Spectral Theory for Systems of Ordinary Differential Equations with Distributional Coefficients

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I am reporting on joint work with

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Introduction

Spectral theory and the Fourier transform

- To describe heat conduction Fourier (1822) considered the problem

$$\phi_t = \phi_{xx}, \quad \phi'(0,t) = \phi'(L,t) = 0, \quad \phi(x,0) = \phi_0(x)$$

- Separating variables and introducing the separation constant λ leads the boundary value problem

$$-y'' = \lambda y, \quad y(0) = y'(L) = 0$$

with eigenfunctions $y_n = \cos(k_n x)$ and eigenvalues $\lambda_n = k_n^2 = (n\pi/L)^2$.

- Then

$$\phi_0(x) = \sum_{n=0}^{\infty} c_n \cos(k_n x)$$

for appropriate Fourier coefficients whenever $\phi_0 \in L^2((0,L), dx)$.

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- Savchuk and Shkalikov (1999) studied a Schrödinger equation with distributional potential v.
- Eckhardt, Gesztesy, Nichols, and Teschl (2013) generalized further and developed a spectral theory for the equation

$$-(p(y' - sy))' - sp(y' - sy) + vy = \lambda ry$$

on an interval (a, b) when $1/p$, v, s, and r are real-valued and locally integrable and $r > 0$.

Systems

- It is useful to note that any of these equations can be realized as a system:
$$Ju' + qu = \lambda wu$$
where $u_1 = y$, $u_2 = p(y' - sy)$ and
$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} v & -s \\ -s & -1/p \end{pmatrix}, \quad \text{and} \quad w = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}.$$

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  - Consider graphs: $(u, f) \in \mathcal{T}_{\max}$ if and only if $u \in \mathrm{BV}_{\mathrm{loc}}$ and $Ju' + qu = wf$
  - Fortunately, there is an abstract spectral theory for linear relations (Arens 1961, Orcutt 1969).

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- If u were even rougher one can not define qu anymore.
- In the presence of discrete components of q and w existence and uniqueness of solutions become an issue.

Hypotheses for this work

We consider the equation $Ju' + qu = wf$ posed on (a, b) and require the following:

- System size is $n \times n$.
- J is constant, invertible, and skew-hermitian.
- q and w are hermitian distributions of order 0 (measures).
- w non-negative (giving rise to the Hilbert space $L^2(w)$ with scalar product $\langle f, g \rangle = \int f^* wg$.
- Additional conditions to be discussed later (probably only technical).

Differential equations

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- $u \in \mathrm{BV}_{\mathrm{loc}}$ implies qu and wu are distributions of order 0.
- Thus each term in
$$Ju' + qu = \lambda wu + wf$$
is a distribution of order 0.

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- The last term disappears unless F and G jump at the same place and if $t = 1/2$.
- We call $(F^+ + F^-)/2$ balanced.

Existence and uniqueness of solutions

- If Q or W have a jump at x the differential equation requires

$$J(u^+(x) - u^-(x)) + (\Delta_q(x) - \lambda\Delta_w(x))\frac{u^+(x) + u^-(x)}{2} = \Delta_w(x)f(x)$$

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- Equivalently, $B_+(\lambda, x)u^+(x) - B_-(\lambda, x)u^-(x) = \Delta_w(x)f(x)$ where

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- Without an existence and uniqueness theorem there is no variation of constants formula.

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- If there are only finitely many such points, a solution of $Ju' + qu = wf$ exists when

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where

$$B = \begin{pmatrix} -B_-(x_1)U_0(x_1) & B_+(x_1) & 0 & \cdots & 0 \\ 0 & -B_-(x_2)U_1(x_2) & B_+(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -B_-(x_N)U_{N-1}(x_N) & B_+(x_N) \end{pmatrix}$$

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- One has to check whether $F(f) \in \operatorname{ran} B$.

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- No technical conditions is needed for this result.

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- $\langle f, v\rangle = \langle u, g\rangle$ and partial integration give

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- $[v - v_1] \in K_0$ and hence $Jv' + qv = wg$ on (ξ_1, ξ_2).

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- For the converse we need to construct a solution u of $Ju' + qu = wf$ if $f \in L^2(w|_{[\xi_1,\xi_2]}) \ominus K_0$.
- This time $f \perp K_0$ allows to show existence of the sought solution.

Spectral theory (expansion in eigenfunctions)

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- $(u, f) \in \ker A$ if and only if $0 = (g_j^* Ju)^-(b) - (g_j^* Ju)^+(a) = 0$ for $j = 1, ..., n_\pm$.

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- Each component of $f \mapsto (R_\lambda f)(x)$ is a bounded linear functional.
- Green's function: $(R_\lambda f)(x) = \langle G(x,\cdot,\lambda)^*, f\rangle = \int G(x,\cdot,\lambda) w f$.

Properties of Green's function I

- The variation of constants formula: if $\lambda \not\in \Lambda$ and $x > x_0$

$$(R_\lambda f)^-(x) = U^-(x,\lambda)\Big(u_0 + J^{-1} \int_{(x_0,x)} U(\cdot,\overline{\lambda})^* wf\Big)$$

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- This gives rise to a (rectangular) linear system
$$F(\lambda)u_0 = \int \big(b_-(\lambda)\chi_{(a,x_0)} + b_+(\lambda)\chi_{(x_0,b)}\big)U(\cdot,\overline{\lambda})^* wf.$$

Properties of Green's function II

- F has a left inverse $F^\dagger$.

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- On $\operatorname{span}(B_+ \cup B_-)^\perp = N_0$ we set $M = 0$.

Properties of Green's function III

- Then

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\begin{aligned}
(R_\lambda f)(x) = U(x,\lambda)M(\lambda)\int_{(a,b)} U(\cdot,\overline{\lambda})^* wf \\
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- All singularities and hence all spectral information is contained in M.

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- Such a function cannot have isolated singularities (except removable ones).
- M is a Herglotz-Nevanlinna function

$$M(\lambda) = A\lambda + B + \int \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right)\nu(t)$$

where $\nu = N'$ and N a non-decreasing matrix.

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- $(u, f) \in T$ if and only if $(\mathcal{F}f)(t) = t(\mathcal{F}u)(t)$.

Thank you