Spectral Theory for Systems of Ordinary Differential Equations with Distributional Coefficients

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Results in Contemporary Mathematical Physics

Santiago

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I am reporting on joint work with

- Ahmed Ghatasheh (UAB)

The Sturm-Liouville equation

- On the real interval (a, b) consider the equation

$$-(pu')' + qu = \lambda wu.$$

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- This guarantees existence and uniqueness of solutions for initial-value problems and symmetry of the resulting minimal operator

$$u \mapsto \frac{1}{w}(-(pu')' + qu)$$

in $L^2(w\ dx)$.

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on an interval (a, b) when $1/p$, v, and s are real-valued and locally integrable and $r > 0$.
- It is useful to note that this equation is equivalent to the system
$$Ju' + qu = \lambda wu$$
where $u_1 = y$ and
$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} v & -s \\ -s & -1/p \end{pmatrix}, \quad \text{and} \quad w = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}.$$

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  - Three additional conditions to be discussed later

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- Fortunately, there is an abstract spectral theory for linear relations.
- The definiteness condition

$$Ju' + qu = 0 \text{ and } wu = 0 \text{ (or } \|u\| = 0) \text{ implies } u \equiv 0$$

is not required.

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- $u \in \mathrm{BV}_{\mathrm{loc}}$ implies qu and wu are distributions of order 0.
- Thus each term in
$$Ju' + qu = \lambda wu + wf$$
is a distribution of order 0.

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- If u right-continuous, i.e., $u(0) = u_r$ implies $(1 + i\alpha)u_r = u_\ell$
- If u balanced, i.e., $u(0) = (u_\ell + u_r)/2$ implies $(2 + i\alpha)u_r = (2 - i\alpha)u_\ell$.

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- Therefore we want our $\mathrm{BV}_{\mathrm{loc}}$ functions balanced.

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- In our case $r = \lambda w - q$ and $g = wf$.
- $\lambda = 0$: $2J \pm \Delta_q(x)$ invertible.
- $\Lambda = \{\lambda : \exists x : \det(2J \pm (\Delta_q(x) - \lambda\Delta_w(x))) = 0\}$, the bad set, is countable.

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- $(u, f) \in \ker A$ if and only if $0 = (g_j^* Ju)^-(b) - (g_j^* Ju)^+(a) = 0$ for $j = 1, \ldots, n_\pm$.

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- Green's function: $(E_\lambda f)(x) = \langle G(x,\cdot,\lambda)^*, f\rangle = \int G(x,\cdot,\lambda) wf$.

Properties of Green's function I

- The variation of constants formula: if $\lambda \not\in \Lambda$ and $x > x_0$
$$(E_\lambda f)^-(x) = U(x,\lambda)\big(u_0 + J^{-1}\int_{(x_0,x)} U(\cdot,\overline{\lambda})^* wf\big)$$
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- Assume that f is compactly supported so that u satisfies the homogeneous equation near a and b. Then u_0 has to be chosen so that
  - $E_\lambda f$ is in $L^2(w)$ near both a and b,
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Properties of Green's function I

- The variation of constants formula: if $\lambda \notin \Lambda$ and $x > x_0$

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- This gives rise to a (rectangular) linear system

$$F(\lambda)u_0 = \int (b_-(\lambda)\chi_{(a,x_0)} + b_+(\lambda)\chi_{(x_0,b)})U(\cdot,\overline{\lambda})^* wf.$$

Properties of Green's function II

- F has a left inverse $F^\dagger$.

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- On $\operatorname{span}(B_+ \cup B_-)^\perp = N_0$ we set $M = 0$.

Properties of Green's function III

- Then

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(E_\lambda f)(x) = U(x,\lambda)M(\lambda)\int_{(a,b)} U(\cdot,\overline{\lambda})^* wf \\
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- M is a Herglotz-Nevanlinna function

$$M(\lambda) = A\lambda + B + \int \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right)\nu(t)$$

where $\nu = N'$ and N a non-decreasing matrix.

The Fourier Transform

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- $(u, f) \in T$ if and only if $(\mathcal{F}f)(t) = t(\mathcal{F}u)(t)$.

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