Spectral Gap for the Symmetric Simple Exclusion Process on Trees and "Higher Spin Trees"

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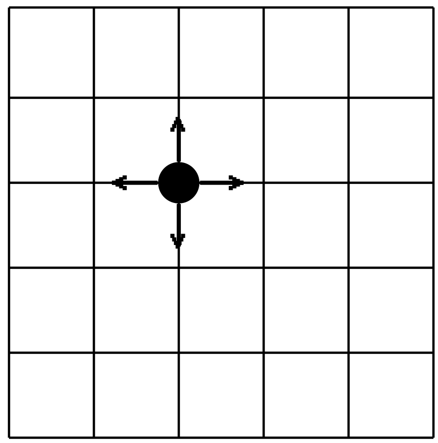
"Ferromagnetic Ordering of Energy Levels"
Bruno Nachtergaele, Wolfgang Spitzer and S*
J. Stat. Phys. (to appear)
`http://front.math.ucdavis.edu/math-ph/0308006`

"A Ferromagnetic Lieb-Mattis theorem"
Bruno Nachtergaele and S* (in preparation)

Symmetric Random Walks



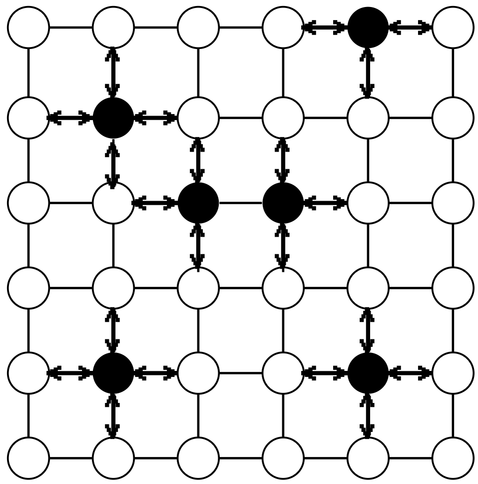
(Λ, E) a *finite* graph.
Λ equals vertex set; E equals edge set.
Assume $x \sim y \Rightarrow y \sim x$.
Assume (Λ, E) is *connected*.

Let $p : E \to (0, \infty)$: $\forall x \sim y$, $p(x, y) = p(y, x)$.
Continuous-time Markov process on Λ,
with generator on $\ell^2(\Lambda)$

$$(\Omega^{\mathrm{rw}} f)(x) \;=\; \sum_{y\,:\;y\sim x} p(x,y)\Big[f(x) - f(y)\Big]\,.$$

! invariant measure by Perron-Frobenius.
Since $\Omega^{\mathrm{rw}} 1 = 0$, μ equals uniform measure.

Symmetric Simple Exclusion Process



Non-ergodic random walk on $\{0,1\}^{\Lambda}$.
Let $\eta : \Lambda \to \{0,1\}$.
$\forall x \neq y \in \Lambda$, define $\eta_{x,y} \in \{0,1\}^{\Lambda}$,

$$\eta_{x,y}(z) = \begin{cases} \eta(y) & z = x\,; \\ \eta(x) & z = y\,; \\ \eta(z) & \text{otherwise}\,. \end{cases}$$

Continuous time Markov process on $\{0,1\}^{\Lambda}$,
with generator on $\ell^2(\{0,1\}^{\Lambda})$

$$(\Omega^{\mathrm{ssep}} f)(\eta) = \sum_{x,y:\ x\sim y} p(x,y)\Big[f(\eta) - f(\eta_{x,y})\Big]\,.$$

Particle number: $|\eta| = \sum_{x\in\Lambda} \eta(x)$.
$\binom{\Lambda}{n} := \{\eta \,:\, |\eta| = n\}$.
Ω^{ssep} *is* ergodic on $\mathcal{H}(n) = \ell^2(\binom{\Lambda}{n})$.

Conjectures

$E_1^{\rm rw}$:= spectral gap of $\Omega^{\rm rw}$.
$E_1^{\rm ssep}(n)$:= spectral gap of $\Omega^{\rm ssep} \restriction \mathcal{H}(n)$ $(1 \le n \le |\Lambda| - 1)$.
$\Omega^{\rm ssep} \restriction \mathcal{H}(1) \cong \Omega^{\rm rw}$.

Conjecture 1. $\min_{1 \le n \le |\Lambda|-1} E_1^{\rm ssep}(n) = E_1^{\rm rw}$.

SU(2)

Given n-particle state, $\mu \in \mathcal{M}_{+,1}(\binom{\Lambda}{n})$, remove one particle, uniformly at random: $\mu^{(n-1)}$.
$S^- : \mathcal{H}(n) \to \mathcal{H}(n-1)$ extends: $n^{-1} S^- \mu = \mu^{(n-1)}$.
Then $[S^-, \Omega^{\rm ssep}] = 0$.
Reverse is to, uniformly at random, add one particle (i.e., remove one hole): $\mu^{(n+1)}$.
$S^+ : \mathcal{H}(n) \to \mathcal{H}(n+1)$, extends: $(|\Lambda| - n)^{-1} S^+ \mu = \mu^{(n+1)}$
$[S^-, S^+] = 2S^{(3)} = [\#(\text{particles}) - \#(\text{holes})]$ is conserved.

The n-particle "lowest-weight vectors", for $n \le |\Lambda|/2$, functions $f \in \mathcal{H}(n)$: $S^- f = 0$.
E.g. $\delta_{(\circ\bullet)} - \delta_{(\bullet\circ)}$.
Define $\mathcal{H}(|\Lambda| - n, n)$ to be their span.
$E_0(|\Lambda| - n, n) \equiv \min \text{spec}\, \Omega^{\rm ssep} \restriction \mathcal{H}((|\Lambda| - n, n))$.

Conjecture 2. (FOEL)

$$E_0(|\Lambda|, 0) \le E_0(|\Lambda| - 1, 1) \le \cdots \le E_0(\lceil |\Lambda|/2 \rceil, \lfloor |\Lambda|/2 \rfloor)\,.$$

Background

Theorem (Koma and Nachtergaele, 1997 L.M.P.)
Conjecture 1 is true for open chains.

A result for rather general graphs, which supports the conjecture is

Theorem (Lu and Yau, 1993 C.M.P.)
The spectral gap of the SSEP on a cube (or other regular figure) in $\mathbb{Z}^d$ of linear size L decays with order $\Theta(L^{-2})$.

Applications

The spectral gap determines L^2 relaxation for the SSEP. Given an initial measure $\nu \in \mathcal{M}_{+,1}(\binom{\Lambda}{n})$, let ν_t be the time-evolved distribution.
Let μ be the uniform measure on $\binom{\Lambda}{n}$.
$\forall \nu \in \mathcal{M}_{+,1}(\binom{\Lambda}{n})$

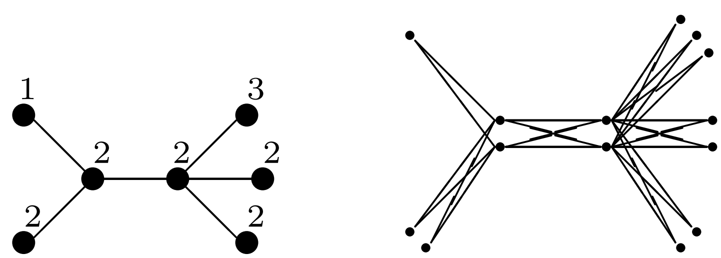
$$\left\| \frac{d\nu_t}{d\mu} - 1 \right\|_{L^2(d\mu)} \leq e^{-t\, E_1^{\mathrm{ssep}}(n)} \left\| \frac{d\nu}{d\mu} - 1 \right\|_{L^2(d\mu)}$$

and for almost all $\nu \in \mathcal{M}_{+,1}(\binom{\Lambda}{n})$ (all but codim 1)

$$\lim_{t\to\infty} t^{-1} \log \frac{\left\| \frac{d\nu_t}{d\mu} - 1 \right\|_{L^2(d\mu)}}{\left\| \frac{d\nu}{d\mu} - 1 \right\|_{L^2(d\mu)}} = -E_1^{\mathrm{ssep}}(n)\,.$$

Theorem 1. (Nachtergaele, Spitzer, S*)
If (Λ, E) is a tree, then Conjecture 1 is true: the spectral gap of the SSEP equals the spectral gap of the RW.
If (Λ, E) is an open chain, Conjecture 2 is true.

Higher Spin Trees



Consider a graph, which we call "higher-spin tree", obtained as follows:

1. Start with a finite tree (Λ_T, E_T);

2. For every vertex $x \in T$, give a positive integer $n(x)$;

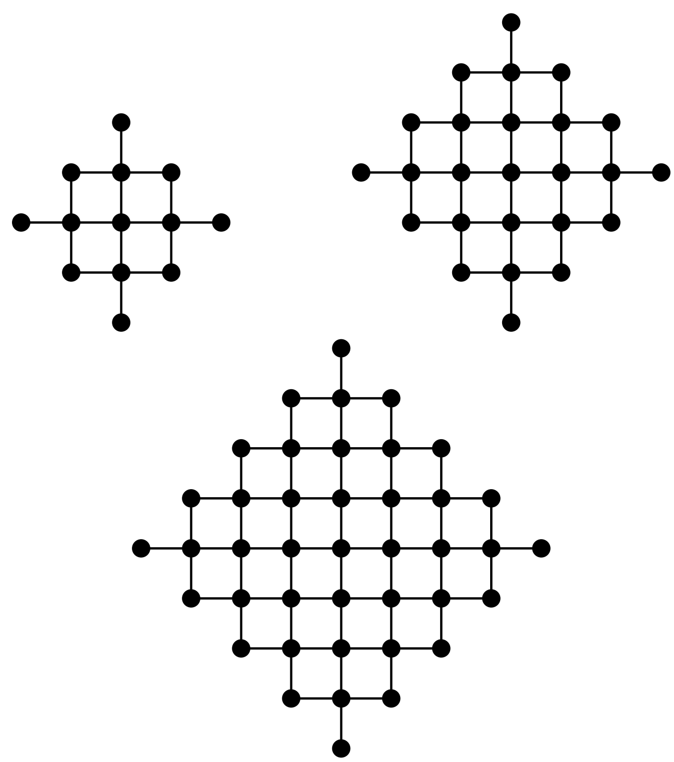
3. Make a new graph labelled (Λ, E) such that:
$\Lambda = \{(x,k) \,:\, x \in \Lambda_T\,,\; k = 1, \ldots, n(x)\}$;
and $(x,k) \sim (y,l)$ in E iff $x \sim y$ in E_T.

4. Assume $p((x,k),(y,l))$ is only a function of x and y, independent of k and l.

Theorem 2. (Nachtergaele, S*)
The same results as Theorem 1 hold if one replaces "tree" or "chain" by "higher spin tree" or "higher spin chain".

Other Examples

By numerically diagonalizing the discrete Laplacian, $\Omega_\Lambda^{\mathrm{rw}}$ (with $p(x,y)$ constant for $x \sim y$) we can also verify Conjecture 1 for the following graphs.



Idea of Proof

We believe conjecture 2, that

$$E_0(|\Lambda|, 0) \leq E_0(|\Lambda| - 1, 1) \leq \cdots \leq E_0(\lceil |\Lambda|/2 \rceil, \lfloor |\Lambda|/2 \rfloor)\,,$$

where $E_0(|\Lambda| - n, n) \equiv \min \operatorname{spec} \Omega^{\mathrm{ssep}} \restriction \mathcal{H}(|\Lambda| - n, n)$
$\mathcal{H}(|\Lambda| - n, n)$ are the n-particle lowest-weight vectors.
We know $E_0(|\Lambda|, 0) = 0$ and this is the minimum (in fact strictly less using PF).
Suppose we could prove

$$E_0(|\Lambda| - 1, 1) \leq \min_{2 \leq n \leq \lfloor |\Lambda|/2 \rfloor} E_0(|\Lambda| - n, n)\,, \qquad (*)$$

Generally $\mathcal{H}(n) \cong \sum_{k=0}^{n} \mathcal{H}((|\Lambda| - k, k))$ for $n \leq \lfloor |\Lambda|/2 \rfloor$.
So

$$E_1^{\mathrm{ssep}}(n) = \min_{1 \leq k \leq n} E_0(|\Lambda| - k, k)\,.$$

Therefore

$$E_0(|\Lambda| - 1, 1) = E_1^{\mathrm{ssep}}(1) = E_1^{\mathrm{rw}}$$

and for $2 \leq n \leq \lfloor |\Lambda|/2 \rfloor$, we have

$$E_1^{\mathrm{ssep}}(n) \geq E_1^{\mathrm{ssep}}(1) \iff E_0(|\Lambda| - 1, 1) \leq \min_{2 \leq k \leq n} E_0(|\Lambda| - k, k)\,.$$

So (*) is actually equivalent to Conjecture 1.

Consider a sequence of induced subgraphs
$\Lambda_2 \subset \Lambda_3 \subset \cdots \subset \Lambda_N = \Lambda$.
Define, for $2 \le M \le N$,

$$(\Omega_M^{\text{ssep}} f)(\eta) \; = \sum_{x,y\in\Lambda_M \,:\, x\sim y} p(x,y)\Big[f(\eta) - f(\eta_{x,y})\Big] \,.$$

Define $E_{1,M}^{\text{ssep}}(n)$ and $E_{0,M}(|\Lambda_M| - n, n)$ relative to Ω_M^{ssep}.

Theorem (Koma and Nachtergaele Induction Step)
If $2 \le M \le N-1$, *and if*

$$E_{0,M}(|\Lambda_M| - n, n) \le \min_{n+1\le k\le \lfloor|\Lambda_M|/2\rfloor} E_0(|\Lambda_M| - k, k) \,,$$

and if

$$E_{0,M+1}(|\Lambda_{M+1}| - n, n) \le E_{0,M}(|\Lambda_M| - n, n) \,,$$

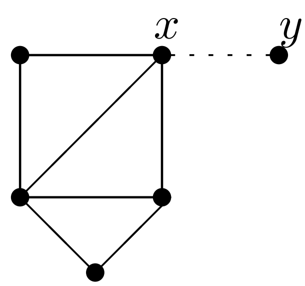
then

$$E_{0,M+1}(|\Lambda_{M+1}| - n, n) \le \min_{n+1\le k\le \lfloor|\Lambda_{M+1}|/2\rfloor} E_0(|\Lambda_{M+1}| - k, k) \,,$$

Therefore, to prove (*) it suffices to prove that $E_{1,M}^{\text{rw}}$ is decreasing as a function of M.

Simple Fact

The eigenvalues of the graph Laplacian, Ω_N^{rw}*, go down when adding a new vertex with only one edge.*



Proof. *When adding one extra vertex without adding an edge, one has the new eigenvalues*

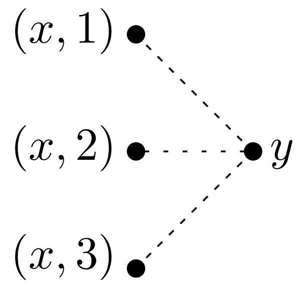
$$0 \leq 0 = E_0(\Omega_N^{rw}) \leq E_1(\Omega_N^{rw}) \leq \cdots \leq E_{N-1}(\Omega_N^{rw})\,.$$

But the edge Hamiltonian is a rank-one operator

$$\Omega_{(x,y)}^{rw} \;=\; p(x,y)\langle \cdot, \delta_x - \delta_y\rangle(\delta_x - \delta_y)\,.$$

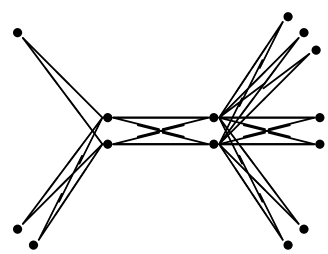
Therefore, by the min-max formulation of the eigenvalues, the new eigenvalues interlace the old ones:

$$0 \leq E_0(\Omega_{N+1}^{rw}) \leq 0 \leq E_1(\Omega_{N+1}^{rw}) \leq E_1(\Omega_N^{rw}) \leq \ldots$$



If y connects to $k > 1$ vertices, $(x,1),\dots,(x,k)$, not rank 1. But if one projects/restricts to vectors symmetric with respect to interchange of $(x,1),\dots,(x,k)$ it is

$$P\,\Omega^{\mathrm{rw}}_{(x,.),y}\,P^{\dagger} \;=\; p(x,y)\left\langle .,\sum_{j=1}^{k}\delta_x - k\delta_y\right\rangle\left(\sum_{j=1}^{k}\delta_x - k\delta_y\right)\,.$$



In general for a higher spin tree, the Markov generator commutes with the symmetric group at a "higher-spin vertex".

Two cases: **1.** Eigenvector is trivial w.r.t. all "single-higher-spin-vertex" permutations.

Rank-one argument works.

2. Eigenvector has a particular sequence of irreps, one for each "higher-spin vertex" such that they are not all trivial. It is a ground state problem to explicitly calculate minimum possible energy for each list of irreps.