J. Math. Anal. Appl. 423 (2015) 1753–1773

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The inverse resonance problem for left-definite Sturm–Liouville operators

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ARTICLE INFO

Article history:
Received 26 July 2013
Available online 4 November 2014
Submitted by S.A. Fulling

Keywords:
Resonances
Scattering theory
Inverse scattering theory
Left-definite problems

ABSTRACT

We address inverse spectral and scattering problems for the half-line, left-definite Sturm–Liouville equation, $-u'' + qu = \lambda wu$. These problems have been considered recently as they are critical in integrating the Camassa–Holm equation. Previous results required that the support of w be free of gaps, relatively open intervals on which $w = 0$ almost everywhere. We relax this condition and prove an inverse spectral theorem that tells to what extent the Weyl–Titchmarsh m-function or the spectral measure determines the coefficients of the equation. Note that, unlike the Schrödinger equation, knowing the spectral measure is not the same as knowing the m-function. We also prove an inverse resonance theorem that explains to what extent the eigenvalues and resonances determine the spectral measure (and, thus, the coefficients). Again, unlike the Schrödinger case, these data are not sufficient; the presence of w multiplying the spectral parameter complicates the analysis. However, we show that, in most cases, only one other number is needed to fully recover the spectral measure.

1. Introduction

We consider here the inverse spectral and the inverse scattering problem for the differential equation

$$-u'' + qu = \lambda wu \tag{1.1}$$

posed on a finite or infinite interval $[0, b)$ in the case where w may change sign but q is assumed to be non-negative. The problem was considered in [3] and we improve here on those results in two directions: first, the inverse spectral result in [3] required $\operatorname{supp} w = [0, b)$ and we will do away with that assumption, see Theorems 2.3 and 2.4; second, we will establish an inverse resonance result, stating to what extent the location of the eigenvalues and resonances of the associated operator determines the coefficients q and w, see Theorem 3.8. The corresponding result for the discrete version of this problem was attained in [1]. We refer to [3] for information on context and applications of left-definite Sturm–Liouville problems.

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http://dx.doi.org/10.1016/j.jmaa.2014.10.078
0022-247X/

The minimal assumptions on the coefficients made here are that q and w are locally integrable and that, as mentioned before, $q \geq 0$ but $q \not\equiv 0$. We then define the Hilbert space $\mathcal{H}_1$ as the set of all locally absolutely continuous complex-valued functions u defined on $[0, b)$ for which u' and $\sqrt{q}u$ are square integrable and the closed linear relation $T_1 \subset \mathcal{H}_1 \oplus \mathcal{H}_1$ as the set of all $(u, f) \in \mathcal{H}_1 \oplus \mathcal{H}_1$ for which $-u'' + qu = wf$ (for more details on this and the following facts see [3]). Now, if $T' \subset T_1$ is a self-adjoint relation, define $\mathcal{H}_\infty = \{g \in \mathcal{H}_1 : (0, g) \in T'\}$ and its orthogonal complement $\mathcal{H}$. Then $T = T' \cap (\mathcal{H} \oplus \mathcal{H})$ is defined on a dense subset of $\mathcal{H}$ and is a self-adjoint operator in $\mathcal{H}$.

From now on we shall assume that b is a singular point, i.e., $b = \infty$ or q not integrable near b (the limit-point case). With this assumption no boundary condition at b is necessary and

$$T' = \big\{(u, f) \in T_1 : f(0) \cos\alpha + u'(0) \sin\alpha = 0\big\}$$

is a self-adjoint relation in $\mathcal{H}_1$ for any $\alpha \in [0, \pi)$. We also note that finite functions[1] are dense in $\mathcal{H}_1$ when b is a singular point, a fact which simplifies some of our arguments.

The spectral theory for T begins with the definition of functions $\theta(\cdot, \lambda)$ and $\varphi(\cdot, \lambda)$ as the (unique) solutions of Eq. (1.1) satisfying the initial conditions

$$\theta'(0, \lambda) = -\lambda\varphi(0, \lambda) = \sin\alpha \quad \text{and} \quad \lambda\theta(0, \lambda) = \varphi'(0, \lambda) = \cos\alpha.$$

The functions $\lambda\theta(x, \cdot)$, $\lambda\varphi(x, \cdot)$, $\theta'(x, \cdot)$, and $\varphi'(x, \cdot)$ are entire functions of growth order 1/2. Since there is, up to constant multiples and for non-real λ, one and only one solution u of Eq. (1.1) for which $\int_0^b (|u'|^2 + q|u|^2)$ is finite, there is a unique function $m : \mathbb{C} \setminus \mathbb{R}$ such that $\psi(\cdot, \lambda) = \theta(\cdot, \lambda) + m(\lambda)\varphi(\cdot, \lambda)$ is in $\mathcal{H}_1$. The function $\psi(\cdot, \lambda)$ is called the Weyl–Titchmarsh solution of (1.1). The function m, called the Weyl–Titchmarsh m-function, is a Herglotz–Nevanlinna function so that it has a unique representation

$$m(\lambda) = A\lambda + B + \int\limits_{\mathbb{R}} \left(\frac{1}{t - \lambda} - \frac{t}{t^2 + 1}\right) d\rho \tag{1.2}$$

where $A \geq 0$, B is real, and $d\rho$ is a positive measure (called the spectral measure) such that $\int_{\mathbb{R}} d\rho/(t^2 + 1) < \infty$. The map $\mathcal{F} : u \mapsto \langle u, \varphi(\cdot, \overline{\lambda})\rangle$, a generalized Fourier transform, initially defined for finite functions $u \in \mathcal{H}_1$ may be extended to $\mathcal{H}_1$ by continuity and takes its values in $L^2(\rho)$. In fact, $\mathcal{F}|_{\mathcal{H}_\infty} = 0$ and $\mathcal{F}|_{\mathcal{H}} : \mathcal{H} \to L^2(\rho)$ is a unitary map. The inverse spectral result (Theorem 2.3) is then that two operators T and $\breve{T}$, respectively associated with coefficients (q, w, α) and $(\breve{q}, \breve{w}, \breve{\alpha})$, have the same m-function only if the coefficients are in specific relationships. Requiring only the equality of the spectral functions gives a somewhat weaker conclusion (Theorem 2.4). The converse statements hold true, too (Theorem 2.5). This is the subject of Section 2.

In Section 3 we consider the inverse resonance problem. To define the concept of resonances and prove the associated results we require the existence of some non-negative constant q_0 such that the functions $q - q_0$ and $w - 1$ are compactly supported. In contrast to the Schrödinger case, the location of all eigenvalues and resonances does not determine the operator uniquely, due to the fact that the presence of a variable w multiplying the spectral parameter λ causes much more complicated asymptotic behavior when λ tends to infinity. Theorem 3.8 describes the set of all operators sharing the same eigenvalues and resonances.

Finally, we remark that we use $[f, g]$ to denote the Wronskian $fg' - f'g$ of two locally absolutely continuous functions and χ_S to denote the characteristic function of the set S.

[1] By this term we will denote functions whose support is compact in $[0, b)$.

2. Inverse spectral theory in the presence of gaps in the support of the weight function

In the presence of two coefficients, q and w, in a Sturm–Liouville expression it cannot be expected that the spectral measure determines them both. Instead two such operators may be unitarily equivalent without being equal. In [3] an inverse spectral result was established when the support[2] of w equals $[0, b)$. Theorem 2.3 generalizes that result to the case where the assumption on $\operatorname{supp} w$ is dropped. In this case $\mathcal{H}_\infty$ is not trivial, and we begin by establishing a characterization of both $\mathcal{H}_\infty$ and $\mathcal{H}$. The case when $w \equiv 0$ is of course extreme and we treat it briefly in Section 2.2 but not before we discuss, in Section 2.1, some basic facts about Liouville transforms which are the manifestations of the operator establishing the unitary equivalence mentioned above. Theorem 2.3 is stated and proved in Section 2.4. It relies essentially on a Paley–Wiener theorem which is proved in Section 2.3. Several proofs in these last two sections rely at least in spirit on those of [3] and [4]. But sometimes we are taking a somewhat different approach or provide a slightly simpler proof.

The complement of $\operatorname{supp} w$ in $[0, b)$ is a disjoint union of (at most countably many) relatively open intervals, which we call *gaps*. If the point a is in the closure of a gap we define a_- and a_+ as the left and right endpoint of that gap. Otherwise we set $a_- = a_+ = a$.

Theorem 2.1. *Suppose q and w are locally integrable, $q \geq 0$ but $q \not\equiv 0$, and $\alpha \in [0, \pi)$. Then*

$$\mathcal{H}_\infty = \{g \in \mathcal{H}_1 : wg = 0 \text{ a.e. and } g(0) \cos \alpha = 0\}.$$

Moreover, if $0 \in \operatorname{supp} w$ or $\cos \alpha \neq 0$,

$$\mathcal{H} = \{u \in \mathcal{H}_1 : -u'' + qu = 0 \text{ on every gap in the support of } w\}$$

while otherwise, if 0 is in a gap and $\cos \alpha = 0$,

$$\mathcal{H} = \{u \in \mathcal{H}_1 : u'(0) = 0 \text{ and } -u'' + qu = 0 \text{ on every gap in the support of } w\}.$$

In particular, $u \in \mathcal{H}_1$ coincides with its projection onto $\mathcal{H}$ outside the gaps.

Proof. The first statement on $\mathcal{H}_\infty$ is immediate from its definition.

Assume $u \in \mathcal{H}_1$ and $g \in \mathcal{H}_\infty$. Since g vanishes on $\operatorname{supp} w$ and since the complement of $\operatorname{supp} w$ is a countable union of (relatively) open intervals we get

$$\langle u, g \rangle = \sum_{n \in N} \int_{a_{n;-}}^{a_{n;+}} (u' \bar{g}' + q u \bar{g})$$

denoting the gaps by $(a_{n;-}, a_{n;+})$, $n \in N$, where N is an appropriate index set.

We begin by considering the case where $0 \in \operatorname{supp} w$ or $\cos \alpha \neq 0$. Assume first that $-u'' + qu = 0$ on every gap. Then integration by parts, the fact that g vanishes at the endpoints of gaps, and statement (3) of Lemma A.3 show that $\langle u, g \rangle = 0$, i.e., $u \in \mathcal{H}$. Conversely, if $u \in \mathcal{H}$ and (a_-, a_+) is a gap, we set $Q(x) = \int_0^x qu$ and $g(x) = \int_0^x (u' - Q - A)\chi_{[c,d]}$. Then g is in $\mathcal{H}_\infty$ if $[c, d] \in (a_-, a_+)$ and A is chosen so that $g(d) = 0$. Thus

$$0 = \int_{a_-}^{a_+} (u' \overline{g'} + q u \bar{g}) = \int_c^d (u' - Q) \overline{g'}$$

[2] This refers to the support of w, denoted by $\operatorname{supp} w$, in the sense of distributions (essential support), i.e., the complement of the union of all open sets V for which $\int_V |w| = 0$.

so that $u' = Q + A$ and $u'' = qu$ on $[c, d]$. This completes the characterization of $\mathcal{H}$ except when $(0, 0_+)$ is a gap and $\cos\alpha = 0$. In this case the condition $u'(0) = 0$ becomes necessary, too, since it is not required that $g(0) = 0$ when $g \in \mathcal{H}_\infty$. It follows as above that $-u'' + qu = 0$ on the gaps and $u'(0) = 0$ are sufficient to have $u \in \mathcal{H}$. □

2.1. Liouville transforms

A *Liouville transform* maps functions defined on an interval $[\breve{a}, \breve{b})$ to functions defined on an interval $[a, b)$ via the prescription $\breve{u} \mapsto u = r\breve{u} \circ s$. We are here interested only in the case where a and $\breve{a}$ are finite (but b and $\breve{b}$ may be infinite). Moreover, we consider only Liouville transforms where r and s have the following properties:

(1) $s, r, r' : [a, b) \to \mathbb{R}$ are locally absolutely continuous;
(2) s maps $[a, b)$ bijectively to $[\breve{a}, \breve{b})$;
(3) r is strictly positive; and
(4) $r^2 s'$ is a positive constant.

We shall denote such a transform by $\mathcal{L}_{r,s}$. The transforms satisfying these conditions form, by definition, the class $\mathcal{S}(a, b; \breve{a}, \breve{b})$. It is clear that the inverse of a transform in $\mathcal{S}(a, b; \breve{a}, \breve{b})$ is in $\mathcal{S}(\breve{a}, \breve{b}; a, b)$. Also, the composition of two transforms in $\mathcal{S}(a_2, b_2; a_1, b_1)$ and $\mathcal{S}(a_1, b_1; a_0, b_0)$, respectively, is in $\mathcal{S}(a_2, b_2; a_0, b_0)$.

Given two potentials q and $\breve{q}$ our next goal is to construct Liouville transforms which map solutions of $-\breve{u}'' + \breve{q}\breve{u} = 0$ bijectively to those of $-u'' + qu = 0$. We begin with the case when q and $\breve{q}$ are defined (and non-negative and integrable) on finite intervals. We denote by $\phi_\pm$ the solutions of $-u'' + qu = 0$ on $[a, b]$ satisfying the boundary conditions $\phi_+(a) = \phi_-(b) = 0$ and $\phi_+(b) = \phi_-(a) = 1$. These are positive on (a, b) and strictly monotone. We also define the analogous solutions $\breve{\phi}_\pm$ of $-\breve{u}'' + \breve{q}\breve{u} = 0$ on $[\breve{a}, \breve{b}]$. If $r_\pm$ are given positive numbers we are looking for a transform $\mathcal{L}_{r,s}$ such that $\mathcal{L}_{r,s}\breve{\phi}_\pm = r_\pm \phi_\pm$.

The derivative of ϕ_+/ϕ_- is $\phi_+'(a)/\phi_-^2 > 0$. Thus ϕ_+/ϕ_- is strictly increasing on $[a, b)$ with range $[0, \infty)$. Since a similar statement holds true for $\breve{\phi}_+/\breve{\phi}_-$, there is an absolutely continuous, strictly increasing function s mapping $[a, b]$ onto $[\breve{a}, \breve{b}]$ such that

$$(\breve{\phi}_+/\breve{\phi}_-) \circ s = r_+\phi_+/(r_-\phi_-).$$

We also define r on $[a, b]$ by $r = r_+\phi_+/(\breve{\phi}_+ \circ s) = r_-\phi_-/(\breve{\phi}_- \circ s)$. It is clear that $r > 0$ and that r and r' are absolutely continuous. One computes

$$r^2 s' = r_+ r_- \phi_+'(a)/\breve{\phi}_+'(\breve{a}) \tag{2.1}$$

showing that $\mathcal{L}_{r,s}$ is in $\mathcal{S}(a, b; \breve{a}, \breve{b})$. Finally, taking two derivatives in $r\breve{\phi}_+ \circ s = r_+\phi_+$ gives

$$r{s'}^2 \breve{q} \circ s = -r'' + qr. \tag{2.2}$$

A similar result holds true when b and $\breve{b}$ are singular points[3] but the proof is a little different. First assume $\breve{q} = 0$ and $[\breve{a}, \breve{b}) = [0, \infty)$. We always have a positive solution r of $-u'' + qu = 0$ such that $\int_a^b 1/r^2 = \infty$. If $q = 0$ we may choose $r = \mu$ where μ is a positive number and if $q \neq 0$ we may choose $r = -\mu\psi_0$, where ψ_0 is defined in Lemma A.4 (an integration by parts shows that ψ_0 cannot have any zeros), but we need

[3] Recall that b is a singular point if it is infinite or if q fails to be integrable near b.

to show that indeed $\int_a^b 1/r^2 = \infty$. To see this set $s(x) = \nu \int_a^x 1/r^2$. Since rs is a solution of $-u'' + qu = 0$ independent of r we have (cf. Lemma A.3)

$$\infty = \int_a^b \left((rs)'^2 + q(rs)^2\right) \le \int_a^b \left(2s^2\left(r'^2 + qr^2\right) + \frac{2\nu^2}{r^2}\right) \le 2s(b)^2\|r\|^2 + 2\nu s(b)$$

so that $s(b)$ must be infinite. Thus we have constructed a Liouville transform $\mathcal{L}_{r,s} \in \mathcal{S}(a, b; 0, \infty)$ which maps the constant function 1 to $-\mu\psi_0$ (or μ) and for which $r^2 s'$ is constant, namely $r^2 s' = \nu$.

In the general case we construct transforms $\mathcal{L}_{r_1,s_1} \in \mathcal{S}(a, b; 0, \infty)$ and $\mathcal{L}_{r_2,s_2} \in \mathcal{S}(\breve{a}, \breve{b}; 0, \infty)$ as above to get a transform $\mathcal{L}_{r,s} = \mathcal{L}_{r_1,s_1} \circ \mathcal{L}_{r_2,s_2}^{-1} \in \mathcal{S}(a, b; \breve{a}, \breve{b})$. We emphasize that $s = s_2^{-1} \circ s_1$ and $r = r_1/(r_2 \circ s)$. In particular, choosing $s_1(x) = \nu \int_a^x 1/\psi_0^2$, $s_2(x) = \int_{\breve{a}}^x 1/\breve{\psi}_0^2$ and $r = \mu\psi_0/(\breve{\psi}_0 \circ s)$, the transform $\mathcal{L}_{r,s}$ maps $\breve{\psi}_0$ to $\mu\psi_0$ so that $r^2 s' = \mu^2\nu$. Eq. (2.2) holds again.

2.2. The case of vanishing w

In [3] the case when w vanishes identically was excluded because it leads to a rather trivial situation, as we will see shortly. However, as all results apply to this situation we ask the reader's indulgence when we now briefly consider this case.

When $w = 0$ we get that $\mathcal{H}_\infty = \{g \in \mathcal{H}_1 : g(0)\cos\alpha = 0\}$. Thus, if $\cos\alpha = 0$, we have $\mathcal{H}_\infty = \mathcal{H}_1$ and $\mathcal{H}$ is trivial. Otherwise, according to Theorem 2.1, $\mathcal{H}$ is the one-dimensional space spanned by ψ_0 defined in Lemma A.4. In either case the Weyl–Titchmarsh solution $\psi(\cdot, \lambda)$ is a multiple of ψ_0. Thus, recalling that $\psi_0'(0) = 1$, we obtain

$$m(\lambda) = \frac{\cos\alpha - \lambda\psi_0(0)\sin\alpha}{\sin\alpha + \lambda\psi_0(0)\cos\alpha}.$$

If $\cos\alpha = 0$ we have $m(\lambda) = -\psi_0(0)\lambda$ which implies that $d\rho = 0$ giving rise to a trivial transform space $L^2(\rho)$ and a trivial Fourier transform.

If $\cos\alpha \ne 0$ the m-function has a pole at $\lambda_0 = -\tan\alpha/\psi_0(0)$ which is the eigenvalue of $T = \{(u, f) \in \mathcal{H} \oplus \mathcal{H} : f(0)\cos\alpha + u'(0)\sin\alpha = 0\}$, i.e., $T\psi_0 = \lambda_0\psi_0$. We also get $\rho_0 = \rho(\{\lambda_0\}) = -\psi_0(0)^{-1}(\cos\alpha)^{-2}$ and $\rho = \rho_0\chi_{\{\lambda_0\}}$. The Fourier transform of a finite function $u \in \mathcal{H}_1$ is $-u(0)\cos\alpha$. By Lemma A.1 this persists for all $u \in \mathcal{H}_1$. In particular

$$(\mathcal{F}\psi_0)(\lambda_0) = -\psi_0(0)\cos\alpha = \begin{cases} \sin\alpha/\lambda_0 & \text{if } \alpha \ne 0 \\ \rho_0^{-1} & \text{if } \alpha = 0 \end{cases}$$

in accordance with Lemma A.9. Moreover, the constants A and B in the Nevanlinna representation of the m-function are $A = 0$ and $B = \lambda_0\rho_0/(1 + \lambda_0^2) - \tan\alpha$.

Now let two operators T and $\breve{T}$, respectively associated with the coefficients (q, w, α) and $(\breve{q}, \breve{w}, \breve{\alpha})$, be given. Suppose that the associated spectral measures $d\rho$ and $d\breve{\rho}$ are identical and have finite support. Then $\mathcal{H}$ and $\breve{\mathcal{H}}$ are finite-dimensional, but this cannot happen unless w and $\breve{w}$ vanish identically in which case we actually have, as we saw above, that $\mathcal{H}$ and $\breve{\mathcal{H}}$ are both trivial or both one-dimensional. In either case we define the Liouville transform $\mathcal{L}_{r,s} \in \mathcal{S}(0, b; 0, \breve{b})$ as in the previous section choosing μ and ν so that $r(0) = \sin\alpha/\sin\breve{\alpha}$ (if $\breve{\alpha} = 0$ we must have $\alpha = 0$ in which case we want $r(0) = 1$) and $r'(0) = 0$ and these choices give $\gamma_0 = r^2 s' = r(0)^2\breve{\psi}_0(0)/\psi_0(0)$.

In the case, when $\mathcal{H}$ and $\breve{\mathcal{H}}$ are trivial, we must have $\alpha = \breve{\alpha} = \pi/2$. It also follows trivially that $T\mathcal{L}_{r,s} = 0 = \mathcal{L}_{r,s}\breve{T}$. If we even require the equality of the m-functions, then $\psi_0(0) = \breve{\psi}_0(0)$ which implies $\gamma_0 = 1$ and, using (2.2), $\breve{q} \circ s = r^3(-r'' + qr)$.

When $\mathcal{H}$ and $\breve{\mathcal{H}}$ are one-dimensional, the equality of the spectral measures means that $\lambda_0 = \breve{\lambda}_0$ and $\rho_0 = \breve{\rho}_0$. These two conditions imply that $\sin(2\alpha) = \sin(2\breve{\alpha})$, $\gamma_0 = 1$, $\breve{q} \circ s = r^3(-r'' + qr)$, and $T\mathcal{L}_{r,s} = \mathcal{L}_{r,s}\breve{T}$. We also note that the requirement $m = \breve{m}$ gives $\tan\alpha = \tan\breve{\alpha}$ so that $\alpha = \breve{\alpha}$.

2.3. The Paley–Wiener theorem

Section 3 of [3] discusses the Fourier transform $\mathcal{F}$ defined in the Introduction. It maps $\mathcal{H}_1$ to $L^2(\rho)$ for a suitable measure ρ. The transform depends on the weight function w which was required to be locally integrable. In Section 5 of that paper a Paley–Wiener theorem, setting up a connection between support properties of functions in $\mathcal{H}_1$ and growth properties of their transforms, was established under the additional assumption that the support of w equals $[0, b)$. This additional assumption is not necessary as will be shown in the following. Nevertheless, our proof here is close to the one in [3] and we refer to it for more details.

We denote the projection of $\psi(\cdot, \lambda)$ onto $\mathcal{H}$ by $\omega(\cdot, \lambda)$. In fact, by Lemma A.5, $\psi(\cdot, \lambda)$ is already in $\mathcal{H}$ unless 0 is in a gap of $\operatorname{supp} w$ and $\cos\alpha = 0$. If we are in that situation we still know from Theorem 2.1 that $\psi(\cdot, \lambda)$ and $\omega(\cdot, \lambda)$ coincide beyond the first gap.

Theorem 2.2. *Assume $u \in \mathcal{H}_1$ is supported in $[0, a]$ where $a < b$. Then $\hat{u} = \mathcal{F}u$ has an entire continuation with growth order at most $1/2$ satisfying*

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left|\hat{u}\left(t^2\lambda\right)\right| \le \int_0^a \operatorname{Re}\sqrt{-\lambda w} \tag{2.3}$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Here the square root is that with positive real part.

Conversely, if $\hat{u} \in L^2(\rho)$ has an entire continuation of order at most $1/2$ satisfying (2.3) for two non-real values of λ on different rays emanating from the origin, then the support of $u = \mathcal{F}^*\hat{u}$ is contained in $[0, a_+]$. In particular, $u = 0$ if $a_+ = 0$.

Proof. The easier part of the proof is where one assumes $\operatorname{supp} u \subset [0, a]$. Integration by parts gives

$$\hat{u}(\lambda) = -u(0)\cos\alpha + \int_0^{a_-} uw\lambda\varphi(\cdot, \lambda).$$

This is an entire function of growth order at most $1/2$ since $\lambda \mapsto \lambda\varphi(x, \lambda)$ has this property for all x. If $a_- = 0$ the validity of (2.3) is immediate interpreting, if necessary, $\ln 0$ as $-\infty$. We may thus turn to the case where $a_- > 0$. By Corollary 6.2 of [2], we have

$$\varphi\left(x, t^2\lambda\right) = \exp\left[t\int_0^x \sqrt{-\lambda w} + o(t)\right] \tag{2.4}$$

as $t \to \infty$ if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and the error is locally uniform in x. Given any $\varepsilon > 0$ we obtain

$$\left|\hat{u}\left(t^2\lambda\right)\right| \le \left|u(0)\cos\alpha\right| + \left|t^2\lambda\right| \exp\left(\varepsilon t + t\int_0^a \operatorname{Re}\sqrt{-\lambda w}\right)\int_0^a |uw|$$

using that $\int_0^x \operatorname{Re}\sqrt{-\lambda w}$ is non-decreasing as a function of x. Since ε is arbitrary this establishes the first part of the theorem.

For the converse, the harder part of the proof, assume now that (2.3) holds for two non-real values of λ on two different rays. There is nothing to prove if $w = 0$ on $[a, b)$ and so we assume that $a_+ < b$. Define $H(\cdot, \lambda) = R_\lambda u - \hat{u}(\lambda)\omega(\cdot, \lambda) \in \mathcal{H}$ and $v = G_0(wf)$ where $f \in \mathcal{H}_1$ is compactly supported in $(a_+ + \varepsilon, b)$ and where G_0 is defined in Lemma A.2. Note that v is in the domain of T (even in the domain of T_1^*, which is a restriction of T) and thus its Fourier transform $\hat{v}$ is integrable with respect to $d\rho$ by Lemma A.8. Now set

$$F(\lambda) = \int_0^b H(\cdot, \lambda) w \overline{f} = \langle H(\cdot, \lambda), v \rangle = \int_{\mathbb{R}} \frac{\hat{u}(t) - \hat{u}(\lambda)}{t - \lambda} \overline{\hat{v}(t)} d\rho(t)$$

where the last equality follows since $\mathcal{F}\omega(\cdot, \lambda) = \mathcal{F}\psi(\cdot, \lambda) = 1/(t - \lambda)$, cf. Lemma A.9. We may now repeat the proof of Lemma 5.3 in [3] to show that F is entire and of growth order at most $1/2$. Since $a_+ \geq 0_+$ we have $\omega(x, \lambda) = \psi(x, \lambda)$ when $x \geq a_+ + \varepsilon$ and we may use again the proof of Lemma 5.3 in [3] to show that F tends to zero in the two directions given by the two values of λ mentioned in the hypothesis while relying on the facts that $\int_{a_+}^{a_+ + \varepsilon} \operatorname{Re} \sqrt{-\lambda w} > 0$ for every positive ε (smaller than $b - a_+$) and that

$$\psi(x, t^2 \lambda) = \exp\left[-t \int_0^x \sqrt{-\lambda w} + o(t)\right] \tag{2.5}$$

as $t \to \infty$ with a locally uniform error in x (cf., Theorem 6.1 of [2]). By Phragmén–Lindelöf's principle and Liouville's theorem it follows first that F is bounded everywhere and then that it is constant. Since the constant must be 0 we get that F is identically 0. Hence,

$$0 = F(\lambda) = \int_0^b H(\cdot, \lambda) w \overline{f}$$

for every $f \in \mathcal{H}_1$ compactly supported in $(a_+ + \varepsilon, b)$. Since this is so for every $\varepsilon \in (0, b - a_+)$ we get that $H(\cdot, \lambda) w$ vanishes almost everywhere in (a_+, b) and hence that $H(\cdot, \lambda)$ vanishes on $\operatorname{supp} w \cap (a_+, b)$. But, since it is in $\mathcal{H}$ it satisfies the equation $-y'' + qy = 0$ on any gap (c_-, c_+) in (a_+, b). Hence $H(\cdot, \lambda)$ vanishes there identically in view of the boundary conditions $H(c_\pm, \lambda) = 0$ and the fact that $q \geq 0$. Applying now the differential equation to H gives

$$-H'' + qH = w(u + \lambda H). \tag{2.6}$$

So we also have $wu = 0$ and, repeating the argument just given, $u = 0$ in $[a_+, b)$. □

2.4. The inverse spectral problem

The inverse spectral theorem in [3] relied on the assumption that $\operatorname{supp} w = [0, b)$ and $\operatorname{supp} \breve{w} = [0, \breve{b})$. It is our goal to relax this assumption and establish an inverse spectral result assuming only local integrability of w and $\breve{w}$ thus allowing for gaps in $\operatorname{supp} w$ and $\operatorname{supp} \breve{w}$. A complication arises when $[0, b)$ starts with a gap and $\alpha = \pi/2$ (or $[0, \breve{b})$ starts with a gap and $\breve{\alpha} = \pi/2$). In the following we shall call such a gap *exceptional*. If α (or $\breve{\alpha}$) is different from $\pi/2$ or if a gap begins to the right of zero, we call the gap *regular*. The trouble caused by exceptional gaps is due to the fact that in their presence the Weyl–Titchmarsh solutions are not in $\mathcal{H}$ (or $\breve{\mathcal{H}}$), see Lemma A.5.

Our main result in this section is the following uniqueness theorem.

Theorem 2.3. *Suppose T and $\breve{T}$ are two self-adjoint operators, respectively associated with the coefficients (q, w, α) and $(\breve{q}, \breve{w}, \breve{\alpha})$, and have the same m-function. Then $\alpha = \breve{\alpha}$ and there is a unitary Liouville transform*

$\mathcal{L}_{r,s} \in \mathcal{S}(0,b;0,\breve{b})$ such that $T\mathcal{L}_{r,s} = \mathcal{L}_{r,s}\breve{T}$, $\breve{q} \circ s = r^3(-r'' + qr)$, and $\breve{w} \circ s = r^4 w$. Moreover $r^2 s' \equiv 1$, $r'(0) = 0$, $r(0) = 1$.

The bulk of the information contained in the m-function is, of course, contained in the spectral measure but not all. Knowing only the spectral measure makes for a somewhat weaker theorem which we also state and prove. In fact, the proof of Theorem 2.3 is very short once one has Theorem 2.4 in hand.

Theorem 2.4. *Suppose T and $\breve{T}$ are two self-adjoint operators, respectively associated with the coefficients (q,w,α) and $(\breve{q},\breve{w},\breve{\alpha})$, and have the same spectral measure $d\rho$. Then $\sin(2\alpha) = \sin(2\breve{\alpha})$ and there is a unitary Liouville transform $\mathcal{L}_{r,s}$ such that $T\mathcal{L}_{r,s} = \mathcal{L}_{r,s}\breve{T}$, $rs'^2\breve{q} \circ s = -r'' + qr$, and $s'^2\breve{w} \circ s = w$. Moreover $\mathcal{L}_{r,s}$ has the following properties:*

- Case 1 *If neither $\operatorname{supp}\breve{w}$ nor $\operatorname{supp} w$ has exceptional gaps, then $\mathcal{L}_{r,s} \in \mathcal{S}(0,b;0,\breve{b})$, $r^2 s' \equiv 1$, $r'(0) = 0$, and $r(0) = \sin\alpha/\sin\breve{\alpha}$ interpreting the latter quotient as 1 if $\alpha = \breve{\alpha} = 0$.*
- Case 2 *If $\operatorname{supp}\breve{w}$ has an exceptional gap but $\operatorname{supp} w$ does not, then $\alpha = \breve{\alpha} = \pi/2$, $\mathcal{L}_{r,s} \in \mathcal{S}(0,b;\breve{0}_+,\breve{b})$, $r^2 s' \equiv 1$, $r'(0) = \breve{\varphi}_0'(\breve{0}_+)$, and $r(0) = -1/\breve{\varphi}_0(\breve{0}_+)$.*
- Case 3 *If both $\operatorname{supp}\breve{w}$ and $\operatorname{supp} w$ have exceptional gaps, then $\alpha = \breve{\alpha} = \pi/2$, $r > 0$, r, s are locally absolutely continuous, r', s' are locally absolutely continuous on both $[0,0_+)$, and $(0_+,b)$, $r'(0) = 0$, $r(0) = 1$, $r^2 s' \equiv 1$ on $(0_+,b)$, and $r^2 s' \equiv \gamma_0$ on $[0,0_+)$ where*

$$(\gamma_0 - 1)\breve{\varphi}_0'(\breve{0}_+) = r(0_+)\breve{\varphi}_0(\breve{0}_+) \lim_{\varepsilon\downarrow 0}\big(r'(0_+ + \varepsilon) - r'(0_+ - \varepsilon)\big).$$

Theorem 2.5. *The converses of Theorems 2.3 and 2.4 are also true.*

Proof. The assumptions on the functions r and s and the coefficients of T and $\breve{T}$ in Theorem 2.4 are such that $\mathcal{L}_{r,s}$ maps solutions of $-\breve{u}'' + \breve{q}\breve{u} = \lambda\breve{w}\breve{u}$ to functions solving the equation $-u'' + qu = \lambda wu$ on both $[0,0_+)$ and $(0_+,b)$. In fact, $\mathcal{L}_{r,s}\breve{\varphi} = \varphi$ on all of $[0,b)$ (but, in Case 3, the images of other solutions may have a kink at 0_+). Moreover $\mathcal{L}_{r,s}$ is a unitary operator from $\breve{\mathcal{H}}$ to $\mathcal{H}$. It follows that $\mathcal{L}_{r,s}\breve{\omega}$ is a constant multiple of ω. In Case 1 one computes

$$(\mathcal{L}_{r,s}\breve{\psi})(x,\lambda) = \theta(x,\lambda) + \big(\cot\alpha - \cot\breve{\alpha} + \breve{m}(\lambda)\big)\varphi(x,\lambda)$$

which shows that $\mathcal{L}_{r,s}\breve{\psi} = \psi$ so that $m = \cot\alpha - \cot\breve{\alpha} + \breve{m}$ and hence that $d\rho = d\breve{\rho}$. Similarly, in Case 2,

$$(\mathcal{L}_{r,s}\breve{\psi})(x,\lambda) = \theta(x,\lambda) + \left(\frac{\lambda}{\breve{\phi}_-'(0)} + \breve{m}(\lambda)\right)\varphi(x,\lambda)$$

where $\breve{\phi}_- \in \breve{\mathcal{H}}_\infty$ is supported on $[0,\breve{0}_+]$ where it solves the boundary value problem $-\breve{u}'' + \breve{q}\breve{u} = 0$, $\breve{\phi}_-(\breve{0}_+) = 0$, and $\breve{\phi}_-(0) = 1$. Now we have $m(\lambda) = \lambda/\breve{\phi}_-'(0) + \breve{m}(\lambda)$ and again $d\rho = d\breve{\rho}$. Case 3 is a little more complicated since the derivative of $\mathcal{L}_{r,s}\breve{\psi}(\cdot,\lambda)$ is not continuous at 0_+. Nevertheless we find that $\mathcal{L}_{r,s}\breve{\theta} - \theta$ is a multiple of φ on $[0_+,b)$ which shows again that $\mathcal{L}_{r,s}\breve{\omega} = \omega$. Therefore, as long as $x \in [0,0_+]$,

$$\left(\frac{\lambda}{\phi_-'(0)} + m(\lambda)\right)\varphi(x,\lambda) = \omega(x,\lambda) = (\mathcal{L}_{r,s}\breve{\omega})(x,\lambda) = \left(\frac{\lambda}{\breve{\phi}_-'(0)} + \breve{m}(\lambda)\right)\varphi(x,\lambda)$$

defining ϕ_- analogously to $\breve{\phi}_-$. Thus once more $d\rho = d\breve{\rho}$.

If the stronger hypotheses of Theorem 2.3 are satisfied we even have $m = \breve{m}$. □

The results of Section 2.2 show the validity of Theorems 2.3 and 2.4 when $d\rho$ has finite support. We may therefore assume in the following that this is not the case and hence that neither w nor $\breve{w}$ vanishes identically. Since the Fourier transforms $\mathcal{F}$ and $\check{\mathcal{F}}$ are unitary and have, by assumption, the same target space, we have that $\mathcal{U} = \mathcal{F}^* \circ \check{\mathcal{F}}$ is a unitary map between $\mathcal{H}$ and $\check{\mathcal{H}}$. We will show that $\mathcal{U}$ is a Liouville transform, i.e., that there are functions r and s so that $\mathcal{U} = \mathcal{L}_{r,s}$. In the absence of gaps the function s may be defined as $\check{h}^{-1} \circ h$ where

$$h(x) = \int_0^x \sqrt{|w|} \quad \text{and} \quad \check{h}(x) = \int_0^x \sqrt{|\check{w}|}. \tag{2.7}$$

Our strategy is to show that this is still the right approach away from the gaps. For this to work we need that $h(b) = \check{h}(\check{b})$ (Lemma 2.8) and that the gaps in $\operatorname{supp} w$ are in one-to-one correspondence with those of $\operatorname{supp} \check{w}$ (Lemma 2.9). The key ingredient for the proof of these claims is the control of the support of $\mathcal{U}\check{u}$ when the support of $\check{u}$ is known (Lemma 2.7). This, in turn, is possible because of the Paley–Wiener Theorem 2.2. We will patch the definitions of r and s in the gaps in Lemma 2.10 and then prove that these functions have the necessary properties.

Recall that we defined $\omega(\cdot, \lambda)$ to be the projection of $\psi(\cdot, \lambda)$ onto $\mathcal{H}$ (these differ only when there is an exceptional gap). We also define ω_0 as the projection of ψ_0 onto $\mathcal{H}$ and, analogously, $\check{\omega}(\cdot, \lambda)$ and $\check{\omega}_0$ as elements of $\check{\mathcal{H}}$. By Lemma A.9 we have

$$\mathcal{U}\check{\psi}_0 = \mathcal{U}\check{\omega}_0 = \sigma\omega_0 \quad \text{where } \sigma = \begin{cases} \sin\check{\alpha}/\sin\alpha & \text{if } \alpha \neq 0 \neq \check{\alpha} \\ 1 & \text{if } \alpha = 0 = \check{\alpha} \end{cases} \tag{2.8}$$

(α and $\check{\alpha}$ can vanish only simultaneously since $\alpha = 0$ implies that 0 is an eigenvalue of T and $\check{T}$ which implies that $\check{\alpha} = 0$). If $\check{u} \in \check{\mathcal{H}}$ it follows from (2.8) and integration by parts that

$$\check{u}(0) = -\langle \check{u}, \check{\psi}_0 \rangle = -\langle \mathcal{U}\check{u}, \mathcal{U}\check{\psi}_0 \rangle = -\sigma\langle \mathcal{U}\check{u}, \omega_0 \rangle = -\sigma\langle \mathcal{U}\check{u}, \psi_0 \rangle = \sigma(\mathcal{U}\check{u})(0). \tag{2.9}$$

Lemma 2.6. *Suppose $u \in \mathcal{H}_1$ is supported in $[0, a]$ and $\varepsilon > 0$. Then there exists $u_\varepsilon \in \mathcal{H}_1$ which is compactly supported in $[0, a)$ and satisfies $\|u - u_\varepsilon\| < \varepsilon$.*

Proof. Suppose $\delta > 0$ (but smaller than $a/2$). We choose $\tilde{u}$ to be zero in $[a - \delta, a]$, linear in $[a - 2\delta, a - \delta]$, and equal to u elsewhere. Then

$$\|u - \tilde{u}\|^2 \leq 2 \int_{a-2\delta}^{a} \left(\left|u'\right|^2 + q|u|^2\right) + 2\frac{|u(a - 2\delta)|^2}{\delta^2} \int_{a-2\delta}^{a-\delta} \left(1 + q\delta^2\right).$$

Here, both terms on the right vanish as δ tends to zero; for the second observe that the fundamental theorem of calculus and Schwarz's inequality imply that $|u(a - 2\delta)|^2$ tends to zero faster than δ. Thus we choose $u_\varepsilon = \tilde{u}$ for a sufficiently small δ. $\square$

Lemma 2.7. *Suppose $h(a) = \check{h}(\check{a})$. Let $\check{u} \in \check{\mathcal{H}}_1$ and set $u = \mathcal{U}\check{u}$. Then the following statements hold:*

1. *If the support of $\check{u}$ is in $[0, \check{a}]$, then the support of u is in $[0, a_+]$ and even in $[0, a_-]$, if $\check{a} = \check{a}_-$. In particular, $u = 0$ if $a_+ = 0$.*
2. *If $\check{u}$ is supported in $[\check{a}, \check{b})$, then u is supported in $[a_-, b)$ and even in $[a_+, b)$, if $\check{a} = \check{a}_+$.*

Analogous statements hold with the roles of u and $\check{u}$ reversed.

Proof. The first claim in (1) follows immediately from the Paley–Wiener theorem, applying first the easy direction to $\breve{u}$ and then the hard direction to $\breve{\mathcal{F}}\breve{u} = \mathcal{F}u$ for $\lambda = \pm i$ using the fact that

$$\int_0^{\breve{a}} \operatorname{Re}\sqrt{\mp i\breve{w}} = \int_0^{\breve{a}} \sqrt{|\breve{w}|/2} = \int_0^{a} \sqrt{|w|/2} = \int_0^{a} \operatorname{Re}\sqrt{\mp iw}$$

in the process.

To prove the second claim in (1) we employ Lemma 2.6 to find, for any $\varepsilon > 0$, a function $\breve{u}_\varepsilon \in \breve{\mathcal{H}}_1$ such that $\operatorname{supp}\breve{u}_\varepsilon \subset [0, \breve{a}_-)$ and $\|\breve{u} - \breve{u}_\varepsilon\| < \varepsilon$. Thus $\mathcal{U}\breve{u}_\varepsilon$ is supported in $[0, a_-]$. Since point evaluations are continuous it follows that $u = \mathcal{U}\breve{u}$ vanishes at a_- and therefore throughout $[a_-, a_+]$, so $\operatorname{supp} u \subset [0, a_-]$. This completes the proof of (1).

To prove the first part of statement (2) we may assume $a_- > 0$ (otherwise there is nothing to do). Note that statement (1) implies that $\operatorname{supp}\mathcal{U}^* v \subset [0, \breve{a}_-]$ whenever $v \in \mathcal{H}_1$ is supported in $[0, a_-]$. Hence $0 = \langle \breve{u}, \mathcal{U}^* v\rangle = \langle u, v\rangle$ and this holds still true, when we replace $\breve{u}$ by its projection onto $\breve{\mathcal{H}}$ allowing us in the following to assume that $\breve{u} \in \breve{\mathcal{H}}$. We first prove that $u'' = qu$ in $[0, a_-]$ in the spirit of the du Bois-Reymond lemma: Set $Q(x) = \int_0^x qu$, $C = \int_0^{a_-} (u' - Q)/a_-$, and $v(x) = \int_0^x (u' - Q - C)\chi_{[0,a_-]}$ so that $v(x) = 0$ for $x \geq a_-$ and $v \in \mathcal{H}_1$. An integration by parts gives $0 = \langle u, v\rangle = \int_0^{a_-} |u' - Q - C|^2$ from which our claim follows immediately. Next, choosing $v = (a_- - x)\chi_{[0,a_-]}(x)$ and integration by parts gives $0 = \langle u, v\rangle = -u'(0)a_-$ so that $u'(0) = 0$. Eq. (2.9) shows that $u(0) = \sigma^{-1}\breve{u}(0) = 0$. Thus u satisfies the initial value problem $-u'' + qu = 0$, $u(0) = u'(0) = 0$ on $[0, a_-]$ and hence vanishes there. The remaining claim follows simply by replacing a_- by a_+ and $\breve{a}_-$ by $\breve{a}_+$ in the previous argument. □

Lemma 2.8. *The functions h and $\breve{h}$, defined in (2.7), tend to the same limit, possibly infinity, as their arguments tend to b and $\breve{b}$, respectively.*

Proof. We prove our claim by contradiction. Without loss of generality assume $\breve{h}(\breve{b}) = \lim_{x \to \breve{b}} \breve{h}(x)$ is finite and that there exists an interval $(x, y) \subset [0, b)$ such that $h(y) > h(x) > \breve{h}(\breve{b})$. Then w has essential support in $[x, y]$. Now, if $v \in \mathcal{H}_1$ is supported in $[x, b)$ and $\breve{u} \in \breve{\mathcal{H}}_1$ is a finite function, the Paley–Wiener theorem shows that the support of $\mathcal{U}\breve{u}$ is contained in $[0, x]$ so that $\langle \mathcal{U}\breve{u}, v\rangle = 0$. Since the set $\{\mathcal{U}\breve{u} : \breve{u} \in \breve{\mathcal{H}}_1$ is finite$\}$ is dense in $\mathcal{H}$, we get $v \in \mathcal{H}_\infty$ and hence $wv = 0$ almost everywhere, which is impossible, since v may be chosen non-zero in (x, y). □

The key to the construction of the Liouville transform is that gaps of $\operatorname{supp} w$ and $\operatorname{supp}\breve{w}$ are in a one-to-one correspondence. However, if $\alpha = \breve{\alpha} = \pi/2$ it may be the case that 0 is in a gap of $\operatorname{supp}\breve{w}$ but not in a gap of $\operatorname{supp} w$ or vice versa. For example, suppose that $0 \in \operatorname{supp} w$ and that $\breve{q} = \breve{w} = 0$ on $[0, 1]$ while to the right of 1 we have $\breve{q}(x) = q(x - 1)$, and $\breve{w}(x) = w(x - 1)$. In this case we have $\breve{m} = m + \lambda$ so that indeed $d\rho = d\breve{\rho}$.

Lemma 2.9. *The regular gaps in* $\operatorname{supp} w$ *and* $\operatorname{supp}\breve{w}$ *are in a one-to-one correspondence respecting the ordering of the gaps. Specifically, if $\breve{a}$ is in a regular gap of* $\operatorname{supp}\breve{w}$ *and $h(a) = \breve{h}(\breve{a})$, then a is in a regular gap of* $\operatorname{supp} w$ *and vice versa.*

Proof. By way of contradiction assume $\breve{a}_- < \breve{a}_+$ but $a = a_- = a_+$. If $[\breve{a}_-, \breve{a}_+)$ does not include 0 and does not extend to $\breve{b}$, the restrictions of elements of $\breve{\mathcal{H}}$ to $[\breve{a}_-, \breve{a}_+]$ span a two-dimensional space. This is still the case when $[\breve{a}_-, \breve{a}_+)$ contains 0 but $\breve{\alpha} \neq \pi/2$. In either of these situations let $\breve{\phi}_+ \in \breve{\mathcal{H}}$ be a function which vanishes in $[0, \breve{a}_-]$ and assumes the value 1 in $\breve{a}_+$ while $\breve{\phi}_- \in \breve{\mathcal{H}}$ is a function which vanishes in $[\breve{a}_+, \breve{b})$ and assumes the value 1 in $\breve{a}_-$. Lemma 2.7 shows then that $(\mathcal{U}\breve{\phi}_+)(a) = (\mathcal{U}\breve{\phi}_-)(a) = 0$ and this implies that $(\mathcal{U}\breve{u})(a) = 0$ for all $\breve{u} \in \breve{\mathcal{H}}_1$, which is impossible.

Now suppose that $(\breve{a}_-, \breve{b})$ is a gap of $\operatorname{supp}\breve{w}$ but that no gap of $\operatorname{supp} w$ extends to b so that $a_+ < b$ for any $a \in [0, b)$. Let $u = \mathcal{U}\breve{\psi}_0$. According to Lemma A.3 we may approximate u by functions $u_n \in \mathcal{H}_1$ supported in $[0, a_n]$. Since $(a_n)_+ < b$ we get $h(a_n) < h(b) = \breve{h}(\breve{a}_-)$. This shows, with the help of Lemma 2.7, that each of the $\mathcal{U}^* u_n$ has its support in $[0, \breve{a}_-]$. By Theorem 2.1 the function $\breve{\psi}_0$ coincides with its projection onto $\breve{\mathcal{H}}$ beyond $\breve{0}_+$. Thus the continuity of point evaluations (Lemma A.1) gives, for an appropriate constant C,

$$0 \neq \left|\breve{\psi}_0(\breve{a})\right| = \left|\left(\mathcal{U}^*(u_n - u)\right)(\breve{a})\right| \leq C\left\|\mathcal{U}^*(u_n - u)\right\| \leq C\|u_n - u\|$$

contradicting the fact that u_n approximates u. □

Lemma 2.10. *Suppose (a_-, a_+) and $(\breve{a}_-, \breve{a}_+)$ are corresponding gaps of the supports of w and $\breve{w}$. Then there is a Liouville transform $\mathcal{L}_{r,s}$ in $\mathcal{S}(a_-, a_+; \breve{a}_-, \breve{a}_+)$ such that $\mathcal{U}\breve{u} = \mathcal{L}_{r,s}\breve{u} = r\breve{u} \circ s$ on $[a_-, a_+]$ for all $\breve{u} \in \breve{\mathcal{H}}$. Moreover, if the gaps are regular then we have $r^2 s' = 1$ while $r^2 s' = \phi_-'(0)/\breve{\phi}_-'(0)$ if they are exceptional.*

Proof. We begin by considering gaps which are regular and bounded. Recall the functions $\breve{\phi}_\pm \in \breve{\mathcal{H}}$ associated with the gap $(\breve{a}_-, \breve{a}_+)$ from the proof of Lemma 2.9 and let $\phi_\pm \in \mathcal{H}$ be their analogues for the gap (a_-, a_+). By Lemma 2.7 the function $\mathcal{U}\breve{\phi}_-$ vanishes on $[a_+, b)$ and therefore coincides with a multiple of ϕ_- on $[a_-, a_+]$. Lemma 2.7 gives also that $\mathcal{U}\breve{\phi}_+(a_-) = 0$ if $a_- > 0$. If $a_- = 0$ (and hence $\breve{a}_- = 0$) we arrive at this conclusion, too, upon applying Eq. (2.9). In any case we get that the restrictions of $\mathcal{U}\breve{\phi}_+$ and ϕ_+ to $[a_-, a_+]$ are multiples of each other. Specifically, there are numbers $r_\pm$ such that $r_\pm\phi_\pm = \mathcal{U}\breve{\phi}_\pm$ on $[a_-, a_+]$. Since $\breve{\psi}_0 - \breve{\psi}_0(\breve{a}_-)\breve{\phi}_- - \breve{\psi}_0(\breve{a}_+)\breve{\phi}_+$ vanishes at $\breve{a}_\pm$, it vanishes in $(\breve{a}_-, \breve{a}_+)$ and is therefore equal to a sum of two functions $\breve{u}_\pm \in \breve{\mathcal{H}}$ supported in $[0, \breve{a}_-]$ and $[\breve{a}_+, \breve{b})$, respectively. Using Lemma 2.7 this shows that, on $[a_-, a_+]$,

$$\sigma\psi_0 = \mathcal{U}\breve{\psi}_0 = \breve{\psi}_0(\breve{a}_-)r_-\phi_- + \breve{\psi}_0(\breve{a}_+)r_+\phi_+.$$

Evaluating this at $a_\pm$ shows that $r_\pm$ are positive. For these values of $r_\pm$ let $\mathcal{L}_{r,s}$ be the Liouville transform for closed intervals described in Section 2.1. Since, by Theorem 2.1, $\breve{\phi}_\pm$ span the space of restrictions of $\breve{u} \in \breve{\mathcal{H}}$ to $[\breve{a}_-, \breve{a}_+]$, we get $\mathcal{L}_{r,s}\breve{u} = \mathcal{U}\breve{u}$ on $[a_-, a_+]$. In Eq. (2.1) we computed $r^2 s' = r_+ r_- \phi_+'(a_-)/\breve{\phi}_+'(\breve{a}_-)$. But this equals 1 since $\mathcal{U}$ is unitary, implying

$$-\breve{\phi}_+'(\breve{a}_-) = \langle\breve{\phi}_+, \breve{\phi}_-\rangle = r_+ r_- \langle\phi_+, \phi_-\rangle = -r_+ r_- \phi_+'(a_-).$$

Now we consider the case when there are gaps extending to b and $\breve{b}$ denoting their left endpoints by b_- and $\breve{b}_-$. Restrictions of functions in $\mathcal{H}$ or $\breve{\mathcal{H}}$ to the gaps are multiples of ψ_0 and $\breve{\psi}_0$, respectively. Again Lemma 2.7 shows that, on $[b_-, b)$, we have $\mathcal{U}\breve{\psi}_0 = \sigma\psi_0$. We have shown in Section 2.1 how to construct a Liouville transformation $\mathcal{L}_{r,s} \in \mathcal{S}(b_-, b; \breve{b}_-, \breve{b})$ with the same effect. We note that here, too, we have $r^2 s' = 1$.

It remains to deal with the case where both $\operatorname{supp} w$ and $\operatorname{supp}\breve{w}$ have exceptional gaps. Note that $\breve{\omega}_0 = \breve{\omega}_0(0)(\breve{\phi}_- - \breve{\phi}_-'(0)\breve{\phi}_+/\breve{\phi}_+'(0))$. Choosing $r_- = 1$ and $r_+ = \phi_-'(0)\breve{\phi}_+'(0)/(\breve{\phi}_-'(0)\phi_+'(0))$ the Liouville transform for closed intervals described in Section 2.1 will (using proper restrictions) map $\breve{\omega}_0$ to $\omega_0 = \mathcal{U}\breve{\omega}_0$ since, by Eq. (2.9), $\omega_0(0) = \breve{\omega}_0(0)$. Moreover, Eq. (2.1) gives $r^2 s' = \phi_-'(0)/\breve{\phi}_-'(0)$. □

Lemma 2.11. *Suppose $a = a_- = a_+$ and let $s(a) = \breve{h}^{-1}(h(a))$. If two functions $\breve{z}$, $\breve{y}$ are in $\breve{\mathcal{H}}$ and have value 1 at a point $s(a)$ then $(\mathcal{U}\breve{z})(a) = (\mathcal{U}\breve{y})(a)$. This number, which we denote by $r(a)$, is positive and we have*

$$(\mathcal{U}\breve{u})(a) = r(a)\breve{u}\big(s(a)\big) \tag{2.10}$$

for any function $\breve{u} \in \breve{\mathcal{H}}$.

Proof. We may write $\breve{z} - \breve{y} = \breve{v}_- + \breve{v}_+$ where $\breve{v}_-$ is supported in $[0, s(a)]$ and $\breve{v}_+$ is supported in $[s(a), \breve{b})$. It follows from Lemma 2.7 or, if $s(a) = 0$, by Eq. (2.9) that $\mathcal{U}(\breve{z} - \breve{y})$ vanishes at a. Similarly, for any $\breve{u} \in \breve{\mathcal{H}}$ we write $\breve{u} = \breve{u}_- + \breve{u}(s(a))\breve{z} + \breve{u}_+$ where $\breve{u}_\mp$ are supported in $[0, s(a)]$ and $[s(a), \breve{b})$, respectively. Eq. (2.10) follows again from Lemma 2.7. Choosing $\breve{z} = \breve{\psi}_0/\breve{\psi}_0(s(a))$ shows that $r(a) = \sigma\psi_0(a)/\breve{\psi}_0(s(a)) > 0$. □

Proof of Theorem 2.4. The first facts to observe are that $\mathcal{U}\breve{u} = r\breve{u} \circ s$ for any $\breve{u} \in \breve{\mathcal{H}}$, s is a strictly increasing function on $[0, b)$ onto $[0, \breve{b})$ and hence continuous, and $r > 0$ and continuous on $[0, b)$.

If $\operatorname{supp} w$ has a gap containing zero, we denote its right endpoint by 0_+; if a gap extends to b we denote its left endpoint by b_-. The numbers $\breve{0}_+$ and $\breve{b}_-$ have the analogous meaning for gaps in $\operatorname{supp} \breve{w}$. We know that $b_- < b$ if and only if $\breve{b}_- < \breve{b}$. Also $0_+ > 0$ if and only if $\breve{0}_+ > 0$ except in the case when α or $\breve{\alpha}$ is equal to $\pi/2$ in which case 0 may be in the support of one of the weight functions but not the other. Should this be the case we assume without loss of generality that it is $\operatorname{supp} w$ which does not have a gap including zero. We will determine the properties of r and s first on the intervals $[0, 0_+)$, $(0_+, b_-)$, and (b_-, b) and postpone the discussion of their behavior at 0_+ and b_-. Of course, in a given case, the first or the last or both of these intervals may be absent. The middle interval is always present, since we have already dealt with the case where $\operatorname{supp} w = \emptyset$ in Section 2.2.

We will assume, until further notice, that the gaps $[0, 0_+)$ and $[0, \breve{0}_+)$ are not exceptional if present. We have from Lemma 2.10 that s, r, and r' are locally absolutely continuous and that $r^2 s' = 1$ on (b_-, b) and on $[0, 0_+)$.

We now consider the open interval $(0_+, b_-)$. Fix $x_0 \in (0_+, b_-)$ and let $\breve{\vartheta}$ and $\breve{\phi}$ be elements of $\breve{\mathcal{H}}$ which satisfy the equation $\breve{u}'' = \breve{q}\breve{u}$ and the initial conditions

$$\breve{\vartheta}\big(s(x_0)\big) = \breve{\phi}'\big(s(x_0)\big) = 1 \quad \text{and} \quad \breve{\vartheta}'\big(s(x_0)\big) = \breve{\phi}\big(s(x_0)\big) = 0$$

in a (relatively) open interval containing $s(x_0)$ which we may choose small enough for $\breve{\vartheta}$ not to vanish there. The function $\breve{\phi}/\breve{\vartheta}$ has a positive derivative in this interval and is therefore strictly increasing and absolutely continuous there. Define $\vartheta = \mathcal{U}\breve{\vartheta}$ and $\phi = \mathcal{U}\breve{\phi}$. Both of these are in $\mathcal{H}$ and ϑ does not vanish at x_0. Therefore $\phi/\vartheta = (\breve{\phi}/\breve{\vartheta}) \circ s$ is strictly increasing and absolutely continuous in some neighborhood of x_0. It follows that $s = (\breve{\varphi}/\breve{\vartheta})^{-1} \circ (\varphi/\vartheta)$ is locally absolutely continuous. Also $r = \vartheta/(\breve{\vartheta} \circ s)$ is locally absolutely continuous.

For $x \in (0_+, b_-)$ we get from Theorem 2.1 and Lemma A.9 that

$$\psi(x, \lambda) = \big(\mathcal{U}\breve{\psi}(\cdot, \lambda)\big)(x) = r(x)\breve{\psi}\big(s(x), \lambda\big)$$

and, upon differentiation

$$\psi'(x, \lambda) = r'(x)\breve{\psi}\big(s(x), \lambda\big) + \big(rs'\big)(x)\breve{\psi}'\big(s(x), \lambda\big). \tag{2.11}$$

The function $\lambda \mapsto \breve{m}_0(x, \lambda) = \breve{\psi}'(s(x), \lambda)/(\lambda\breve{\psi}(s(x), \lambda))$ is the Weyl–Titchmarsh m-function for $-\breve{u}'' + \breve{q}\breve{u} = \lambda\breve{w}\breve{u}$ posed on $[s(x), \breve{b})$ with $\breve{\alpha} = 0$. Writing Eq. (2.11) for $\lambda = i$ and a general λ gives rise to the linear system

$$\big(r', rs'\big)\begin{pmatrix} 1 & 1 \\ \lambda\breve{m}_0(s(\cdot), \lambda) & i\breve{m}_0(s(\cdot), i) \end{pmatrix} = \left(\frac{\psi'(\cdot, \lambda)}{\breve{\psi}(s(x), \lambda)}, \frac{\psi'(\cdot, i)}{\breve{\psi}(s(x), i)}\right).$$

The determinant of the matrix occurring here cannot be identically equal to zero, since this would mean that m_0 is analytic in the plane save for a pole at zero and hence that the corresponding spectral measure would be supported only in zero. However, this is precluded by the fact that $\breve{w}$ does not vanish on $[s(x), \breve{b})$. It follows that r' and rs' are locally absolutely continuous on $(0_+, b_-)$.

Taking another derivative in (2.11) and making use of the differential equations gives

$$\big(r'' - rq + rs'^2\breve{q} \circ s\big) + \big(rw - rs'^2\breve{w} \circ s\big)\lambda + \big(2r's' + rs''\big)\lambda\breve{m}_0\big(s(\cdot), \lambda\big) = 0.$$

We now use three instances of this equation, say for $\lambda = i$, $\lambda = 2i$, and a general λ. Again the vanishing of the resulting 3×3 determinant would mean that $\breve{m}_0$ is analytic except for a pole at zero and since this is not the case we get to conclude that

$$2r's' + rs'' = \big(r^2 s'\big)'/r = 0,$$
$$-r'' + rq = rs'^2\breve{q} \circ s,$$

and

$$w = s'^2\breve{w} \circ s.$$

The first of these equations shows that $r^2 s'$ is equal to a constant γ on $(0_+, b_-)$. It follows that $\mathcal{L}_{r,s} \in \mathcal{S}(0_+, b_-; \breve{0}_+, \breve{b}_-)$.

In order to understand the behavior of r and s near 0_+ and b_- we consider now finite functions $\breve{u}$ in $\breve{\mathcal{H}}$ and set $u = \mathcal{U}\breve{u} = r\breve{u} \circ s$. Then

$$\big|u'\big|^2 + q|u|^2 = r^2 s'^2\big|\breve{u}' \circ s\big|^2 + rr'\big(|\breve{u} \circ s|^2\big)' + \big(r'^2 + qr^2\big)|\breve{u} \circ s|^2.$$

We need to integrate this expression over the three intervals $(0, 0_+)$, $(0_+, b_-)$, and (b_-, b). In each case we integrate the middle term by parts. This gives

$$\begin{aligned}\|u\|^2 = &\int_0^{0_+} \Big(r^2 s'^2\big|\breve{u}' \circ s\big|^2 + \big(-rr'' + qr^2\big)|\breve{u} \circ s|^2\Big) + \big(rr'|\breve{u} \circ s|^2\big)\Big|_0^{0_+} \\ &+ \int_{0_+}^{b_-} \Big(r^2 s'^2\big|\breve{u}' \circ s\big|^2 + \big(-rr'' + qr^2\big)|\breve{u} \circ s|^2\Big) + \big(rr'|\breve{u} \circ s|^2\big)\Big|_{0_+}^{b_-} \\ &+ \int_{b_-}^{b} \Big(r^2 s'^2\big|\breve{u}' \circ s\big|^2 + \big(-rr'' + qr^2\big)|\breve{u} \circ s|^2\Big) + \big(rr'|\breve{u} \circ s|^2\big)\Big|_{b_-}^{b}.\end{aligned}$$

Since $-rr'' + qr^2 = r^2 s'^2\breve{q} \circ s$ and using the fact that $r^2 s'$ is constant on each of the three intervals, we obtain after changing variables

$$\begin{aligned}\|u\|^2 = &\gamma_0 \int_0^{\breve{0}_+} \big(\big|\breve{u}'\big|^2 + \breve{q}|\breve{u}|^2\big) + \gamma \int_{\breve{0}_+}^{\breve{b}_-} \big(\big|\breve{u}'\big|^2 + \breve{q}|\breve{u}|^2\big) + \int_{\breve{b}_-}^{\breve{b}} \big(\big|\breve{u}'\big|^2 + \breve{q}|\breve{u}|^2\big) \\ &+ \big(rr'|\breve{u} \circ s|^2\big)\Big|_0^{0_+} + \big(rr'|\breve{u} \circ s|^2\big)\Big|_{0_+}^{b_-} + \big(rr'|\breve{u} \circ s|^2\big)\Big|_{b_-}^{b}. \qquad (2.12)\end{aligned}$$

We note first that $(rr')(x)|\breve{u}(s(x))|^2 = (u\overline{u'})(x) - r^2 s'(\breve{u}\overline{\breve{u}'})(s(x))$ tends to zero as x tends to b according to Lemma A.3. Next, picking a nontrivial $\breve{u} \in \breve{\mathcal{H}}$ supported in $(\breve{0}_+, \breve{b}_-)$ shows that $\gamma = 1$ since $\|u\| = \|\breve{u}\|$. Picking $\breve{u} \in \breve{\mathcal{H}}$ supported in $(\breve{0}_+, \breve{b})$ but different from zero at $\breve{b}_-$, shows that r' must be continuous at $\breve{b}_-$. Since the gap $[0, \breve{0}_+)$ is regular (so that we have $\gamma_0 = 1$) we may choose $\breve{u}$ to have support only in $[0, \breve{0}_+]$

(i.e., as a multiple of $\breve{\phi}_-$ for that gap) and this shows $r'(0) = 0$, a fact which follows similarly when 0 is not contained in a gap. Finally, choosing $\breve{u}$ as a function which does not vanish at $\breve{0}_+$ shows that r' must be continuous at $\breve{0}_+$.

That $r(0) = 1/\sigma$ follows from (2.8) and (2.9) for $\breve{u} = \breve{\psi}_0$. Using the initial conditions satisfied by $\psi(\cdot, \lambda)$ and $\breve{\psi}(\cdot, \lambda)$ in the equation $\psi(\cdot, \lambda) = \mathcal{U}\breve{\psi}(\cdot, \lambda) = r\breve{\psi}(s(\cdot), \lambda)$ and its derivative gives

$$\cos\alpha - m(\lambda)\sin\alpha = r(0)\big(\cos\breve{\alpha} - \breve{m}(\lambda)\sin\breve{\alpha}\big) \quad \text{and}$$
$$\sin\alpha + m(\lambda)\cos\alpha = r(0)^{-1}\big(\sin\breve{\alpha} + \breve{m}(\lambda)\cos\breve{\alpha}\big).$$

If it were possible to solve this system simultaneously for m and $\breve{m}$ these would be constant. Since they are not we must have that the associated determinant vanishes giving $\sin\alpha\cos\alpha = \sin\breve{\alpha}\cos\breve{\alpha}$. Moreover,

$$m(\lambda) - \cot\alpha = \breve{m}(\lambda) - \cot\breve{\alpha}. \tag{2.13}$$

This proves Theorem 2.4 in Case 1.

We now turn to the case of exceptional gaps. First suppose that $[0, \breve{0}_+)$ is exceptional ($\breve{\alpha} = \pi/2$) but that $0 \in \operatorname{supp} w$. The missing gap for $\operatorname{supp} w$ entails that our construction of s gives a map from $[0, b)$ to $[s(0), \breve{b}) = [\breve{0}_+, \breve{b})$ but after this modification we still have $(\mathcal{U}\breve{u})(x) = r(x)\breve{u}(s(x))$. The condition $\|\mathcal{U}\breve{u}\| = \|\breve{u}\|$ gives $(rr')(0) = -\breve{\varphi}_0'(\breve{0}_+)/\breve{\varphi}_0(\breve{0}_+)$ using that $\breve{u}$ is a multiple of $\breve{\varphi}_0$ on $[0, \breve{0}_+]$. Next (2.8) and (2.9) give $r(0)\sigma\breve{\varphi}_0(\breve{0}_+) = -1$ and we note that $\sigma = 1/\sin\alpha$ in this case. Finally, we evaluate $\psi(\cdot, \lambda) = \mathcal{U}\breve{\psi}(\cdot, \lambda) = r\breve{\psi}(s(\cdot), \lambda)$ and its derivative at zero. Expressing $\breve{\psi}(\cdot, \lambda)$ in terms of $\breve{\varphi}_0$ and $\breve{\phi}_-$ on the gap we obtain from $\psi(0, \lambda) = r(0)\breve{\psi}(s(0), \lambda)$ that

$$m(\lambda) - \cot\alpha = \breve{m}(\lambda) + \lambda/\breve{\phi}_-'(0). \tag{2.14}$$

Additionally, $r(0)\psi'(0, \lambda) = (rr')(0)\breve{\psi}(s(0), \lambda) + \breve{\psi}'(s(0), \lambda)$ gives $\psi'(0, \lambda)\sin\alpha = 1$ which shows that $\alpha = \pi/2$ since m cannot be constant.

This same strategy works also in the remaining case where both $[0, 0_+)$ and $[0, \breve{0}_+)$ are exceptional gaps. We have already that $\alpha = \breve{\alpha} = \pi/2$. From Eq. (2.8) we have $\omega_0 = r\breve{\omega}_0 \circ s$. This and Eq. (2.9) give $r(0) = 1$ while taking a derivative shows $r'(0) = 0$. Using Eq. (2.12) for a function $\breve{u} \in \breve{\mathcal{H}}$ supported in $[0, \breve{b}_-)$ for which $\breve{u}(\breve{0}_+) = 1$ and using that $\breve{u}$ is a multiple of $\breve{\varphi}_0$ in $[0, \breve{0}_+)$ gives

$$0 = (\gamma_0 - 1)\overline{\big(\breve{u}'/\breve{u}\big)}(\breve{0}_+) - r(0_+)\Delta r'(0_+) = (\gamma_0 - 1)\big(\breve{\varphi}_0'/\breve{\varphi}_0\big)(\breve{0}_+) - r(0_+)\Delta r'(0_+)$$

where $\Delta r'(0_+)$ denotes the jump of r' at 0_+. Moreover, $\omega_0(0) = \breve{\omega}_0(0)$ shows that

$$m(\lambda) + \lambda/\phi_-'(0) = \breve{m}(\lambda) + \lambda/\breve{\phi}_-'(0). \qquad \square \tag{2.15}$$

Proof of Theorem 2.3. We now use our additional hypothesis, that $m = \breve{m}$ in Eqs. (2.13)–(2.15). In the absence of exceptional gaps the first of these gives $\cot\alpha = \cot\breve{\alpha}$, i.e., $\alpha = \breve{\alpha}$. If both $\operatorname{supp} w$ and $\operatorname{supp}\breve{w}$ have an exceptional gap the last of these gives $\phi_-'(0) = \breve{\phi}_-'(0)$ which in conjunction with Lemma 2.10 gives that γ_0, the value of $r^2 s'$ on the gap, equals 1. This, in turn, forces that r' is continuous at 0_+. Lastly, Eq. (2.14) shows that it is impossible for $\operatorname{supp}\breve{w}$ to have an exceptional gap if $\operatorname{supp} w$ does not and m equals $\breve{m}$. $\square$

3. The inverse resonance problem

We turn now to the inverse resonance problem for (1.1) on $[0, \infty)$. This is a problem in inverse scattering theory, so we begin with a summary of that. Specifically, we recall the results of [3] in this area.

The standard condition placed on the coefficients is that for some $q_0 \geq 0$, $w-1$ and $q-q_0$ are integrable. Writing $\lambda = k^2 + q_0$ with $\operatorname{Im} k \geq 0$, $k \neq 0$, there exists a solution, $f(\cdot, k)$, of (1.1) such that $f(x,k) \sim e^{ikx}$ as $x \to \infty$; it and its derivative are analytic in the upper half plane when x is fixed; see Lemma 3.2 below. We call f the *Jost solution*.

When $\operatorname{Im} k > 0$, $f(\cdot, k) \in \mathcal{H}_1$. If $\operatorname{Re} k \neq 0$ also, then $f(\cdot,)$ and $\psi(\cdot, \lambda)$ are proportional. That is, there is a function F such that

$$f(x,k) = F(k)\psi(x,\lambda).$$

We call F the *Jost function*, and it will be the main object we investigate.

Recalling that Wronskians of solutions of (1.1) are independent of x, we find

$$F(k) = \left[f(\cdot, k), \lambda\varphi(\cdot, \lambda)\right]. \tag{3.1}$$

This relationship allows one to extend F analytically to the positive imaginary axis and continuously to the positive and negative real half-lines.

The Jost function contains most of the spectral information. If $F(k) = 0$, then f and φ are linearly dependent, and $\lambda = k^2 + q_0$ is an eigenvalue. Also, F determines the absolutely continuous part of the spectral measure via

$$\pi\rho'(\lambda) = \frac{k\lambda}{|F(k)|^2}, \quad k > 0. \tag{3.2}$$

The only part of the measure unknown so far is the set $\{\rho(\{\lambda\}) : \lambda \text{ is an eigenvalue}\}$. According to Corollary A.7 and Parseval's formula these numbers are given by the reciprocals of the *norming constants*, $\|\varphi(\cdot, \lambda)\|^2$ if $\lambda \neq 0$ and $\|\psi_0\|^2$ if $\lambda = 0$.

Thus, given the eigenvalues, norming constants, and $|F(k)|$ on the positive half-line, the spectral measure is determined, and we can apply Theorem 2.4 to complete the uniqueness of the inverse scattering problem.

We are interested in a more specific problem. By making a more restrictive assumption on the coefficients (Assumption 3.1), the Jost function extends analytically to the entire complex plane. The zeros of F in the lower half plane are called *resonances*. It is also possible (Lemma 3.6) that $F(0) = 0$, but $\lambda = q_0$ is not an eigenvalue; we include zero as a resonance in this case. The resonances and eigenvalues become our scattering data, and Theorem 3.8 tells to what extent these determine the spectral measure.

3.1. The Jost function determines the spectral measure

We make the following assumption which will be in force throughout this and the next section.

Assumption 3.1. There exist $q_0 \geq 0$ and $x_0 > 0$ such that $\operatorname{supp}(q - q_0)$ and $\operatorname{supp}(w-1)$ are contained in $[0, x_0]$ and $q, w \in L^1(0, x_0)$.

With this assumption, the Jost solution exists for all $k \in \mathbb{C}$ and is entire when x is fixed.

Lemma 3.2. *For every $k \in \mathbb{C}$ there exists a solution $f(\cdot, k)$ of (1.1) with $\lambda = k^2 + q_0$ such that $f(\cdot, k)$ and $f'(\cdot, k)$ are entire with growth order at most one, and $f(x,k) = e^{ikx}$ and $f'(x,k) = ike^{ikx}$ for $x \geq x_0$.*

Proof. Set $g(x,k) = e^{-ikx} f(x,k)$. The differential equation for g is $g'' + 2ikg' = Qg$ where $Q(\cdot,k) = q - q_0 w + k^2(1-w)$. In order for f to satisfy the given boundary conditions, g must satisfy $g(x) = 1$ and $g'(x) = 0$ for $x \geq x_0$. These facts lead us to the integral equation

$$g(x,k) = 1 + \int_x^{x_0} \frac{e^{2ik(t-x)} - 1}{2ik} Q(t,k) g(t,k)\, dt, \quad x \leq x_0,\ k \in \mathbb{C} \tag{3.3}$$

with the exponential factor reducing to $t - x$ when $k = 0$. Using the standard iteration technique (see, e.g., Deift and Trubowitz [5]), we set $g_0 = 1$ and

$$g_n(x,k) = \int_x^{x_0} \frac{e^{2ik(t-x)} - 1}{2ik} Q(t,k) g_{n-1}(t,k)\, dt. \tag{3.4}$$

Since the interval of integration is finite, we can estimate $g_n(\cdot,k)$ for all $k \in \mathbb{C}$ (not just for $\operatorname{Im} k > 0$) by

$$\left|g_n(x,k)\right| \leq \frac{1}{n!} e^{2|k|(x_0-x)} \left[|k|^{-1} \int_x^{x_0} \left|Q(\cdot,k)\right| \right]^n, \quad k \neq 0, \tag{3.5}$$

and

$$\left|g_n(x,k)\right| \leq \frac{1}{n!} e^{2|k|(x_0-x)} \left[(x_0 - x) \int_x^{x_0} \left|Q(\cdot,k)\right| \right]^n. \tag{3.6}$$

Therefore, the sum $g = \sum g_n$ converges uniformly in x and on compact subsets of $\mathbb{C}$ when x is fixed. This sum is the solution of (3.3). Moreover, the derivative of g satisfies

$$g'(x,k) = -\int_x^{x_0} e^{2ik(t-x)} Q(t,k) g(t,k)\, dt. \tag{3.7}$$

For fixed x, $g_n(x,\cdot)$ is entire, and since $\sum g_n(x,\cdot)$ converges uniformly on compact subsets, it is entire too. By (3.7), $g'(x,\cdot)$ is also entire. Because

$$|k|^{-1} \int_x^{x_0} \left|Q(\cdot,k)\right| \leq \|q - q_0\|_1 + |k| \|w - 1\|_1$$

for $|k \geq |1$, (3.5) implies that

$$\left|g(x,k)\right| \leq C e^{(2x_0 + \|w-1\|_1)|k|},$$

for some constant C. Therefore, $g(x,\cdot)$ and $g'(x,\cdot)$ (by (3.7)) have growth order at most one. □

Since the Wronskian of two solutions of (1.1) is constant in x, we evaluate (3.1) at $x = 0$ to obtain

$$F(k) = \lambda f(0,k) \cos\alpha + f'(0,k) \sin\alpha. \tag{3.8}$$

By Lemma 3.2, F is entire and has growth order at most one. Its derivative will be written $\dot{F}$. The next lemma shows that the norming constants are now determined by F.

Lemma 3.3. *Let k_0 be a zero of F in the upper half-plane such that $\lambda_0 = k_0^2 + q_0 \neq 0$. Then*

$$\frac{1}{\rho(\{\lambda_0\})} = \left\|\varphi(\cdot,\lambda_0)\right\|^2 = -\frac{\dot F(k_0)F(-k_0)}{4ik_0^2\lambda_0}.$$

Proof. We begin by observing that

$$\int_0^x \left[\left(\varphi'(\cdot,\lambda)\right)^2 + q\left(\varphi(\cdot,\lambda)\right)^2\right] = \left(\varphi'\varphi\right)(x,\lambda) + \frac{1}{\lambda}\sin\alpha\cos\alpha + \int_0^x \lambda w\left(\varphi(\cdot,\lambda)\right)^2 \tag{3.9}$$

where we have integrated by parts and used that φ solves (1.1).

To simplify the last term in (3.9), let $\dot\varphi(x,\cdot)$ be the derivative of φ with respect to the second variable. Then, differentiating (1.1) with respect to λ, we find $-\dot\varphi'' + q\dot\varphi = w\varphi + \lambda w\dot\varphi$. Using this equation and that φ solves (1.1), we calculate

$$\left[\dot\varphi(\cdot,\lambda),\varphi(\cdot,\lambda)\right]' = w\left(\varphi(\cdot,\lambda)\right)^2.$$

Therefore, the last term in (3.9) becomes

$$\lambda\left(\left[\dot\varphi(\cdot,\lambda),\varphi(\cdot,\lambda)\right](x) - \left[\dot\varphi(\cdot,\lambda),\varphi(\cdot,\lambda)\right](0)\right).$$

Using the boundary conditions at $x = 0$, $\lambda[\dot\varphi(\cdot,\lambda),\varphi(\cdot,\lambda)](0) = \sin\alpha\cos\alpha/\lambda$. Hence, (3.9) becomes

$$\int_0^x \left[\left(\varphi'(\cdot,\lambda)\right)^2 + q\left(\varphi(\cdot,\lambda)\right)^2\right] = \left(\varphi'\varphi\right)(x,\lambda) + \lambda\left[\dot\varphi(\cdot,\lambda),\varphi(\cdot,\lambda)\right](x). \tag{3.10}$$

Now, if $\lambda_0 = k_0^2 + q_0 \neq 0$ is an eigenvalue, then $\varphi(\cdot,\lambda_0)$ is proportional to $f(\cdot,k_0)$ and is therefore in $\mathcal{H}$. So its norm is given by taking the limit of (3.10) as $x \to \infty$. The first term vanishes in the limit since f and f' do, so our goal is to write the Wronskian of φ and $\dot\varphi$ in terms of F.

To that end, we write, for $k \in \mathbb{C}^+ \setminus i\mathbb{R}^+$ (and, thus, $\lambda \notin \mathbb{R}$),

$$f(x,k) = F(k)\psi(x,\lambda) = F(k)\theta(x,\lambda) + G(k)\varphi(x,\lambda) \tag{3.11}$$

where $G(k) = m(\lambda)F(k)$. The last expression of (3.11) can be extended to all $k \neq \pm i\sqrt{q_0}$ by observing that G can be extended there because

$$m(\lambda) = \cos(\alpha)\psi'(0,\lambda) - \lambda\sin(\alpha)\psi(0,\lambda) = \cos(\alpha)\frac{f'(0,k)}{F(k)} - \lambda\sin(\alpha)\frac{f(0,k)}{F(k)}.$$

The first equation is an immediate consequence of the boundary conditions satisfied by θ and φ, while the second uses the first equality of (3.11).

Differentiating (3.11) with respect to k and using that $f(\cdot,k_0) = G(k_0)\varphi(\cdot,\lambda_0)$, $F(k_0) = 0$, and $[\theta(\cdot,\lambda),\varphi(\cdot,\lambda)] = \lambda^{-1}$, we find

$$\left[\dot f(\cdot,k_0), f(\cdot,k_0)\right](x) = \lambda_0^{-1}\dot F(k_0)G(k_0) + 2k_0G^2(k_0)\left[\dot\varphi(\cdot,\lambda_0),\varphi(\cdot,\lambda_0)\right](x).$$

Because $f(\cdot,k_0) = e^{ik_0x}$ for $x \geq x_0$, $[\dot f(\cdot,k_0), f(\cdot,k_0)](x) = ie^{2ik_0x}$ for such x. Since $\operatorname{Im} k_0 > 0$, the left hand side above vanishes as x tends to infinity, and we get

$$\left\|\varphi(\cdot,\lambda_0)\right\|^2 = \lim_{x\to\infty}\lambda_0\left[\dot\varphi(\cdot,\lambda_0),\varphi(\cdot,\lambda_0)\right](x) = -\frac{\dot F(k_0)}{2k_0G(k_0)}. \tag{3.12}$$

To complete the proof, we need to show

$$G(k_0) = \frac{2ik_0\lambda_0}{F(-k_0)}.$$

So, observe that

$$-2ik_0 = \big[f(\cdot, k_0), f(\cdot, -k_0)\big] = G(k_0)\big[\varphi(\cdot, \lambda_0), f(\cdot, -k_0)\big] = -\frac{G(k_0)}{\lambda_0}F(-k_0)$$

because the Wronskian is linear in the first argument and $\lambda_0\varphi(\cdot, \lambda_0)$ is invariant under changing the sign of k_0. □

If $\alpha = 0$, then $\lambda_0 = 0$ is an eigenvalue with eigenfunction ψ_0 and $F(k) = \lambda f(0, k)$. In this case φ and $\lambda\theta$ are entire solutions of (1.1). The limit of $\lambda\theta(\cdot, \lambda)$ as λ tends to 0, which we denote by θ_0, is a solution of (1.1) for $\lambda = k_0^2 + q_0 = 0$. The initial conditions satisfied by ψ_0, θ_0, and $\varphi(\cdot, 0)$ show that $\psi_0 = \psi_0(0)\theta_0 + \varphi(\cdot, 0)$. Eq. (3.11) gives $f(\cdot, k_0) = f(0, k_0)\theta_0 + G(k_0)\varphi(\cdot, 0)$. Since ψ_0 and $f(\cdot, k_0)$ are linearly dependent this shows that $f(\cdot, k_0) = G(k_0)\psi_0$ and hence

$$\frac{1}{\rho(\{0\})} = \|\psi_0\|^2 = -\psi_0(0) = -\frac{f(0, k_0)}{G(k_0)} = -\frac{\dot{F}(k_0)}{2k_0 G(k_0)}.$$

This is the analogue of (3.12) but the previous argument to determine $G(k_0)$ fails in this case. While we do not have an example showing that $G(k_0)$ is independent of F, we note that this is the case in the discrete analogue of our problem, see [1].

Theorem 3.4. *Let q and w satisfy Assumption 3.1. Then the Jost function determines the spectral measure unless $\lambda = 0$ is an eigenvalue. Then, we also need $\rho(\{0\})$.*

Proof. The zeros of F in $\mathbb{C}^+$ give the eigenvalues. Lemma 3.3 shows that F determines the norming constants of the non-zero eigenvalues. Thus, F and, if needed, $\rho(\{0\})$ determine the discrete part of the spectral measure. On the other hand, we already know that F determines the continuous part of it by (3.2). □

3.2. The inverse resonance problem

Let $E(z) = (1 - z)e^z$ be the Weierstrass elementary factor, and $\{k_n : n \in \mathbb{N}\}$ the set of non-zero zeros of F listed according to multiplicity and by increasing modulus. Because F is an entire function of growth order at most one, its Hadamard factorization is

$$F(k) = k^\ell \exp(Ak + B) \prod_{n=1}^{\infty} E(k/k_n). \tag{3.13}$$

Thus, F is determined by its zeros (the eigenvalues, resonances, and possibly zero) up to the constants A and B.

Unlike the Schrödinger case (see [6]), the following example shows that, in general, the eigenvalues and resonances do *not* determine the Jost function and, hence, the spectral measure.

Example 3.5. Let

$$w(x) = \begin{cases} 0 & \text{if } x < x_0, \\ 1 & \text{if } x > x_0, \end{cases}$$

and

$$q(x) = \begin{cases} a^2 & \text{if } x < x_0, \\ 1 & \text{if } x > x_0, \end{cases}$$

for some $a \neq 0$.

One then computes

$$\varphi(x,\lambda) = \frac{\cos(\alpha)}{a}\sinh(ax) - \frac{\sin(\alpha)}{\lambda}\cosh(ax), \quad \text{if } x \leq x_0.$$

Evaluating (3.1) at $x = x_0$ we get

$$F(k) = \frac{1}{a}e^{ikx_0}\big(\lambda\cos(\alpha)\big(a\cosh(ax_0) - ik\sinh(ax_0)\big) + a\sin(\alpha)\big(ik\cosh(ax_0) - a\sinh(ax_0)\big)\big).$$

In particular, if $\alpha = \pi/2$,

$$F(k) = e^{ikx_0}\cosh(ax_0)\big(ik - a\tanh(ax_0)\big).$$

Thus any such problem has no eigenvalue and precisely one resonance. The sign of a is irrelevant (as it should be), and we assume now that it is positive.

We see that knowing the resonance gives us the value of $a\tanh(ax_0)$ but not the value of $\cosh(ax_0)$. Thus, by (3.2), we have the same resonance, but many spectral measures.

Let $\tilde{F}(k) = k^{-\ell}F(k)$. Then, $\tilde{F}(0) = e^B$ and $\dot{\tilde{F}}(0)/\tilde{F}(0) = A$. We will need the following facts.

Lemma 3.6. *In (3.13), $\operatorname{Re} A = 0$ and $\ell \in \{0,1,2\}$. Moreover, $\ell = 2$ if and only if $\alpha = 0 = q_0$.*

Proof. Since q and w are real-valued and $\lambda = k^2 + q_0$, we have $Q(x,-\bar{k}) = \overline{Q(x,k)}$ and, by (3.4), $g_n(x,-\bar{k}) = \overline{g_n(x,k)}$. Therefore, $g(x,-\bar{k}) = \overline{g(x,k)}$ and $F(-\bar{k}) = \overline{F(k)}$. This property means that F is real and its derivative $\dot{F}$ is purely imaginary on the imaginary axis. Differentiating $\tilde{F}$, we find

$$\frac{\dot{\tilde{F}}(k)}{\tilde{F}(k)} = \frac{\dot{F}(k)}{F(k)} - \frac{\ell}{k}$$

which is purely imaginary on the imaginary axis. Because $\tilde{F}(0) \neq 0$, $\dot{\tilde{F}}/\tilde{F}$ is analytic (in particular, continuous) near zero. Therefore, A is purely imaginary.

Differentiating $[f(\cdot,k), f(\cdot,-k)] = -2ik$ with respect to k and evaluating at $k = 0$ gives

$$f(0,0)\dot{f}'(0,0) - \dot{f}(0,0)f'(0,0) = i \neq 0. \tag{3.14}$$

On the other hand, differentiating (3.8) with respect to k gives

$$\dot{F}(k) = \big(\lambda\dot{f}(0,k) + 2kf(0,k)\big)\cos\alpha + \dot{f}'(0,k)\sin\alpha. \tag{3.15}$$

Now suppose $F(0) = \dot{F}(0) = 0$, i.e., $\ell \geq 2$. Then (3.8) and (3.15) imply

$$\begin{pmatrix} f(0,0) & f'(0,0) \\ \dot{f}(0,0) & \dot{f}'(0,0) \end{pmatrix}\begin{pmatrix} q_0\cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By (3.14) the determinant is not zero. Hence $q_0\cos\alpha = \sin\alpha = 0$. Thus, $\ell \leq 1$ unless $\alpha = 0$ and $q_0 = 0$.

If $\alpha = 0 = q_0$, we have $F(k) = k^2 f(0,k)$ and $\ell \geq 2$ by (3.8). When $k = 0$, we have $-f'' + qf = 0$ so that f is proportional to ψ_0. But, $\psi_0(0) \neq 0$. Therefore, $f(0,0) \neq 0$ and $\ell = 2$. □

Lemma 3.7. *Let $\lambda_0 = k_0^2 + q_0 \neq 0$ be an eigenvalue. Then, the norming constant associated with λ_0 is determined by the eigenvalues, resonances, and $\tilde{F}(0)$.*

Proof. By Lemma 3.3, the norming constant associated with λ_0 is given by the formula

$$\frac{\dot{F}(k_0)F(-k_0)}{4ik_0^2\lambda_0}.$$

We calculate the numerator using the factorization of F. If $\Pi(k)$ is the canonical product in (3.13), then

$$\dot{\tilde{F}}(k_0) = \frac{\dot{F}(k_0)}{k_0^\ell} = e^{Ak_0+B}\dot{\Pi}(k_0),$$

because $\Pi(k_0) = 0$. Thus,

$$\dot{F}(k_0)F(-k_0) = (-1)^\ell k_0^{2\ell} e^{2B}\dot{\Pi}(k_0)\Pi(-k_0).$$

Since $e^{2B} = (\tilde{F}(0))^2$, and $\Pi(k)$, k_0, and ℓ are determined by the eigenvalues and resonances, the proof is complete. $\square$

Theorem 3.8. *The eigenvalues, resonances, and $\tilde{F}(0)$ determine the spectral measure unless $\lambda = 0$ is an eigenvalue. Then, $\rho(\{0\})$ is also needed.*

Proof. By Lemma 3.7, the given data determine the norming constants and, consequently, the discrete part of the spectral measure. On the other hand, Lemma 3.6 implies that $|F(k)|^2$ does not depend on A from (3.13) when $k \in \mathbb{R}$. Therefore, the given data also determine the continuous part of the spectral measure by (3.2). $\square$

Appendix A. A compendium of known results

Here we collect those results from [3] which are needed in the body of the paper about left-definite problems set on the half-line $[0,\infty)$. We emphasize, though, that all results have analogs for the case where the left endpoint is a number different from 0.

Let $\mathcal{H}_1$ be the set of locally absolutely continuous functions u defined in $[0,b)$ such that $u' \in L^2(0,b)$ and $q|u|^2 \in L^1(0,b)$. $\mathcal{H}_1$ is a Hilbert space with scalar product

$$\langle u, v\rangle = \int_0^b \big(u'\overline{v'} + qu\bar{v}\big).$$

A major feature of $\mathcal{H}_1$ is that point evaluations are continuous as the following lemma implies.

Lemma A.1. *For any $a \in [0,b)$ there exists a constant C_a such that*

$$\big|u(x)\big| \le C_a\|u\| \tag{A.1}$$

for any $x \in [0,a]$ and any $u \in \mathcal{H}_1$.

Denote the set of integrable functions with compact support in $(0,b)$ by L_0 and the set of solutions of $-u'' + qu = 0$ which lie in $\mathcal{H}_1$ by D_0.

Lemma A.2. *There is a bounded linear operator* $G_0 : L_0 \to \mathcal{H}_1$ *so that* $\langle u, G_0 v\rangle = \int_0^b u\bar{v}$ *for all* $u \in \mathcal{H}_1$ *and all* $v \in L_0$.

Lemma A.3.

(1) *The set* D_0 *is the orthogonal complement in* $\mathcal{H}_1$ *of* $L_0 \cap \mathcal{H}_1$. *It has dimension* 1 *or* 2.
(2) $\dim D_0 = 2$ *if and only if* $b < \infty$ *and* $q \in L^1[0, b)$.
(3) *If* $\dim D_0 = 1$ *and* $D_0 \ni v \not\equiv 0$, *then* $v(0)\overline{v'(0)} < 0$ *and* $u(x)\overline{v'(x)} \to 0$ *as* $x \to b$ *for any* $u \in \mathcal{H}_1$.
(4) *Finite functions are dense in* $\mathcal{H}_1$ *if and only if* $\dim D_0 = 1$.

Lemma A.4. *There are real-valued functions* ψ_0 *and* φ_0 *which solve* $-u'' + qu = 0$ *such that, if* $u \in \mathcal{H}_1$, *then*

(1) $\psi_0 \in \mathcal{H}_1$, $\psi_0'(0) = 1$, *and* $\psi_0'(x)u(x) \to 0$ *as* $x \to b$; *and*
(2) $\varphi_0(0) = -1$ *and* $\varphi_0'(0) = 0$.

Lemma A.5. *Unless* $\alpha = \pi/2$ *and* 0 *is in a gap of* $\operatorname{supp} w$ *the functions* ψ_0 *and* $\psi(\cdot, \lambda)$ *are in* $\mathcal{H}$.

Lemma A.6. *If* $\alpha \neq 0$, $u \in \mathcal{H}$ *and* $\hat{u} = \mathcal{F}u$, *then*

$$u(x, \lambda) = \int_0^b \hat{u}\varphi(x, \cdot)d\rho.$$

If $\alpha = 0$ *the same is true, except that we must replace* $\varphi(x, 0)$ *by* ψ_0.

Corollary A.7. *If* $\lambda \neq 0$ *is an eigenvalue of* T, *the Fourier transform of the eigenfunction* $\varphi(\cdot, \lambda)$ *is a multiple of the characteristic function of the set* $\{\lambda\}$ *and* $(\mathcal{F}\varphi(\cdot, \lambda))(\lambda) = 1/\rho(\{\lambda\})$. *The same is true for* $\lambda = 0$, *if* $\varphi(\cdot, 0)$ *is replaced by* ψ_0.

Lemma A.8. *If* u *is in the domain of* T, *then* $\mathcal{F}u \in L^1(\rho)$.

Lemma A.9. *If* $\lambda \notin \mathbb{R}$ *the Fourier transform of* $\psi(\cdot, \lambda)$ *is* $1/(t - \lambda)$. *Furthermore, the Fourier transform of* ψ_0 *equals* $\sin\alpha/t$ *for* $\alpha \neq 0$ *and* $\rho(\{0\})^{-1}\chi_{\{0\}}$ *for* $\alpha = 0$.

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