

But superadditivity gives, for any $m, N \in \mathbb{N}_+$,

$$\frac{X(m+N)-X(m)}{N} \;\geq\; \frac{X(N)}{N}\,. \tag{17}$$

Hence we have

$$\liminf_{M\to\infty} \frac{X(M)}{M} \geq \liminf_{m\to\infty} \frac{X(m+N)-X(m)}{N} \;\geq \frac{X(N)}{N} \tag{18}$$

for every $N \in \mathbb{N}_+$. Since generally $X(N)/N \in \mathbb{R}$, this immediately implies that the liminf of $X(M)/M$ is greater than $-\infty$, so that if the limit of $(X(N)/N\,:\,N \in \mathbb{N}_+)$ exists, it does not equal $-\infty$. Taking the limit-supremum, as $N \to \infty$, we deduce that

$$\liminf_{M\to\infty} \frac{X(M)}{M} \;\geq \limsup_{N\to\infty} \frac{X(N)}{N}\,. \tag{19}$$

This proves that the sequence $(X(N)/N\,:\,N \in \mathbb{N}_+)$ does have a limit in the extended sense. On the other hand, the sequence

$$Y(N) \;=\; \liminf_{m\to\infty} \frac{X(m+N)-X(m)}{N} \tag{20}$$

is sandwiched by a convergent subsequence

$$\lim_{M\to\infty} \frac{X(M)}{M} \;\geq\; Y(N) \;\geq\; \frac{X(N)}{N}\,. \tag{21}$$

Hence, we conclude that $(Y(N)\,:\,N \in \mathbb{N}_+)$ also converges and that it has the same limit as $(X(N)/N\,:\,N \in \mathbb{N}_+)$. ■

Considering the sequence $X(N) = Np_N(\beta, h)$, this proves that $p(\beta, h)$ exists. Moreover, the limit is not $+\infty$, because of an application of Jensen's inequality. Namely,

$$\begin{aligned} p_N(\beta,h) \;&=\; \mathbb{E}\left\{N^{-1}\log\sum_{\sigma\in\mathbb{Z}_2^N} e^{-\beta H_N(\sigma,h,\omega)}\right\} \\ &\leq\; N^{-1}\log\sum_{\sigma\in\mathbb{Z}_2^N} 2^{-N}\mathbb{E}\left\{e^{-\beta H_N(\sigma,h,\omega)}\right\} \\ &=\; N^{-1}\log\sum_{\sigma\in\mathbb{Z}_2^N} 2^{-N}\mathbb{E}\left\{\exp\left(-\beta\sqrt{N/2}\,K_N(\sigma,\omega)+\beta h\sum_{j=1}^N \sigma_j\right)\right\}\,. \end{aligned} \tag{22}$$

But since $K_N(\sigma, \omega)$ is a centered, Gaussian random variable, with *variance* equal to 1 (although the covariance is different) we know that

$$\mathbb{E}\left\{\exp(-\beta\sqrt{N/2}\,K_N(\sigma,\omega)\Big)\right\} \;=\; e^{\beta^2 N/4}\,. \tag{23}$$

Hence we have the uniform upper bound

$$\begin{aligned} p_N(\beta,h) \;&\leq\; N^{-1}\log\sum_{\sigma\in\mathbb{Z}_2^N} e^{\beta^2 N/4}\exp\left(\beta h\sum_{j=1}^N \sigma_j\right) \\ &=\; \frac{\beta^2}{4} \,+\, \log(\cosh(\beta h))\,. \end{aligned} \tag{24}$$

Therefore, $p(\beta,h)$ satisfies the same upper bound. Since $(Np_N(\beta,h)\,:\,N\in\mathbb{N}_+)$ is a superadditive sequence, this means that the limit is not $-\infty$.

3 Proof of Superadditivity

We will now prove Guerra and Toninelli's theorem. The proof is identical to theirs. However, we will break the presentation up as Talagrand does in his book. Specifically, we begin by introducing Slepian's lemma. Talagrand refers to [4]. We would refer to [3], which seems more general, and is an earlier reference. The proof is quite simple, and we will include it here.

Lemma 3.1 *Suppose that $\mathcal{A}$ is a (possibly infinite) index set, and that $(X(\alpha):\alpha\in\mathcal{A})$ and $(Y(\alpha):\alpha\in\mathcal{A})$ are two real, centered Gaussian processes such that*

$$\begin{aligned} &\mathbb{E}\{X(\alpha)^2\} = \mathbb{E}\{Y(\alpha)^2\}\,, \quad \forall \alpha\in\mathcal{A}\,; \\ &\mathbb{E}\{X(\alpha)X(\alpha')\} \leq \mathbb{E}\{Y(\alpha)Y(\alpha')\}\,, \quad \forall \alpha,\alpha'\in\mathcal{A}\,:\;\alpha\neq\alpha'\,. \end{aligned} \tag{25}$$

Suppose that $F(x)=F(x(\alpha):\alpha\in\mathcal{A})$ is a real-valued function, which is in $\mathcal{C}^2(\mathbb{R}^{\mathcal{A}})$ and such that

$$\frac{\partial^2 F}{\partial x(\alpha)\partial x(\alpha')} \leq 0\,, \quad \forall \alpha,\alpha'\in\mathcal{A}\,:\;\alpha\neq\alpha'\,. \tag{26}$$

Suppose furthermore that F and its first two derivatives satisfy an $\|x\|^N$ growth condition for some finite N. Then

$$\mathbb{E}\{F(X(\alpha):\alpha\in\mathcal{A})\} \geq \mathbb{E}\{F(Y(\alpha):\alpha\in\mathcal{A})\}\,. \tag{27}$$

Proof. We may take an independent coupling of X and Y. For each $t\in\mathbb{R}$, we define $Z_t=\sqrt{1-t}\,X+\sqrt{t}\,Y$. Then for $0<t<1$ the derivative exists, and

$$\frac{\partial}{\partial t}Z_t \;=\; -\frac{1}{2\sqrt{1-t}}X \,+\, \frac{1}{2\sqrt{t}}Y\,. \tag{28}$$

In particular, this implies that

$$\mathbb{E}\left\{Z_t(\alpha')\frac{\partial}{\partial t}Z_t(\alpha)\right\} \;=\; -\frac{1}{2}\mathbb{E}\{X(\alpha)X(\alpha')\} + \frac{1}{2}\mathbb{E}\{Y(\alpha)Y(\alpha')\}\,. \tag{29}$$

Generally, this is nonnegative, but in case $\alpha = \alpha'$, it equals zero. We also have

$$\frac{\partial}{\partial t}\mathbb{E}\{F(Z_t)\} \;=\; \sum_{\alpha\in\mathcal{A}} \mathbb{E}\left\{\frac{\partial F}{\partial x(\alpha)}(Z_t)\,\frac{\partial}{\partial t} Z_t(\alpha)\right\}\,. \tag{30}$$

Then, using Wick's rule, we have

$$\frac{\partial}{\partial t}\mathbb{E}\{F(Z_t)\} \;=\; \sum_{\alpha,\alpha'\in\mathcal{A}} \mathbb{E}\left\{\frac{\partial^2 F}{\partial x(\alpha)\partial x(\alpha')}(Z_t)\right\} \mathbb{E}\left\{Z_t(\alpha')\,\frac{\partial}{\partial t} Z_t(\alpha)\right\}\,. \tag{31}$$

We can remove the diagonal terms since the second factor is zero

$$\frac{\partial}{\partial t}\mathbb{E}\{F(Z_t)\} \;=\; \sum_{\substack{\alpha,\alpha'\in\mathcal{A}\\ \alpha\neq\alpha'}} \mathbb{E}\left\{\frac{\partial^2 F}{\partial x(\alpha)\partial x(\alpha')}(Z_t)\right\} \mathbb{E}\left\{Z_t(\alpha')\,\frac{\partial}{\partial t} Z_t(\alpha)\right\}\,. \tag{32}$$

Since the mixed partial of F is uniformly nonpositive, and the covariance term is uniformly nonnegative, we deduce that $\partial_t \mathbb{E}\{F(Z_t)\} \leq 0$. Since $\mathbb{E}\{F(Z_0)\} = \mathbb{E}\{F(X)\}$ and $\mathbb{E}\{F(Z_1)\} = \mathbb{E}\{F(Y)\}$, we obtain the result. ■

An example of a function F is as follows. Suppose that $(\xi(\alpha) : \alpha \in \mathcal{A})$ is any summable sequence of nonnegative numbers, and take

$$F(x) \;=\; \log \sum_{\alpha\in\mathcal{A}} \xi(\alpha) e^{-\beta x(\alpha)}\,. \tag{33}$$

It is easily checked that the off-diagonal components of the second derivative matrix are all nonpositive, even though the same may not be true of the diagonal components. (This just follows from concavity of the logarithm.) We will use this fact in the case that $\mathcal{A} = \mathbb{Z}_2^N$, whence we call the elements σ instead of α, and

$$\xi(\sigma) \;=\; 2^{-N} \exp\left(\beta h \sum_{j=1}^N \sigma_j\right)\,. \tag{34}$$

Proof of Theorem 2.1. To prove superadditivity means that we must show that for any $N_1, N_2 \in \mathbb{N}_+$ that

$$N p_N(\beta,h) \geq N_1 p_{N_1}(\beta,h) + N_2 p_{N_2}(\beta,h)\,, \tag{35}$$

where $N = N_1 + N_2$. If we write for each $\sigma \in \mathbb{Z}_2^N$ that $\sigma^{(1)} \in \mathbb{Z}_2^{N_1}$ is equal to the first N_1 elements of σ, and that $\sigma^{(2)} \in \mathbb{Z}_2^{N_2}$ is equal to the last N_1 elements, then this inequality is equivalent to proving (where K_{N_1} and K_{N_2} are independent)

$$\begin{aligned}\mathbb{E}&\left\{\log \sum_{\sigma\in\mathbb{Z}_2^N} 2^{-N} \exp\left(\beta h \sum_{j=1}^N \sigma_j\right) e^{\beta\sqrt{N/2}\, K_N(\sigma,\omega)}\right\}\\ &\geq \mathbb{E}\left\{\log \sum_{\sigma\in\mathbb{Z}_2^N} 2^{-N} \exp\left(\beta h \sum_{j=1}^N \sigma_j\right) e^{\beta[\sqrt{N_1/2}\, K_{N_1}(\sigma^{(1)},\omega) + \sqrt{N_2/2}\, K_{N_2}(\sigma^{(2)},\omega)]}\right\}\,.\end{aligned} \tag{36}$$

Using the definition of F from above, we see that this is equivalent to the inequality

$$\mathbb{E}\{F(X)\} \,\geq\, \mathbb{E}\{F(Y)\} \tag{37}$$

where $X = (X(\sigma)\,:\,\sigma \in \mathbb{Z}_2^N)$ and $Y = (Y(\sigma)\,:\,\sigma \in \mathbb{Z}_2^N)$ are two Gaussian centered processes

$$X(\sigma) \,=\, \sqrt{N/2}K_N(\sigma)\,, \quad \text{and} \quad Y(\sigma) \,=\, \sqrt{N_1/2}K_{N_1}(\sigma^{(1)}) + \sqrt{N_2/2}K_{N_2}(\sigma^{(2)})\,. \tag{38}$$

Therefore, the theorem will follow from Slepian's lemma if we can verify that the diagonal entries of the variance of X and Y are equal, while the offdiagonal entries are greater for Y than for X.

We may easily calculate the variance matrices for X and Y. They are

$$\begin{aligned}\mathbb{E}\{X(\sigma)X(\widetilde{\sigma})\} \,&=\, \frac{N}{2}\,\mathcal{R}_N(\sigma,\widetilde{\sigma})^2\,, \quad \text{and}\\ \mathbb{E}\{Y(\sigma)Y(\widetilde{\sigma})\} \,&=\, \frac{N_1}{2}\,\mathcal{R}_{N_1}(\sigma^{(1)},\widetilde{\sigma}^{(1)})^2 \,+\, \frac{N_2}{2}\,\mathcal{R}_{N_2}(\sigma^{(2)},\widetilde{\sigma}^{(2)})^2\,.\end{aligned} \tag{39}$$

In case $\widetilde{\sigma} = \sigma$, for diagonal entries, all three spin-spin overlaps equal 1. Since $N = N_1 + N_2$, we then conclude that both variances have the same diagonal entries, equal to $N/2$. On the other hand, if $\widetilde{\sigma} \neq \sigma$, we observe

$$\mathcal{R}_N(\sigma,\widetilde{\sigma}) \,=\, \frac{1}{N}\sum_{j=1}^{N} \sigma_j \widetilde{\sigma}_j \,=\, \frac{N_1}{N}\mathcal{R}_{N_1}(\sigma^{(1)},\widetilde{\sigma}^{(1)}) + \frac{N_2}{N}\mathcal{R}_{N_2}(\sigma^{(2)},\widetilde{\sigma}^{(2)})\,. \tag{40}$$

Therefore, taking the difference, we can easily calculate that

$$\begin{aligned}&\mathbb{E}\{X(\sigma)X(\widetilde{\sigma})\} - \mathbb{E}\{Y(\sigma)Y(\widetilde{\sigma})\}\\ &\quad= \frac{1}{2(N_1+N_2)}\left[N_1\mathcal{R}_{N_1}(\sigma^{(1)},\widetilde{\sigma}^{(1)}) + N_2\mathcal{R}_{N_2}(\sigma^{(2)},\widetilde{\sigma}^{(2)})\right]^2\\ &\quad- \frac{1}{2}\left[N_1\,\mathcal{R}_{N_1}(\sigma^{(1)},\widetilde{\sigma}^{(1)})^2 \,+ N_2\,\mathcal{R}_{N_2}(\sigma^{(2)},\widetilde{\sigma}^{(2)})^2\right]\\ &\quad= -\frac{N_1N_2}{2(N_1+N_2)}\left[\mathcal{R}_{N_1}(\sigma^{(1)},\widetilde{\sigma}^{(1)}) - \mathcal{R}_{N_2}(\sigma^{(2)},\widetilde{\sigma}^{(2)})\right]^2\\ &\quad\leq 0\end{aligned} \tag{41}$$

This finishes the proof. ■

Another way to argue the final point, that the covariance of Y is greater than that of X, is to observe that the spin-spin overlaps satisfy a convex relation

$$\mathcal{R}_N(\sigma,\widetilde{\sigma}) \,=\, \frac{N_1}{N}\mathcal{R}_{N_1}(\sigma^{(1)},\widetilde{\sigma}^{(1)}) + \frac{N_2}{N}\mathcal{R}_{N_2}(\sigma^{(2)},\widetilde{\sigma}^{(2)})\,, \tag{42}$$

with convex coefficients equal to (N_1/N) and (N_2/N). Since the function $r \mapsto (N/2)r^2$ is a convex function, this necessarily implies that

$$\frac{N}{2}\,\mathcal{R}_N(\sigma,\widetilde{\sigma})^2 \,\leq\, \frac{N}{2}\left[\frac{N_1}{N}\,\mathcal{R}_{N_1}(\sigma^{(1)},\widetilde{\sigma}^{(1)})^2 \,+\, \frac{N_2}{N}\,\mathcal{R}_{N_2}(\sigma^{(2)},\widetilde{\sigma}^{(2)})^2\right]\,, \tag{43}$$

by Jensen's inequality.

Using this argument one can prove the following related result, by exactly the same proof as Guerra and Toninelli's theorem. (For this reason we call it a corollary, although it is a corollary of the proof, not the theorem.)

Corollary 3.2 *Consider a mean-field, Gaussian spin glass, but where $K_N(\sigma)$ is now a different Gaussian centered process*

$$K_N(\sigma) \;=\; \sum_{n=0}^{\infty} a_n \sum_{j_1,\dots,j_n=1}^{N} \frac{J^{(n)}(j_1,\dots,j_n)}{\sqrt{N^n}} \sigma_{j_1}\cdots\sigma_{j_n}\,, \tag{44}$$

where all the couplings $\{J^{(n)}(j_1,\dots,j_n)\,:\, n\in\mathbb{N}_+\,,\ j_1,\dots,j_n\in[1,N]\}$ are i.i.d., and $\mathcal{N}(0,1)$, jointly distributed. We assume furthermore that all the a_n are real, and, defining the complex analytic function

$$f(z) \;=\; \sum_{n=0}^{\infty} |a_n|^2 z^n\,, \tag{45}$$

this function has radius of convergence greater than 1, and that the restriction $f:[-1,1]\to\mathbb{R}$ is a convex function. Then Guerra and Toninelli's result still holds.

We will not give a formal proof, because it follows the previous one so closely, but we will point out why this is true. We observe that we can calculate the covariance of this new Gaussian process. Using the independence of the Gaussian coupling constants, one easily deduces that

$$\begin{aligned} \mathbb{E}\{K_N(\sigma,\omega)K_N(\sigma',\omega)\} \;&=\; \sum_{n=0}^{\infty} |a_n|^2 \mathcal{R}_N(\sigma,\sigma')^n \\ &=\; f(\mathcal{R}_N(\sigma,\sigma'))\,. \end{aligned} \tag{46}$$

Since this function is convex, the argument above, using Jensen's inequality, works to prove that the Gaussian processes we defined as X and Y satisfy the hypotheses of Slepian's lemma.

The generalization above is useful, because it allows one to prove the existence of the pressure for other models of interes. For example, taking $f(z)=z^p$, where p is even, yields a model called Derrida's p-spin model. Guerra and Toninelli did prove their method also for these models. There is a limit point of these even p-spin models, called Derrida's random energy model (REM). By an inequality giving continuity of the pressure in the covariance of K, one can deduce that the pressure for the Derrida random energy model also exists. We will prove that inequality in the next lecture, as it is a necessary part of the proof of the extended variational principle. Of course, in the case of Derrida random energy model, the pressure is already known to exist. In fact one can exactly calculate the pressure using large deviation methods, as it is done in Chapter 1 of [6]. This is, indeed, a very important result for the Sherrington-Kirkpatrick model, as the random limiting Gibbs states of the REM are key in guessing the random limiting Gibbs states of the SK model. We will comment more on this in Lecture 4.

Even though one can explicitly calculate the pressure for the REM, it is nice to know that Guerra and Toninelli's result also applies to that model.

4 The Concentration Estimate

We now present Talagrand's concentration estimate, which is Theorem 2.2.4 in his book [6]. This will, in particular, imply Theorem 1.1, saa special case. Talgrand is a special expert in the area of concentration of measure, and elightening historical remarks can be found on page 196 of [6]. He has certainly proved how important these techniques are for the study of mean-field spin glass models, by now. We will even present the exact same proof as Talagrand gives in his book on page 75 (and we mention that we do *not* have copywrite permission).

Theorem 4.1 (Talagrand) *Consider a Lipschitz function F on $\mathbb{R}^M$, of Lipschitz constant A. If $X = (X(n) : n \in [1, M])$ is a Gaussian, centered process, with i.i.d., $\mathcal{N}(0,1)$ joint distribution, then for each $s \in \mathbb{R}$,*

$$\mathbb{E}\left\{\exp(s[F(X) - \mathbb{E}\{F(X)\}])\right\} \le \exp(s^2 A^2)\,. \tag{47}$$

Using Markov's inequality, this implies that for any $t > 0$

$$\mathbb{P}(|F(X) - \mathbb{E}\{F(X)\}| \ge t) \le 2\exp(-t^2/4A^2)\,. \tag{48}$$

Proof. By convolving with a smooth function, if necessary, we may assume that F is smooth. (Such convolutions can be made so that the sup norm differences are arbitraritly small, which implies that the estimate still holds in the limit.) Given $s \in \mathbb{R}$, one defines a function $G : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}_+$ by $G(x, y) = \exp(s[F(x) - F(y)])$. One considers three independent families of Gaussian random variables U, V, W such that each is of size M, and all are i.i.d., and $\mathcal{N}(0,1)$ distributed. Then, for each $0 \le t \le 1$, define the Gaussian centered processes

$$X_t = \sqrt{1-t}\,U + \sqrt{t}\,W\,, \quad \text{and} \quad Y_t = \sqrt{1-t}\,V + \sqrt{t}\,W\,. \tag{49}$$

One defines a function $\varphi(t) = \mathbb{E}\{G(X_t, Y_t)\}$.

Note that

$$\begin{aligned} \frac{\partial}{\partial t} X_t &= -\frac{1}{2\sqrt{1-t}}\,U + \frac{1}{2\sqrt{t}}\,W\,, \quad \text{and} \\ \frac{\partial}{\partial t} Y_t &= -\frac{1}{2\sqrt{1-t}}\,V + \frac{1}{2\sqrt{t}}\,W\,, \quad \text{and}\,. \end{aligned} \tag{50}$$

Therefore, since U, V and W are i.i.d.., and all components are i.i.d., $\mathcal{N}(0,1)$ distributed, one deduces that

$$\begin{aligned} \mathbb{E}\left\{X_t(k)\,\frac{\partial}{\partial t} X_t(j)\right\} &= 0\,, \\ \mathbb{E}\left\{Y_t(k)\,\frac{\partial}{\partial t} Y_t(j)\right\} &= 0\,, \quad \text{and} \\ \mathbb{E}\left\{X_t(k)\,\frac{\partial}{\partial t} Y_t(j)\right\} &= \mathbb{E}\left\{Y_t(k)\,\frac{\partial}{\partial t} X_t(j)\right\} = \frac{1}{2}\,\delta_{j,k}\,. \end{aligned} \tag{51}$$

Therefore, using Wick's rule, by a calculation like that in the proof of Slepian's lemma, one determines that

$$\varphi'(t) \;=\; -\sum_{j=1}^{M} \mathbb{E}\left\{ \frac{\partial^2 G}{\partial x(j) \partial y(j)}(X_t, Y_t) \right\} . \tag{52}$$

But by an explicit calculation,

$$\frac{\partial^2 G}{\partial x(j) \partial y(j)}(x,y) \;=\; -s^2 \frac{\partial F}{x(j)}(x)\, \frac{\partial F}{x(j)}(y)\, G(x,y)\,. \tag{53}$$

Since F is Lipshcitz, one bounds $\|\nabla F(x)\|^2 \leq A^2$ for all x.

Hence, the Cauchy-Schwartz inequality and equation (52) give that

$$\varphi'(t) \;\geq\; -s^2 A^2 \varphi(t)\,. \tag{54}$$

Therefore, by Gronwall's inequality, backward-in-time, $\varphi(1-t) \leq \varphi(1)\exp(s^2 A^2 t)$. Since

$$\varphi(1) \;=\; \mathbb{E}\{\exp(s[F(W) - F(W)])\}\,, \tag{55}$$

obviously $\varphi(1) = 1$. In particular, taking $t = 0$ gives

$$\mathbb{E}\{\exp(s[F(U) - F(V)])\} \leq \exp(s^2 A^2)\,. \tag{56}$$

Using Jensen's inequality with respect to the random variable V, which is independent of U, one obtains

$$\mathbb{E}\{\exp(s[F(U) - \mathbb{E}\{F(V)\}])\} \;\leq\; \mathbb{E}\{\exp(s[F(U) - F(V)])\} \leq \exp(s^2 A^2)\,. \tag{57}$$

But of course $\mathbb{E}\{F(V)\} = \mathbb{E}\{F(U)\}$. So this proves inequality (47).

By Markov's inequality, a specialization of Chebyshev's inequality, we then conclude that

$$\mathbb{P}(F(U) - \mathbb{E}\{F(U)\} \geq t) \;\leq\; \exp(s^2 A^2 - st)\,. \tag{58}$$

Optimizing in s, we obtain the optimal value at $s = t/2A^2$ and this gives

$$\mathbb{P}(F(U) - \mathbb{E}\{F(U)\} \geq t) \;\leq\; \exp(-t^2/4A^2)\,. \tag{59}$$

One can do exactly the same for F replaced by $-F$, which has the same Lipschitz constant. This gives the opposite bound, which concludes the proof of inequality (48). ■

The following corollary is Corollary 2.2.5 in [6].

Corollary 4.2 *For any β and h, and any $t \geq 0$,*

$$\mathbb{P}(|P_N(\beta,h,\omega) - p_N(\beta,h)| \geq t) \;\leq 2\exp(-Nt^2/2\beta^2)\,. \tag{60}$$

In particular, using the identity,

$$\mathbb{E}\{[P_N(\beta,h,\omega) - p_N(\beta,h)]^2\} \;=\; 2\int_0^\infty t\mathbb{P}(|P_N(\beta,h,\omega) - p_N(\beta,h)| \geq t)\, dt\,, \tag{61}$$

this implies the Pastur and Shcherbina result, in the form,

$$\mathbb{E}\{[P_N(\beta,h,\omega) - p_N(\beta,h)]^2\} \;\leq\; \frac{4\beta^2}{N}\,. \tag{62}$$

Proof. Consider the pressure $P_N(\beta, h, \omega)$ as a function of the N^2 i.i.d., $\mathcal{N}(0,1)$ random coupling constants $J = (J_{i,j} \, : \, 1 \leq i, j \leq N)$. As a function, this is

$$P_N(\beta, h, J) \, = \, \frac{1}{N} \log \sum_{\sigma \in \mathbb{Z}_2^N} 2^{-N} \exp(\beta h \sum_{j=1}^N \sigma_j) \exp\left(\frac{\beta}{\sqrt{2N}} \sum_{j,k=1}^N J_{j,k} \sigma_j \sigma_k \right) . \tag{63}$$

Using Cauchy-Schwarz, one concludes that for any $\sigma \in \mathbb{Z}_2^N$,

$$\left| \frac{\beta}{\sqrt{2N}} \sum_{j,k=1}^N J_{j,k} \sigma_j \sigma_k - \frac{\beta}{\sqrt{2N}} \sum_{j,k=1}^N J'_{j,k} \sigma_j \sigma_k \right|^2 \, \leq \, \frac{\beta^2}{2N} \sum_{j,k=1}^N |J_{j,k} - J'_{j,k}|^2 \, = \, \frac{\beta^2 N}{2} \|J - J'\| . \tag{64}$$

From this one concludes that

$$\begin{aligned} P_N(\beta, h, J') \, &= \, \frac{1}{N} \log \sum_{\sigma \in \mathbb{Z}_2^N} 2^{-N} \exp(\beta h \sum_{j=1}^N \sigma_j) \exp\left(\frac{\beta}{\sqrt{2N}} \sum_{j,k=1}^N J'_{j,k} \sigma_j \sigma_k \right) \\ &\leq \frac{1}{N} \log \sum_{\sigma \in \mathbb{Z}_2^N} 2^{-N} \exp(\beta h \sum_{j=1}^N \sigma_j) \exp\left(\frac{\beta}{\sqrt{2N}} \sum_{j,k=1}^N J_{j,k} \sigma_j \sigma_k \right) e^{\sqrt{\beta^2 N/2}\, \|J - J'\|} \\ &= \, \frac{\beta}{\sqrt{2N}} \|J - J'\| \, + \, P_N(\beta, h, J) . \end{aligned} \tag{65}$$

The opposite inequality follows symmetrically. This proves that $J \mapsto P_N(\beta, h, J)$ is Lipschitz with Lipschitz constant $A = \beta/\sqrt{2N}$. The rest follows from the concentration estimate, Theorem 4.1. ■

5 Replica Symmetry for $h = 0$ and $0 \leq \beta \leq 1$

Recall the Parisi functional, which Parisi conjectured equals the quenched pressure,

$$\Phi(\beta, h) \, := \, \lim_{K \to \infty} \min_{\substack{0 \leq q_0 \leq \cdots \leq q_K \leq 1 \\ 0 < x_1 < \cdots < x_K < 1}} \Phi_0^{(K)}(\beta; h; q_0, \ldots, q_K; x_1, \ldots, x_K) . \tag{66}$$

Let us further define, for $K = 0, 1, 2, \ldots$,

$$\Phi^{(K)}(\beta, h) \, := \, \min_{\substack{0 \leq q_0 \leq \cdots \leq q_K \leq 1 \\ 0 < x_1 < \cdots < x_K < 1}} \Phi_0^{(K)}(\beta; h; q_0, \ldots, q_K; x_1, \ldots, x_K) , \tag{67}$$

so that $\Phi(\beta, h) = \lim_{K \to \infty} \Phi^{(K)}(\beta, h)$. Note that $\Phi^{(K)}(\beta, h)$ is the "K-level replica symmetry breaking" approximation to the pressure. One may wonder why the limit is guaranteed to exist. It is because, by collapsing some of the q_k onto each other, one can always recover any choice for $K - 1$ from the choices for K. Therefore, the sequence $(\Phi^{(K)}(\beta, h) \, : \, K \in \mathbb{N}_+)$ is nonincreasing. Also, one defines $\Phi^{(0)}(\beta, h; q_0) = \lim_{x_1 \uparrow 1} \Phi^{(1)}(\beta, h; q_0, 1; x_1)$. This gives back the Sherrington-Kirkpatrick approximation to the pressure.

It turns out that there is a subset of (β, h) values in $\mathbb{R}_+ \times \mathbb{R}$, where $\Phi(\beta, h) = \Phi^{(0)}(\beta, h)$. We call this region $\mathcal{U} \subset \mathbb{R}_+ \times \mathbb{R}$. It is of the form $\mathcal{U} = \{(\beta, h) \, : \, \beta \leq \beta_c(|h|) \, h \in \mathbb{R}\}$, with an implicit formula for $\beta_c(|h|)$, which we will not give here. One property of this region is that $\beta_c(|h|)$ is supposed to be increasing with $|h|$, with $\lim_{|h|\to\infty} \beta_c(|h|) = +\infty$. This indicates that high magnetic fields help to prevent replica symmetry from being broken. The boundary $\partial\mathcal{U} = \{(\beta_c(|h|), h) \, : \, h \in \mathbb{R}\}$, is a curve, called the Almeida-Thouless line. A good reference for the Almeida-Thouless line is [7], which is a mathematical rederivation of Thouless and Almeida's result, in light of the results of Guerra and Toninelli and Guerra, himself. In [7], it is not proved that in $\mathcal{U}$ that $p(\beta, h) = \Phi^{(0)}(\beta, h)$ (which indeed, according to Talagrand is supposed to be almost as difficult a problem as proving Parisi's ansatz, itself) or even that $\Phi(\beta, h) = \Phi^{(0)}(\beta, h)$, but rather that for all (β, h) in the complementary region $\mathcal{U}^c$, it is proved that $\Phi(\beta, h) \neq \Phi^{(0)}(\beta, h)$.

It is at least psychologically helpful to know that there is an, admittedly small, range of values of (β, h) in $\mathcal{U}$ where one can prove that not only $\Phi(\beta, h) = \Phi^{(1)}(\beta, h)$ but that $p(\beta, h) = \Phi(\beta, h)$. The region is $\{(\beta, 0) \, : \, \beta < 1\}$. In this region, not only is Parisi's ansatz correct, but so is Sherrington and Kirkpatrick's guess. Moreover, and what is actually more important, is that for $h = 0$ and $0 \leq \beta < 1$, the Sherrington-Kirkpatrick approximation to the quenched pressure is the exact value of the annealed pressure

$$\begin{aligned} q^{\mathrm{ann}}(\beta, h) &= q_N^{\mathrm{ann}}(\beta, h) \quad \text{identical for every } N \in \mathbb{N}_+ \\ &= N^{-1} \log \mathbb{E}\{Z_N(\beta, h)\} \, . \end{aligned} \tag{68}$$

It can be calculated, and it is equal to $\beta^2/4$.

The result was first proved by Aizenman, Lebowitz and Ruelle in [1], using expansion techniques. But a simpler proof has been given in Chapter 2.1 of [6]. We will reproduce that argument. Using Toninelli's result from [7], or results of [1], one can easily determine that the replica symmetric solution is not valid for $\beta > 1$. But we will not give that proof, for which one is recommended to consult [7].

Theorem 5.1 (Aizenman, Lebowitz and Ruelle 1987) *For $h = 0$ and $\beta < 1$, one has*

$$p(\beta, 0) \, = \, \Phi(\beta, 0) \, = \, \Phi^{(0)}(\beta, 0; q_0 = 0) \, . \tag{69}$$

We have already proved the upper bound for the quenched pressure

$$p_N(\beta, h) \, = \, \mathbb{E}\{N^{-1} \log \, Z_N(\beta, h, \omega)\} \, \leq \, N^{-1} \log \, \mathbb{E}\{Z_N(\beta, h, \omega)\} \, = \, \frac{\beta^2}{4} + \log(\cosh(\beta h)) \, . \tag{70}$$

This follows just from Jensen's inequality, and proves that

$$p(\beta, 0) \, \leq \, \frac{\beta^2}{4} \, . \tag{71}$$

We want to see that in fact $p(\beta, 0) = \beta^2/4$, when $\beta \leq 1$. The fact that this equals $\Phi^{(0)}(\beta, 0; q_0 = 0)$ can be easily read off from equation (36) of lecture 1, taking the limit $n \downarrow 0$. The fact that this is also equal to $\Phi^{(0)}(\beta, 0)$, and indeed $\Phi(\beta, 0)$ will follows from

Guerra's replica symmetry breaking bound, which we discuss in the next lecture, which proves that $p(\beta, h) \leq \Phi(\beta, h)$ for all β and h. In this case, since we will prove that $p(\beta, 0) = \Phi^{(0)}(\beta, 0; q_0 = 0)$, this must imply that $\Phi(\beta, 0) \geq \Phi^{(0)}(\beta, 0; q_0 = 0)$, which implies equality since $\Phi(\beta, 0)$ is defined as the minimum over all K, q and x choices.

To prove that $p(\beta, 0) = \beta^2/4$, we need only show the opposite bound that $p(\beta, 0) \geq \beta^2/4$. It will be useful to have the following lemma, which is proved by the familiar Hubbard-Stratonovich transformation.

Lemma 5.2 *For $0 < t < 1$, and any $N \in \mathbb{N}_+$,*

$$\sum_{\sigma \in \mathbb{Z}_2^N} 2^{-N} \exp\left(\frac{t^2}{2N}\left[\sum_{j=1}^N \sigma_j\right]^2\right) \leq \frac{1}{\sqrt{1-t^2}}\,. \tag{72}$$

Proof. By the familiar Hubbard-Stratonovich transformation, we obtain

$$\sum_{\sigma \in \mathbb{Z}_2^N} 2^{-N} \exp\left(\frac{t^2}{2N}\left[\sum_{j=1}^N \sigma_j\right]^2\right) = \sum_{\sigma \in \mathbb{Z}_2^N} 2^{-N} \int_{-\infty}^{\infty} \frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}} \exp\left(tzN^{-1/2}\sum_{j=1}^N \sigma_j\right)\,. \tag{73}$$

Interchanging the sum and the integral, this equals

$$\begin{aligned}\sum_{\sigma \in \mathbb{Z}_2^N} 2^{-N} \exp\left(\frac{t^2}{2N}\left[\sum_{j=1}^N \sigma_j\right]^2\right) &= \int_{-\infty}^{\infty} \frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}} \sum_{\sigma \in \mathbb{Z}_2^N} 2^{-N} \exp\left(tzN^{-1/2}\sum_{j=1}^N \sigma_j\right) \\ &= \int_{-\infty}^{\infty} \frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}} [\cosh(tzN^{-1/2})]^N\,.\end{aligned} \tag{74}$$

But we have the elementary inequality,

$$\cosh(x) \leq e^{x^2/2} \tag{75}$$

which can be proved by calculus techniques, for example since $\tanh(x) \leq x$ for $x \geq 0$ (since its derivative $\operatorname{sech}^2(x)$ is bounded by 1), this implies that $\frac{d}{dx}\log(\cosh(x)) \leq x$ for $x \geq 0$, hence that $\log(\cosh(x)) \leq x^2/2$. Therefore, we conclude

$$\begin{aligned}\sum_{\sigma \in \mathbb{Z}_2^N} 2^{-N} \exp\left(\frac{t^2}{2N}\left[\sum_{j=1}^N \sigma_j\right]^2\right) &\leq \int_{-\infty}^{\infty} \frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}} [\exp(t^2 z^2 N^{-1})]^N \\ &\leq \int_{-\infty}^{\infty} \frac{e^{-(1-t^2)z^2/2}\,dz}{\sqrt{2\pi}} \\ &= \frac{1}{\sqrt{1-t^2}}\,.\end{aligned} \tag{76}$$

■

Proof of Theorem 5.1. *The proof is copied (paraphrased) from [6] without copywrite permission.* Suppose that $0 \leq \beta < 1$. Then one can conclude that

$$\mathbb{E}\{Z_N(\beta,0,\omega)^2\} \;\leq\; \frac{1}{\sqrt{1-\beta^2}}\,[\mathbb{E}\{Z_N(\beta,0,\omega)\}]^2\;. \tag{77}$$

Indeed, an explicit calculation shows that

$$\begin{aligned}\mathbb{E}\{Z_N(\beta,0,\omega)^2\} \;&=\; \mathbb{E}\left\{\sum_{\sigma,\sigma'} 2^{-2N} e^{\beta\sqrt{N/2}\,[K_N(\sigma,\omega)+K_N(\sigma',\omega)]}\right\}\\ &=\; \sum_{\sigma,\sigma'} 2^{-2N}\exp\left(\frac{\beta^2 N}{4}\mathbb{E}\{[K_N(\sigma,\omega)+K_N(\sigma',\omega)]^2\}\right)\\ &=\; \sum_{\sigma,\sigma'} 2^{-2N}\exp\left(\frac{\beta^2 N}{4}[2+2\mathcal{R}_N(\sigma,\sigma')]\right)\\ &=\; \sum_{\sigma,\sigma'} 2^{-2N}\exp\left(\frac{\beta^2 N}{4}[2+2\mathcal{R}_N(\sigma,\sigma')]\right)\end{aligned} \tag{78}$$

Since $\mathbb{E}\{Z_N(\beta,0,\omega)\} = \exp(\beta^2 N/4)$, this gives

$$\mathbb{E}\{Z_N(\beta,0,\omega)^2\} \;=\; [\mathbb{E}\,\{Z_N(\beta,0,\omega)\}]^2 \sum_{\sigma,\sigma'} 2^{-2N}\exp\left(\frac{\beta^2 N}{2}\mathcal{R}_N(\sigma,\sigma')\right)\,. \tag{79}$$

It is quite easy to conclude, using symmetry, that this is equal to

$$\mathbb{E}\{Z_N(\beta,0,\omega)^2\} \;=\; [\mathbb{E}\,\{Z_N(\beta,0,\omega)\}]^2 \sum_{\sigma\in\mathbb{Z}_2^N} 2^{-N}\exp\left(\frac{\beta^2 N}{2}\left[\sum_{j=1}^{N}\sigma_j\right]^2\right)\,. \tag{80}$$

This is equal to the one term in the average over $\sigma' \in \mathbb{Z}_2^N$ where $\sigma'_j \equiv 1$ for all spins. At this point the lemma proves the inequality we desire, namely (77).

Talagrand recalls the second moment method of Paley and Zygmund, which states that for any nonnegative random variable,

$$\mathbb{P}\left(X \geq \frac{1}{2}\mathbb{E}\{X\}\right) \;\geq\; \frac{1}{4}\frac{[\mathbb{E}\{X\}]^2}{\mathbb{E}\{X^2\}}\,. \tag{81}$$

The reader may consult any number of references for the proof (or work it out for himself/herself).

From this one concludes that

$$\mathbb{P}\left(Z_N(\beta,0,\omega) \geq \frac{1}{2}\mathbb{E}\{Z_N(\beta,0)\}\right) \;\geq\; \frac{1}{4}\sqrt{1-\beta^2}\,. \tag{82}$$

Hence one has the important identity,

$$\mathbb{P}\left(P_N(\beta,0,\omega) \;\geq\; \frac{\beta^2}{4} - N^{-1}\log(2)\right) \geq \frac{1}{4}\sqrt{1-\beta^2}\,. \tag{83}$$

By the concentration estimate, one knows that

$$\mathbb{P}\left(P_N(\beta,0,\omega) \;\geq\; p_N(\beta,0)+t\right) \;\leq\; 2\exp(-Nt^2/2\beta^2)\,. \tag{84}$$

Therefore,

$$p_N(\beta,0) \;\geq \frac{\beta^2}{4} - N^{-1}\log(2) - c(\beta)N^{-1/2}\,, \tag{85}$$

where

$$c(\beta) \;=\; \sqrt{2\beta^2|\log(2^{-3}\sqrt{1-\beta^2})|}\,. \tag{86}$$

In the limit $N \to \infty$, this implies that $p(\beta,0) \geq \beta^2/4$ as long as $0 \leq \beta < 1$. The fact that the same is true when $\beta = 1$ follows from convexity of $p(\beta,0)$ with respect to β, which implies continuity of $p(\beta,0)$ in β. ■

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