Operator Theory:
Advances and Applications, Vol. 186, 31–48

On Inverse Problems for Finite Trees

B.M. Brown and R. Weikard

Abstract. In this paper two classical theorems by Levinson and Marchenko for the inverse problem of the Schrödinger equation on a compact interval are extended to finite trees. Specifically, (1) the Dirichlet eigenvalues and the Neumann data of the eigenfunctions determine the potential uniquely (a Levinson-type result) and (2) the Dirichlet eigenvalues and a set of generalized norming constants determine the potential uniquely (a Marchenko-type result).

Mathematics Subject Classification (2000). Primary 34A55, Secondary: 05C05.

Keywords. Dirichlet-to-Neumann map; inverse spectral problems; trees.

1. Introduction

Some 60 years ago Borg, Levinson, and Marchenko established the now famous inverse spectral theory for the Schrödinger equation on a compact interval. Denote the solution of the initial value problem $-y'' + qy = \lambda y$, $y(0) = 0$ and $y'(0) = 1$ on the interval $[0, 1]$ by $s(\lambda, \cdot)$. Then the real integrable potential q is uniquely determined by any of the following sets of data:

- Borg [5] (1946): The Dirichlet-Dirichlet eigenvalues and the Dirichlet-Neumann eigenvalues, i.e., the zeros of $s(\cdot, 1)$ and of $s'(\cdot, 1)$.
- Levinson [17] (1949): The Dirichlet-Dirichlet eigenvalues (denoted by λ_n) and the Neumann data of the associated eigenfunctions, i.e., the numbers $s'(\lambda_n, 1)$ (recall that $s'(\lambda_n, 0) = 1$).
- Marchenko [18] (1950): The Dirichlet-Dirichlet eigenvalues and the norming constants $\int_0^1 s(\lambda_n, t)^2 dt$ of the associated eigenfunctions.

Since the 1980s spectral problems on graphs and trees have also been investigated (we are interested here solely in so-called metric trees where edges are homeomorphic to real intervals and may support potentials). We refer the reader

This material is based upon work supported by the National Science Foundation under Grant No. DMS-0304280.

to the excellent surveys [13] and [14] by Kuchment for an overview of these developments and their applications. Investigation of inverse problems on such graphs and trees are not quite so numerous. Without a claim to completeness we list here the works of Gerasimenko [11]; Curtis and Morrow [9], [10]; Carlson [7]; Pivovarchik [21], [20]; Kurasov and Stenberg [15]; Belishev [2]; Brown and Weikard [6]; Harmer [12]; Kurasov and Nowaczyk [16]; Yurko [22]; Belishev and Vakulenko [3]; and Avdonin and Kurasov [1]. We emphasize that the results in [2], [22], and [1] are particularly close to ours.

In this paper we address generalizations of the above mentioned results of Levinson and Marchenko to the case of finite trees with r edges and n_0 boundary vertices. The generalization of Borg's result has been established recently by Yurko [22]: spectra for n_0 (specific) boundary conditions are sufficient to determine the potential on the tree.

In this paper we focus for simplicity on real and bounded potentials on trees whose edge lengths are one. Our methods generalize to the case of integrable complex potentials on trees with varying edge lengths. We plan to address such generalizations in a subsequent paper.

Under the given circumstances the Dirichlet eigenvalues, which we denote by $\lambda_1, \lambda_2, \ldots$, are real and discrete since they are zeros of an entire function which can not vanish away from the real axis. We use the letter Σ to signify the set of all Dirichlet eigenvalues. Geometric multiplicities of Dirichlet eigenvalues may be larger than one. The multiplicity of the eigenvalue E is denoted $\mu(E)$.

The generalization of Levinson's result reads as follows.

Theorem 1.1. *Let q be a real-valued bounded potential supported on a finite metric tree whose edge lengths are one. Then the Dirichlet eigenvalues, their multiplicities, and the Neumann data of an orthonormal set of Dirichlet eigenfunctions uniquely determine the potential almost everywhere on the tree.*

We remark here that eigenfunctions are not automatically orthonormal since eigenspaces may have dimensions larger than one. Theorem 1.1 will be proved in Section 6.

A particularly important role is played in this paper by the Weyl solutions. They are uniquely defined away from the Dirichlet eigenvalues by requiring that they satisfy homogeneous Dirichlet boundary conditions at all but one boundary vertex where they assume the value 1. At the Dirichlet eigenvalues the Weyl solutions cease to exist but multiplying them with the "minimal function" $\chi(\lambda) = \prod_{E\in\Sigma}^{\infty}(1 - \lambda/E)$ (as opposed to the characteristic function which takes the multiplicities of the eigenvalues into account) gives globally defined functions $\omega(k, \lambda, \cdot)$ where $k \in \{1, \ldots, n_0\}$ signifies the boundary vertex at which the function value is prescribed to be $\chi(\lambda)$. If E is a Dirichlet eigenvalue of multiplicity $\mu(E)$ then there will be $\mu(E)$ of the functions $\omega(k, \lambda, \cdot)$ which are linearly independent. Let $K(E) \subset \{1, \ldots, n_0\}$ be a maximal set such that $\{\omega(k, \lambda, \cdot) : k \in K(E)\}$ is linearly independent. We also introduce the quantities $N_{j,k}(E)$ which are the inner products of the functions $\omega(j, E, \cdot)$ and $\omega(k, E, \cdot)$. For $j = k$ these are norming constants of eigenfunctions.

We can now formulate the generalization of Marchenko's result.

Theorem 1.2. *Let q be a real-valued bounded potential supported on a finite metric tree whose edge lengths are one. Then the Dirichlet eigenvalues, their multiplicities, and the quantities $N_{j,k}(E)$, $j \in \{1, \dots, n_0\}$, $k \in K(E)$, $E \in \Sigma$ uniquely determine the potential almost everywhere on the tree.*

This theorem will be proved in Section 7.

The paper is organized as follows. Section 2 gives formal definitions of trees, interface conditions, and operators under consideration. It also provides basic results on initial value and Dirichlet boundary value problems. Weyl solutions and Green's function play (as is to be expected) an important role. They are studied in Section 3 and Section 4, respectively. The Dirichlet-to-Neumann map is introduced in Section 5. Sections 6 and 7 are devoted to the proofs of Theorems 1.1 and 1.2, respectively.

2. Preliminaries

2.1. Trees

A finite tree is given by a Hausdorff space T and a set of r homeomorphisms $\epsilon_j : [0, 1] \to T$, $j = 1, \dots, r$, such that the following conditions are met:

1. $T = \bigcup_{j=1}^{r} \{\epsilon_j(t) : t \in [0, 1]\}$.
2. If $\epsilon_j(t) = \epsilon_k(s)$ and $j \neq k$ then $t, s \in \{0, 1\}$.
3. T is simply connected.

In this paper we will be concerned only with finite trees and we will always mean a finite tree when we speak of a tree.

The set $V = \{\epsilon_1(0), \epsilon_1(1), \dots, \epsilon_r(0), \epsilon_r(1)\}$ has precisely $r+1$ elements called vertices of the tree. The homeomorphisms ϵ_j are called edges of the tree. We say a vertex v belongs to an edge ϵ_j or that edge ϵ_j is attached to v if $v = \epsilon_j(0)$, the initial vertex of ϵ_j, or $v = \epsilon_j(1)$, the terminal vertex of ϵ_j. A vertex is called a boundary vertex if it belongs to only one edge. Such an edge will be called a boundary edge. A vertex which belongs to several edges is called an internal vertex. An edge both of whose endpoints are internal vertices is called an internal edge.

The fact that edges are homeomorphic to the metric space $[0, 1]$ turns the space T in a natural way into a metric space.

Since the interval $[0, 1]$ has an orientation associated with it, so does each of the edges. However, we are not interested in the orientations of the edges, but only in the metric structure provided by the homeomorphisms ϵ_j. Thus, given a finite tree we obtain another tree by replacing the homeomorphism $t \mapsto \epsilon_j(t)$ by $t \mapsto \epsilon_j(1-t)$ for some (or none or all) of the indices j. The new tree is, of course, still associated with the Hausdorff space T and the trees, so related, are called equivalent. This is an equivalence relation and we will henceforth choose the labels and orientations for the edges as convenience suggests. Often times we will call the Hausdorff space T a tree, assuming a tacit understanding of its metric structure.

We take here (for the most part) the point of view that all boundary vertices play identical roles. Assuming there are n_0 boundary vertices we will then assign labels $1, \dots, n_0$ to the boundary vertices and the corresponding boundary edges. We assume also that the boundary edges are oriented so that $v_j = \epsilon_j(0)$ for $j = 1, \dots, n_0$.

However, in some intermediate results one boundary vertex of the tree is singled out. In this case it is convenient to assign labels and orientations in the following way. The special boundary vertex is called the root and is designated as v_0. All other boundary vertices are called branch tips. The edge attached to the root, denoted by ϵ_0, will be called the trunk. All edges are oriented so that their initial vertex is closer to the root than their terminal vertex. In particular, $\epsilon_0(0) = v_0$, the root of the tree. The edges other than the trunk attached to the terminal vertex of the trunk $v_1 = \epsilon_0(1)$ are called limbs. Each one of them is the trunk of a subtree with root v_1. A tree labeled and oriented this way will be called a rooted tree.

We emphasize that all results obtained are independent of the particular way in which edges and vertices are labeled and in which edges are oriented; these designations only serve to communicate proofs more easily.

2.2. The interface conditions

A function y defined on T may be represented as $\vec{y} = (y_1, \dots, y_r)^\top$ where $y_j(t) = y(\epsilon_j(t))$. We say that y is integrable on T (or square integrable on T or $y \in L^p(T)$) if the y_j are integrable on $[0, 1]$ (or square integrable on $[0, 1]$ or $y_j \in L^p([0, 1])$) for $j = 1, \dots, r$.

We define $\mathcal{C}$ to be the set of all functions y defined on T which satisfy the following conditions:

1. For each j the functions y_j and y_j' are absolutely continuous on $[0, 1]$.
2. y is continuous on T.
3. For each internal vertex v the Kirchhoff condition
$$\sum_{\epsilon_j(1)=v} y_j'(1) - \sum_{\epsilon_j(0)=v} y_j'(0) = 0$$
holds.

Conditions 2 and 3 are called interface conditions. Note that $\mathcal{C}$ is independent of the orientation or the labeling of the edges.

In order to deal conveniently with the interface conditions we introduce[1] the operator
$$\mathfrak{I} = E_0\mathfrak{E}_0 + E_1\mathfrak{E}_1 + D_0\mathfrak{D}_0 + D_1\mathfrak{D}_1,$$

[1] This notation is different from the one we used in [6]

where E_0, E_1, D_0 and D_1 are certain $(2r - n_0) \times r$ matrices whose entries are 0 or ± 1 and $\mathfrak{E}_0$, etc. are evaluation operators defined by $\mathfrak{E}_0 \vec{y} = \vec{y}(0)$, $\mathfrak{E}_1 \vec{y} = \vec{y}(1)$, $\mathfrak{D}_0 \vec{y} = \vec{y}\,'(0)$, and $\mathfrak{D}_1 \vec{y} = \vec{y}\,'(1)$. Thus y satisfies the interface conditions if and only if $\mathfrak{I}\vec{y} = 0$.

2.3. The differential equations

If q is an integrable function on T define $Q = \operatorname{diag}(q_1, \ldots, q_r)$. We will consider the differential expression $\vec{y} \mapsto -\vec{y}\,'' + Q\vec{y}$ and the differential equation $-\vec{y}\,'' + Q\vec{y} = \lambda\vec{y}$. Define

$$\mathcal{D} = \{y \in \mathcal{C} : -y_j'' + q_j y_j \in L^2([0, 1])\}.$$

We now define the operator $L : \mathcal{D} \to L^2(T)$ by $(Ly)(\epsilon_j(t)) = -y_j''(t) + q_j(t)y_j(t)$. Again, this definition is independent of the orientation or the labeling of the edges.

If $y \in \mathcal{D}$ satisfies $Ly = \lambda y$ we will call it a solution of $Ly = \lambda y$ signifying that both the differential equations and the interface conditions are satisfied. Functions solving the differential equations form a $2r$-dimensional vector space. Since there are $2r - n_0$ interface conditions it is reasonable to expect that the space of solutions of $Ly = \lambda y$ is n_0-dimensional. This fact will be proved below.

Denote by $c_j(\lambda, \cdot)$ and $s_j(\lambda, \cdot)$ the basis of solutions of $-y_j'' + q_j y_j = \lambda y_j$ defined by initial conditions $c_j(\lambda, 0) = s_j'(\lambda, 0) = 1$ and $c_j'(\lambda, 0) = s_j(\lambda, 0) = 0$. We collect these functions in the $r \times r$-matrices $C(\lambda, t) = \operatorname{diag}(c_1(\lambda, t), \ldots, c_r(\lambda, t))$ and $S(\lambda, t) = \operatorname{diag}(s_1(\lambda, t), \ldots, s_r(\lambda, t))$. A function y satisfying the differential equations may now be expressed as $\vec{y} = (C(\lambda, \cdot), S(\lambda, \cdot))\xi$ for an appropriate $\xi \in \mathbb{C}^{2r}$.

In particular, the function $\vec{y} = (C(\lambda, \cdot), S(\lambda, \cdot))\xi$ satisfies the interface conditions (and hence is a solution of $Ly = \lambda y$) precisely if ξ is in the kernel of the $(2r - n_0) \times 2r$-matrix

$$J(\lambda) = \mathfrak{I}(C(\lambda, \cdot), S(\lambda, \cdot)).$$

2.4. Initial value problems

Initial value problems do, in general, not have unique solutions on trees. This causes the main differences in the treatment of inverse problems on trees when compared to intervals. However, it is still useful to investigate the set of all solutions for the initial value problem.

One of the boundary vertices is being singled out as the "initial point". Thus, it is now convenient to treat the tree as a rooted tree.

Lemma 2.1. *The initial value problem $Ly = \lambda y$, $y_0(0) = a$, and $y_0'(0) = b$ has a solution for any choice of $a, b \in \mathbb{C}$.*

Proof. This is obvious when the tree has only one internal vertex and follows by induction over the number of internal vertices for general trees. In fact, if there are k limbs attached to v_1 (i.e., $k + 1$ edges), then there are k subtrees for which we know the existence of solutions of the initial value problem by induction hypothesis. For subtree j, where $1 \leq j \leq k$, we use initial conditions $a_j = ac_0(\lambda, 1) + bs_0(\lambda, 1)$ and some value for b_j. These provide then for a solution on the full tree provided that $b_1 + \cdots + b_k = ac_0'(\lambda, 1) + bs_0'(\lambda, 1)$. $\square$

Corollary 2.2. *Suppose T has n_0 boundary vertices. The vector space of solutions of $Ly = \lambda y$ satisfying homogeneous initial conditions $y_0(0) = 0$ and $y_0'(0) = 0$ has dimension $n_0 - 2$.*

Proof. Again, this holds obviously for a tree with one internal vertex. Suppose it is true for trees with less than ℓ internal vertices and consider a tree with exactly ℓ internal vertices. Suppose the tree has k limbs. On each of the k subtrees which start at the end of the trunk there is a solution of the initial value problem for function value 0 and derivative 1. These give rise to $k-1$ linearly independent solutions of the homogeneous initial value problem on the full tree. Now, given any solution of the homogeneous initial value problem, one can subtract a suitable linear combination of the $k-1$ basis functions just constructed to obtain a solution on the full tree which is identically zero on the trunk and on the k limbs. It is thus a linear combination of all possible basis functions for each of the homogeneous initial value problems on the subtrees. If subtree j has $n_j + 1$ boundary vertices then we have, by the induction hypothesis, $n_j - 1$ such basis functions. Since $n_0 = 1 + \sum_{j=1}^{k} n_j$ we get for the total number of basis functions on the full tree $k - 1 + \sum_{j=1}^{k}(n_j - 1) = n_0 - 2$. □

These observations give us now immediately the following theorem.

Theorem 2.3. *Suppose T has r edges and n_0 boundary vertices. Then the vector space of solutions of $Ly = \lambda y$ has dimension n_0. Moreover, the matrix $J(\lambda)$ has full rank $2r - n_0$ for every complex λ.*

Proof. There are linearly independent solutions which are equal to $c_0(\lambda, \cdot)$ and $s_0(\lambda, \cdot)$, respectively, when restricted to the trunk of the tree. Let y be any solution and subtract a suitable combination of the functions just mentioned so that the resulting function satisfies homogeneous initial conditions. That function may expressed as a linear combination of the $n_0 - 2$ basis functions constructed in the previous corollary.

The last statement follows now from the fundamental theorem of linear algebra. □

2.5. The Dirichlet boundary value problem

Given a vector $f = (f_1, \dots, f_{n_0})^\top$ in $\mathbb{C}^{n_0}$ we are looking for a solution of $Ly = \lambda y$ satisfying the nonhomogeneous Dirichlet boundary conditions $y(v_j) = f_j$ when $v_1, \dots, v_{n_0}$ denote the boundary vertices. These conditions are expressed as $\mathfrak{A}\vec{y} = f$ where $\mathfrak{A} = A_0 \mathfrak{E}_0$ and $A_0 = (I_{n_0 \times n_0}, 0_{n_0 \times (r - n_0)})$ assuming that the boundary edges are oriented such that $v_j = \epsilon_j(0)$ for $j = 1, \dots, n_0$. Here $I_{n \times n}$ and $0_{n \times m}$ denote the identity matrix and the zero matrix of the given dimensions, respectively. We might subsequently drop the subscripts if the dimensions are clear from the context. We also note that the $n_0 \times 2r$-matrix $\mathfrak{A}(C(\lambda, \cdot), S(\lambda, \cdot)) = (I_{n_0 \times n_0}, 0_{n_0 \times (2r - n_0)})$.

Finally, we introduce the $2r \times 2r$ matrix

$$M(\lambda) = \begin{pmatrix} \mathfrak{A} \\ \mathfrak{J} \end{pmatrix} (C(\lambda, \cdot), S(\lambda, \cdot)) = \begin{pmatrix} I & 0 \\ R(\lambda) & P(\lambda) \end{pmatrix}$$

where R represents the first n_0 columns and P the last $2r - n_0$ columns of the matrix J introduced in Section 2.3. Thus solutions of the equation $M(\lambda)\xi = (f, 0)^\top$ provide solutions of $Ly = \lambda y$ satisfying the given boundary conditions.

The entries of M are entire functions. Therefore the zeros of the determinant of M are isolated unless it is identically equal to zero.

This latter possibility can be ruled out most easily under our assumption of a real potential since the zeros of the determinant are eigenvalues of the Dirichlet operator L_D, the restriction of L to the set of all $y \in \mathcal{D}$ which satisfy homogeneous Dirichlet conditions at the boundary vertices. An integration by parts, using the interface and boundary conditions, shows, that the operator L_D is self-adjoint so that its eigenvalues are real.

Thus, the equation $Ly = \lambda y$ has a unique solution satisfying given (possibly nonhomogeneous) Dirichlet boundary conditions unless λ is one of the countably many real eigenvalues of L_D.

The geometric multiplicity of a Dirichlet eigenvalue is strictly less than n_0 for there is at least one solution of $Ly = \lambda y$ which does not satisfy homogeneous Dirichlet conditions. Since the problem is self-adjoint the geometric multiplicity of a Dirichlet eigenvalue equals its algebraic multiplicity. The algebraic multiplicity in turn equals the order of the eigenvalue as a zero of $\det M$. This may be seen by a generalization of Naimark's argument for a differential equation with scalar coefficients to one with matrix coefficients.

3. Weyl solutions

A solution ψ of $Ly = \lambda y$ is called a Weyl solution if ψ assumes the value one at precisely one of the boundary vertices and the value zero at each of the others. With one of the boundary vertices singled out it is convenient, in this section, to treat the tree as a rooted tree (see Section 2.1). If x and x' are two points in T we denote their distance by $d(x, x')$ (recall that T is a metric space). The number $h = \max\{d(x, v_0) : x \in T\}$ is called the height of the tree with respect to the root v_0. (The height of a tree depends on which boundary vertex is designated as root).

Lemma 3.1. *Let $\psi(\lambda, \cdot)$ be the Weyl solution for a tree T which assumes the value one at the root and the value zero at each of the branch tips. Then*

$$\psi_0'(\lambda, 0) = -\sqrt{-\lambda} + o(1)$$

as λ tends to infinity on the negative real axis. Furthermore, if s is the label of a boundary edge leading to branch tip $\epsilon_s(1)$, then

$$\psi_s'(\lambda, 1) = -2\sqrt{-\lambda}\exp(-d(v_0, \epsilon_s(1))\sqrt{-\lambda})(1 + o(1))$$

again as λ tends to infinity on the negative real axis.

Proof. The proof is by induction on the height of the tree. If the tree has height one, then $\psi_0'(\lambda, 0) = m(\lambda)$ is the Titchmarsh-Weyl m-function for the problem (recall that $\psi_0(\lambda, 0) = 1$). It is well known that $m(\lambda) = iz + o(1)$ as $z = i\sqrt{-\lambda}$

tends to infinity on the positive imaginary axis. Also for the branch tip $\epsilon_0(1)$ we have $\psi_0'(\lambda,1) = -1/s_0(\lambda,1) = 2iz\mathrm{e}^{iz} + O(\mathrm{e}^{iz} + z\mathrm{e}^{3iz})$.

Next assume the validity of our claim for all trees of height at most n and let T be a tree of height $n+1$. Let $v_1 = \epsilon_0(1)$. Then v_1 is the root of several, say k, subtrees, whose height is at most n. These will be labeled $T_1, \dots, T_k$ and their trunks, viewed as limbs of the full tree, are correspondingly labeled $\epsilon_1, \dots, \epsilon_k$. The Weyl solution for the tree T_j is denoted by $\psi(j,\lambda,\cdot)$. Note that the Weyl solution $\psi(\lambda,\cdot)$ for the full tree, restricted to T_j, is a multiple of $\psi(j,\lambda,\cdot)$, in fact

$$\psi_\sigma(\lambda,\cdot) = \psi(\lambda,v_1)\psi_s(j,\lambda,\cdot) \tag{3.1}$$

when s denotes the label of an edge of T_j and σ denotes the label of the same edge when viewed as an edge of T. In particular,

$$\psi_j(\lambda,\cdot) = \psi(\lambda,v_1)\psi_0(j,\lambda,\cdot).$$

Now we will determine $\psi_0(\lambda,\cdot)$, the Weyl solution for T on its trunk, by solving a Riccati equation. To this end we define

$$\mu(\lambda,\cdot) = \frac{\psi_0'(\lambda,\cdot)}{\psi_0(\lambda,\cdot)}.$$

Then $\mu(\lambda,\cdot)$ satisfies the differential equation

$$\mu'(\lambda,x) + \mu(\lambda,x)^2 + \lambda - q_0(x) = 0$$

and the initial condition

$$\mu(\lambda,1) = \frac{\sum_{j=1}^k \psi_j'(\lambda,0)}{\psi(\lambda,v_1)} = \sum_{j=1}^k \psi_0'(j,\lambda,0) = ikz + o(1).$$

This problem will be solved by emulating Bennewitz's approach in [4]. The first step is to find the solution $\mu_0(\lambda,\cdot)$ for vanishing q_0 but with the correct initial condition. Thus

$$\mu_0(\lambda,x) = iz + \frac{2iz(\mu(\lambda,1) - iz)}{(iz + \mu(\lambda,1))\exp(2iz(x-1)) + iz - \mu(\lambda,1)}.$$

Now recall that $q_0 \in L^1([0,1])$ and define

$$\mu_1(\lambda,x) = -\int_x^1 \mathrm{e}^{2\int_x^t \mu_0(\lambda,u)du} q_0(t)dt$$

and

$$\mu_n(\lambda,x) = \int_x^1 \mathrm{e}^{2\int_x^t \sum_{j=0}^{n-1} \mu_j(\lambda,u)du} \mu_{n-1}(\lambda,t)^2 dt, \quad n = 2,3,\dots.$$

Then

$$\begin{aligned}
\mu_1'(\lambda,x) &= q_0(x) - 2\mu_0(\lambda,x)\mu_1(\lambda,x), \\
\mu_n'(\lambda,x) &= -\mu_{n-1}(\lambda,x)^2 - 2\mu_n(\lambda,x)\sum_{j=0}^{n-1}\mu_j(\lambda,x), \quad n = 2,3,\dots,
\end{aligned}$$

and $\mu_n(\lambda,1) = 0$ for all $n \in \mathbb{N}$.

Thus, assuming uniform convergence of $\sum_{n=0}^{\infty} \mu_n$ and $\sum_{n=0}^{\infty} \mu_n'$, we have (after some algebra) that the series $\sum_{n=0}^{\infty} \mu_n(\lambda,\cdot)$ satisfies the same initial value problem (the Riccati equation and the initial condition at $x = 1$) as $\mu(\lambda,\cdot)$ and hence is equal to it, i.e.,

$$\mu(\lambda, x) = \sum_{n=0}^{\infty} \mu_n(\lambda, x). \tag{3.2}$$

In order to investigate convergence we first realize that

$$\frac{\mu_0(\lambda, x)}{iz} = 1 + o(1/z)$$

for $k = 1$ and

$$\frac{\mu_0(\lambda, x)}{iz} = 1 + \frac{2\mathrm{e}^{2iz(1-x)}(k-1)}{k+1-(k-1)\mathrm{e}^{2iz(1-x)}}(1 + o(1/z))$$

for $k > 1$. Since iz is negative we may estimate in either case that

$$\mathrm{Re}(\mu_0(\lambda, x)) \le 3iz/4 = -3\,\mathrm{Im}(z)/4$$

for sufficiently large z. Thus

$$|\mu_1(\lambda, x)| \le \int_x^1 \mathrm{e}^{-3\,\mathrm{Im}(z)(t-x)/2} |q_0(t)| dt.$$

Given $\varepsilon > 0$, there are complex numbers $\alpha_1, \ldots, \alpha_N$ and intervals $I_1, \ldots, I_N$ such that $\tilde{q}_0 = \sum_{j=1}^{N} \alpha_j \chi_{I_j}$ is a step function on $[0, 1]$ with $\|q_0 - \tilde{q}_0\| \le \varepsilon$. Then

$$|\mu_1(\lambda, x)| \le \varepsilon + \sum_{j=1}^{N} |\alpha_j| \int_x^1 \mathrm{e}^{-3\,\mathrm{Im}(z)(t-x)/2} \chi_{I_j}(t) dt \le \varepsilon + \sum_{j=1}^{N} \frac{2|\alpha_j|}{3\,\mathrm{Im}(z)}.$$

This estimate holds regardless of x and proves that $|\mu_1(\lambda, x)|$ tends to zero uniformly in x as $\mathrm{Im}(z)$ tends to infinity.

Now let

$$a(\lambda, x) = \sup\{|\mu_1(\lambda, t)| : x \le t \le 1\}$$

and assume that z is large enough so that $\mathrm{Im}(z) \ge 1$ and $a(\lambda, x)/\,\mathrm{Im}\, z \le 1/2$.

One then shows by induction that

$$|\mu_n(\lambda, x)| \le \left(\frac{a(\lambda, x)}{\mathrm{Im}(z)}\right)^{2^{n-1}} \mathrm{Im}(z)$$

using that $a(\lambda, \cdot)$ is non-increasing and that

$$\sum_{j=1}^{n-1} |\mu_j(\lambda, x)| \le \mathrm{Im}(z) \sum_{j=1}^{n-1} \left(\frac{a(\lambda, x)}{\mathrm{Im}(z)}\right)^{2^{j-1}} \le \mathrm{Im}(z) \sum_{j=1}^{\infty} \left(\frac{a(\lambda, x)}{\mathrm{Im}(z)}\right)^{j} \le 2a(\lambda, 0).$$

These estimates show also that $\sum_{n=0}^{\infty} \mu_n(\lambda, \cdot)$ may be differentiated term by term, thus proving the validity of equation (3.2).

The Weyl solution on the trunk of T is now given as

$$\psi_0(\lambda, x) = \exp\left(\int_0^x \mu(\lambda, t) dt\right).$$

In particular,

$$\psi_0'(\lambda, 0) = \mu(\lambda, 0) = iz + o(1) \tag{3.3}$$

and

$$\psi(\lambda, v_1) = \psi_0(\lambda, 1) = \exp\left(\int_0^1 \mu(\lambda, t) dt\right) = \mathrm{e}^{iz}(1 + o(1)).$$

Finally, consider a boundary edge s on tree T_j, different from the trunk of T_j, and denote the distance of the corresponding boundary vertex from the root of T_j by N. Then, according to our induction hypothesis,

$$\psi_s'(j, \lambda, 1) = 2iz\mathrm{e}^{izN}(1 + o(1)).$$

This edge is also a boundary edge of T with label σ, say. Using (3.1) we find

$$\psi_\sigma'(\lambda, 1) = \psi(\lambda, v_1)\psi_s'(j, \lambda, 1) = 2iz\mathrm{e}^{iz(N+1)}(1 + o(1)).$$

This and equation (3.3) complete our proof. □

4. Green's function

In Section 2.5 we introduced the matrix $M(\lambda)$ to describe both boundary and interface conditions. Now we introduce also the operators $\mathfrak{I}_0 = E_0\mathfrak{E}_0 + D_0\mathfrak{D}_0$ and $\mathfrak{A}_0 = A_0\mathfrak{E}_0$ as well as the matrix

$$M_0(\lambda) = \begin{pmatrix} \mathfrak{A}_0 \\ \mathfrak{I}_0 \end{pmatrix} (C(\lambda, \cdot), S(\lambda, \cdot)).$$

The solution of the nonhomogeneous system of equations $(L - \lambda)u = h$ where $h \in L^2(T)$ and where u is subject to the homogeneous boundary conditions $\mathfrak{A}\vec{u} = 0$ as well as the interface conditions $\mathfrak{I}\vec{u} = 0$ is given by (see [6])

$$\vec{u}(t) = \int_0^1 \Gamma(\lambda, t, t')\vec{h}(t')dt'$$

where

$$\Gamma(\lambda, t, t') = (C(\lambda, t), S(\lambda, t)) \left(M(\lambda)^{-1}M_0(\lambda) - H(t' - t)\right) \begin{pmatrix} S(\lambda, t') \\ -C(\lambda, t') \end{pmatrix}$$

with H being the Heaviside function, i.e., $H(t)$ equals zero or one depending on whether t is negative or positive.

We remark that $\Gamma(\lambda, \cdot, \cdot)$ represents a scalar function $G(\lambda, \cdot, \cdot)$ on $T \times T$ via $G(\lambda, x, y) = \Gamma_{j,k}(t, t')$ when $x = \epsilon_j(t)$ and $y = \epsilon_k(t')$.

Lemma 4.1. *The function Γ has the following properties:*

1. *For any $t, t' \in [0, 1]$ the function $\Gamma(\cdot, t, t')$ is analytic away from the eigenvalues associated with the Dirichlet problem for T.*

2. *If λ is not an eigenvalue associated with the Dirichlet problem for T then the function $\Gamma(\lambda, \cdot, \cdot)$ has the following properties:*
   (a) $\Gamma(\lambda, \cdot, \cdot)$ *is continuous on* $[0, 1]^2$.
   (b) $\Gamma^{(0,0,1)}(\lambda, \cdot, \cdot)$ *is continuous in either of the triangles* $\{(t, t') \in [0, 1]^2 : t < t'\}$ *and* $\{(t, t') \in [0, 1]^2 : t > t'\}$. *On the line* $t = t'$ *there is a jump discontinuity:*
   $$\lim_{\tau\downarrow 0}(\Gamma^{(0,0,1)}(\lambda, t, t + \tau) - \Gamma^{(0,0,1)}(\lambda, t, t - \tau)) = -I.$$
   (c) *For fixed* $t' \in [0, 1]$ *the function* $\Gamma(\lambda, \cdot, t')$ *satisfies the equation*
   $$-\Gamma(\lambda, \cdot, t')'' + (Q - \lambda)\Gamma(\lambda, \cdot, t') = 0$$
   on $(0, t')$ *as well as* $(t', 1)$.
   (d) *For fixed* $t \in [0, 1]$ *the function* $\Gamma(\lambda, x, \cdot)$ *satisfies the equation*
   $$-\Gamma(\lambda, t, \cdot)'' + \Gamma(\lambda, t, \cdot)(Q - \lambda) = 0$$
   on $(0, t)$ *as well as* $(t, 1)$.
   (e) *If* $k \le n_0$ *then* $\Gamma_{k,j}(\lambda, 0, t') = 0$.
   (f) *If* $j \le n_0$ *then* $\Gamma_{k,j}(\lambda, t, 0) = 0$.

Proof. Statements (a) through (d) follow easily from the properties of the Heaviside function and the functions c_j and s_j.

To see the validity of statement (e) note first that M and M_0 are both of the form
$$\begin{pmatrix} I_{n_0\times n_0} & 0_{n_0\times(2r-n_0)} \\ * & ** \end{pmatrix}$$
where $*$ indicates an appropriate $(2r - n_0) \times n_0$-matrix and $**$ indicates an appropriate $(2r - n_0) \times (2r - n_0)$-matrix. This implies that $M^{-1}M_0$ is also of that same form. Hence, when $t' > 0$, the first n_0 rows of $M^{-1}M_0 - H(t')$ are all zero proving (e) in that case. By continuity this is also true when $t' = 0$.

To prove (f) note that, if $t > 0$, we have $H(-t) = 0$. For this case we need to know more about the structure of M_0. Since the bottom $2r - n_0$ rows contain only information about interior vertices we know that the entries in the first n_0 columns and the last $2r - n_0$ rows of $(\mathfrak{A}_0, \mathfrak{I}_0)^\top$ are zero. Hence M_0 has the form
$$\begin{pmatrix} I_{n_0\times n_0} & 0_{n_0\times(r-n_0)} & 0_{n_0\times n_0} & 0_{n_0\times(r-n_0)} \\ 0_{(2r-n_0)\times n_0} & * & 0_{(2r-n_0)\times n_0} & ** \end{pmatrix}$$
where $*$ and $**$ indicate appropriate $(2r - n_0) \times (r - n_0)$-matrices. Thus
$$M_0 \begin{pmatrix} S(\lambda, t') \\ -C(\lambda, t') \end{pmatrix} = \begin{pmatrix} S_e(\lambda, t') & 0_{n_0\times(r-n_0)} \\ 0_{(2r-n_0)\times n_0} & *** \end{pmatrix}$$
where $***$ indicates a $(2r-n_0)\times(r-n_0)$-matrix and where $S_e(\lambda, t') = \mathrm{diag}(s_1(\lambda, t'), \ldots, s_{n_0}(\lambda, t'))$. The first n_0 columns, corresponding to the requirement $j \le n_0$ are zero when $t' = 0$. $\square$

We now express the Weyl solutions in terms of Green's function. Let h be a function in $\mathcal{D}$, the domain of L satisfying the boundary conditions $\mathfrak{A}h = e_\ell$. Next let y be the unique solution of the following inhomogeneous problem

$$(L-\lambda)y = (\lambda - L)h, \quad y \in \mathcal{D}, \quad \mathfrak{A}y = 0.$$

Then $\psi(\ell,\lambda,t) = (y+h)(t)$. We proceed to compute y. By partial integration

$$\begin{aligned}
\vec{y}(t) &= \int_0^1 \Gamma(\lambda,t,t')(\lambda - Q(t'))\vec{h}(t')dt' + \int_0^1 \Gamma(\lambda,t,t')\vec{h}''(t')dt' \\
&= -\int_0^t \Gamma^{(0,0,2)}(\lambda,t,t')\vec{h}(t')dt' - \int_t^1 \Gamma^{(0,0,2)}(\lambda,t,t')\vec{h}(t')dt' \\
&\quad + \int_0^1 \Gamma(\lambda,t,t')\vec{h}''(t')dt' \\
&= -\Gamma^{(0,0,1)}(\lambda,t,\cdot)\vec{h}|_0^t - \Gamma^{(0,0,1)}(\lambda,t,\cdot)\vec{h}|_t^1 + \Gamma(\lambda,t,\cdot)\vec{h}'|_0^t + \Gamma(\lambda,t,\cdot)\vec{h}'|_t^1.
\end{aligned}$$

In particular, for $h_\ell(t') = (1-t')^2$ and $h_j(t') = 0$ for $j \neq \ell$ we get

$$\begin{aligned}
\vec{y}(t) = \lim_{\tau\to 0}&[\Gamma^{(0,0,1)}(\lambda,t,t+\tau)\vec{h}(t+\tau) - \Gamma^{(0,0,1)}(\lambda,t,t-\tau)\vec{h}(t-\tau)] \\
&+ \Gamma^{(0,0,1)}(\lambda,t,0)\vec{h}(0) - \Gamma(\lambda,t,0)\vec{h}'(0).
\end{aligned}$$

Since $\vec{h}(0) = e_\ell$, since column ℓ of $\Gamma(\lambda,t,0)$ is a zero column, and since the term in brackets tends to $-\vec{h}(t)$ we obtain

$$\psi_j(k,\lambda,t) = \Gamma_{j,k}^{(0,0,1)}(\lambda,t,0). \tag{4.1}$$

4.1. Eigenfunction expansion of Green's function

Let L_D be the Dirichlet operator introduced in Section 2.5. The eigenvalues λ_k of L_D may have geometric multiplicity larger than one. We will label them in such a way that they are repeated according to their multiplicity and such that $\lambda_1 \leq \lambda_2 \leq \cdots$. The corresponding orthonormalized eigenfunctions are denoted by φ_1, φ_2 etc. Then (see Coddington and Levinson [8], pp. 298/299 for a similar situation)

$$G(\lambda,x,y) = \sum_{n=1}^\infty \frac{\varphi_n(x)\overline{\varphi_n(y)}}{\lambda_n - \lambda}$$

or, equivalently,

$$\Gamma_{j,\ell}(\lambda,t,s) = \sum_{n=1}^\infty \frac{\varphi_{n;j}(t)\overline{\varphi_{n;\ell}(s)}}{\lambda_n - \lambda}.$$

Integration by parts shows that

$$\lambda_n = \lambda_n \int_T |\varphi_n|^2 = \sum_{j=1}^r \int_0^1 (-\varphi_{n;j}'' + q_j\varphi_{n;j})\overline{\varphi_{n;j}} = \sum_{j=1}^r \int_0^1 (|\varphi_{n;j}'|^2 + q_j|\varphi_{n;j}|^2)$$

since $\sum_{j=1}^r(\varphi'_{n;j}\overline{\varphi_{n;j}})|_0^1 = 0$ due to the boundary and interface conditions satisfied by φ_n. Because q is real and bounded by a constant C this gives

$$\sum_{j=1}^r \int_0^1 |\varphi'_{n;j}|^2 \le C + \lambda_n.$$

Also

$$\sum_{j=1}^r \int_0^1 |\varphi''_{n;j}|^2 \le \sum_{j=1}^r \int_0^1 |(q_j - \lambda_n)\varphi_{n;j}|^2 \le (C + |\lambda_n|)^2$$

again using the boundedness of q. The elementary estimate

$$|f(x)|^2 \le 4\int_0^1 (|f|^2 + |f'|^2),$$

which holds for an absolutely continuous function defined on $[0,1]$, shows then the existence of a constant C' such that

$$|\varphi'_{n;j}(t)| \le C'|\lambda_n|$$

for all sufficiently large n. Thus the series $\sum_{n=1}^\infty \varphi'_{n;j}(t)\overline{\varphi'_{n;\ell}(s)}(\lambda_n - \lambda)^{-m-1}$ is absolutely and uniformly convergent for $m \ge 2$, since the λ_n, as zeros of an entire function of growth order $1/2$ satisfy $\sum_{n=1}^\infty |\lambda_n|^{-1} < \infty$. Therefore

$$\Gamma_{j,\ell}^{(m,1,1)}(\lambda, t, s) = m! \sum_{n=1}^\infty \frac{\varphi'_{n;j}(t)\overline{\varphi'_{n;\ell}(s)}}{(\lambda_n - \lambda)^{m+1}} \tag{4.2}$$

if $m \ge 2$.

5. The Dirichlet to Neumann Map

Recall that there is a unique solution of $Ly = \lambda y$ satisfying the nonhomogeneous Dirichlet boundary conditions $y_j(0) = f_j$ for $j = 1, \dots, n_0$ unless λ is one of the isolated Dirichlet eigenvalues. One may then compute the values $g_j = -y'_j(0)$, $j = 1, \dots, n_0$, (Neumann data). The relationship between the f_j and the g_j is linear and is called the Dirichlet-to-Neumann map. We denote this map by Λ. Recall that $\psi(k, \lambda, \cdot)$ denotes the Weyl solution which assumes the value one at the boundary vertex v_k (and zero at every other boundary vertex). Thus we have

$$\Lambda_{j,k} = -\psi'_j(k, \lambda, 0)$$

and, using equation (4.1),

$$\Lambda_{j,k} = -\Gamma_{j,k}^{(0,1,1)}(\lambda, 0, 0). \tag{5.1}$$

In [6] the following theorem was proved.

Theorem 5.1. *Let q be a complex-valued integrable potential supported on a finite metric tree whose edge lengths are one. Then the associated Dirichlet-to-Neumann map uniquely determines the potential almost everywhere on the tree.*

This theorem is the analogue for trees of a result of Nachman, Sylvester, and Uhlmann [19] who consider a Schrödinger equation on a bounded domain in $\mathbb{R}^n$.

Our proof, given in [6], proceeds in two steps. First, we show that the diagonal elements of the Dirichlet-to-Neumann map determine the potential on the boundary edges. Then we show that the Dirichlet-to-Neumann map of the tree determines the Dirichlet-to-Neumann map for a smaller tree with some of the boundary edges pruned off.

As the case of a finite interval can be considered a tree with two boundary vertices, it is clear that not all the information provided by the Dirichlet-to-Neumann map is necessarily needed. In that case it is sufficient to know one of the diagonal entries. Indeed, as Yurko proves in [22], this generalizes to the present case: knowing $n_0 - 1$ of the diagonal elements of the Dirichlet-to-Neumann map is already sufficient to determine the potential almost everywhere on the tree. Yurko shows this by a more careful investigation of the tree pruning procedure than we had performed.

6. A Levinson-type theorem

In this section we prove Theorem 1.1. The proof is divided in two lemmas following the idea of Nachman, Sylvester, and Uhlmann [19]. Assume two potentials q and $\tilde{q}$ are given both satisfying the hypotheses of the theorem, that is to be real-valued and bounded. To each is associated a Dirichlet-to-Neumann map, denoted by Λ and $\tilde{\Lambda}$, respectively. Both problems have the same Dirichlet eigenvalues and the same Neumann data for an orthonormal set of Dirichlet eigenfunctions. We prove in Lemma 6.1 that, under these circumstances, $\Lambda-\tilde{\Lambda}$ is a polynomial and in Lemma 6.2 that this polynomial must be zero. The conclusion of the theorem follows then by applying Theorem 5.1, i.e., that the Dirichlet-to-Neumann map determines the potential almost everywhere on the tree.

Lemma 6.1. *$\Lambda - \tilde{\Lambda}$ is a polynomial of degree at most one.*

Proof. By equation (5.1) we have $\Lambda_{j,k}(\lambda) = -\Gamma_{j,k}^{(0,1,1)}(\lambda, 0, 0)$. Hence, employing equation (4.2),

$$\Lambda_{j,k}^{(m)}(\lambda) = -\Gamma_{j,k}^{(m,1,1)}(\lambda, 0, 0) = -m! \sum_{n=1}^{\infty} \frac{\varphi'_{n;j}(0)\overline{\varphi'_{n;k}(0)}}{(\lambda_n - \lambda)^{m+1}}$$

provided $m \geq 2$. The right-hand side is determined by the information provided, i.e., the Dirichlet eigenvalues and the Neumann data of the Dirichlet eigenfunctions. Therefore the exact same expression is obtained for $\tilde{\Lambda}_{j,k}^{(m)}$. □

Lemma 6.2. *As λ tends to negative infinity $\Lambda - \tilde{\Lambda}$ tends to zero.*

Proof. By Lemma 3.1 the quantities $\Lambda_{j,k} = -\psi'_j(k, \lambda, 0)$ are exponentially small except for $j = k$ in which case we have $\Lambda_{k,k} = \psi'_k(k, \lambda, 0) = -\sqrt{-\lambda} + o(1)$. But since also $\tilde{\Lambda}_{k,k} = -\sqrt{-\lambda} + o(1)$, we have that $(\Lambda - \tilde{\Lambda})_{k,k} = o(1)$. □

7. A Marchenko-type theorem

In this section we prove Theorem 1.2. The proof relies on establishing a relationship between the eigenfunctions associated with a Dirichlet eigenvalue E and the Weyl solutions for λ near E.

We denote the set of eigenvalues by Σ, i.e., $\Sigma = \{\lambda_n : n \in \mathbb{N}\}$. If $E \in \Sigma$ we denote its geometric multiplicity by $\mu(E)$. Recall that the solution of a Dirichlet boundary value problem is determined by an equation of the form $M\xi = f$. The characteristic function associated with the operator L_D is (a multiple of) the determinant of M, an entire function of growth order 1/2. Hence, by Hadamard's factorization theorem[2]

$$\det(M(\lambda)) = C\prod_{n=1}^{\infty}(1-\lambda/\lambda_n) = C\prod_{E\in\Sigma}(1-\lambda/E)^{\mu(E)}$$

where C is an appropriate constant. Since our problem is self-adjoint geometric and algebraic multiplicities coincide. In particular, every eigenvalue has index one[3]. Therefore we define

$$\chi(\lambda) = \prod_{E\in\Sigma}(1-\lambda/E)$$

the "minimal function" associated with the operator L_D.

Lemma 7.1. *Suppose that E is a Dirichlet eigenvalue. Let*

$$\xi(j,\lambda) = (a_1(j,\lambda),\dots,a_r(j,\lambda),b_1(j,\lambda),\dots,b_r(j,\lambda))^\top$$

be the unique solution of $M(\lambda)\xi = \chi(\lambda)e_j$ for λ near E and $1 \le j \le n_0$. Then the $\xi(j,\lambda)$ have a nonzero limit as λ tends to E and the functions

$$(C(E,\cdot), S(E,\cdot))\xi(j,E), \quad j = 1,\dots,n_0$$

span the space of Dirichlet eigenfunctions for E.

Proof. We know that $\mu(E)$, the multiplicity of the Dirichlet eigenvalue E, is also the multiplicity of E as a zero of $\det M$. Because of the structure of M this means that the lower right $(2r-n_0)\times(2r-n_0)$ block $P(E)$ of $M(E)$ has rank $2r-n_0-\mu(E)$. Without loss of generality, employing elementary row operations, we may assume that the top $2r-n_0-\mu(E)$ rows of $P(E)$ are independent and, consequently, that all entries in the bottom $\mu(E)$ rows of $P(\lambda)$ tend to zero as λ tends to E. This structure is exhibited if we write M in the following way:

$$M = \begin{pmatrix} I & 0 & 0 \\ R_1 & P_1 & P_2 \\ R_2 & P_3 & P_4 \end{pmatrix}$$

where the blocks in the diagonal are of size $n_0 \times n_0$, $(2r-n_0-\mu(E)) \times (2r-n_0-\mu(E))$, and $\mu(E)\times\mu(E)$, respectively. As we just pointed out we know that $P_3(\lambda)$

[2]We are assuming here that zero is not an eigenvalue. This is without loss of generality since a zero eigenvalue would only change the notation but not the essence of the argument.

[3]The index of an eigenvalue is the length of the longest possible Jordan chain.

and $P_4(\lambda)$ tend to zero as λ tends to E. Moreover, $R_2(E)$ has full rank (equal to $\mu(E)$) since, by Theorem 2.3, this is true for

$$J = \begin{pmatrix} R_1 & P_1 & P_2 \\ R_2 & P_3 & P_4 \end{pmatrix}$$

regardless of λ.

We now multiply the equation $M(\lambda)\xi = \chi(\lambda)e_j$ from the left with the invertible matrix

$$T = \begin{pmatrix} I & 0 & 0 \\ -P_1^{-1}R_1 & P_1^{-1} & 0 \\ P_3P_1^{-1}R_1 - R_2 & P_3P_1^{-1} & I \end{pmatrix}$$

(suppressing the dependence on λ wherever it is convenient). Thereby we obtain

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & P_1^{-1}P_2 \\ 0 & 0 & P_4 - P_3P_1^{-1}P_2 \end{pmatrix} \xi = \chi(\lambda)Te_j.$$

Writing $\xi = (x, y, z)^\top$ with appropriately sized columns x and y, and z, we will first consider the equation $(P_4 - P_3P_1^{-1}P_2)z = \chi g_j$ where g_j is the jth column of $P_3P_1^{-1}R_1 - R_2$. Since all entries of the matrix $P_4 - P_3P_1^{-1}P_2$ tend to zero as λ tends to E we may write $(P_4 - P_3P_1^{-1}P_2)(\lambda) = (\lambda - E)F(\lambda)$ and $Fz = \tilde{\chi}g_j$ where $\tilde{\chi}(\lambda) = \chi(\lambda)/(\lambda - E)$. Since $\det(T) = 1/\det(P_1)$ assumes a finite nonzero value at E we obtain that $\det(TM) = \det(P_4 - P_3P_1^{-1}P_2)$ has a zero at E of the same order as $\det M$, i.e., $\mu(E)$. This implies that $\det F(E) \neq 0$. Also, since $\tilde{\chi}(E) \neq 0$ and $g_j(E) \neq 0$ we obtain that $z(j, \lambda)$ tends to a nontrivial vector $z(j, E)$. We now may determine also $x(j, E)$ (which will always be zero) and $y(j, E)$. This proves the first statement of the lemma.

The second statement follows from the observation that $(P_3P_1^{-1}R_1 - R_2)(E) = -R_2(E)$ has rank $\mu(E)$ and hence $\mu(E)$ linearly independent columns $j_1, \ldots, j_{\mu(E)}$, giving rise to linearly independent vectors $\xi(j_1, E), \ldots, \xi(j_{\mu(E)}, E)$ which in turn provide $\mu(E)$ linearly independent Dirichlet eigenfunctions. □

Corollary 7.2. *The functions* $\omega(k, \lambda, \cdot) = \chi(\lambda)\psi(k, \lambda, \cdot)$, $k = 1, \ldots, n_0$ *are defined for all values of* $\lambda \in \mathbb{C}$. *If* λ *is not a Dirichlet eigenvalue these functions are linearly independent and span the space of all solutions of* $Ly = \lambda y$. *If* $\lambda = E$ *is a Dirichlet eigenvalue they span the space of all solutions of* $L_D y = Ey$, *i.e., the eigenspace associated with the Dirichlet eigenvalue* E.

In the following we denote derivatives with respect to the spectral parameter by a dot and (as before) derivatives with respect to the spatial variable with a prime. It is easy to check that $uv = (u'\dot{v} - u\dot{v}')'$ if $u(\lambda, \cdot)$ and $v(\lambda, \cdot)$ both satisfy the equation $-y'' + qy = \lambda y$ for all λ. Hence we get

$$\int_T \omega(k, \lambda, \cdot)\omega(\ell, \lambda, \cdot) = \sum_{j=1}^{r} \left(\omega_j'(k, \lambda, \cdot)\dot{\omega}_j(\ell, \lambda, \cdot) - \omega_j(k, \lambda, \cdot)\dot{\omega}_j'(\ell, \lambda, \cdot)\right)\Big|_0^1.$$

Note that both $\omega(k,\lambda,\cdot)$ and $\dot\omega(\ell,\lambda,\cdot)$ satisfy the interface conditions. Moreover, the boundary conditions, $\omega_j(k,\lambda,0)=\chi(\lambda)\delta_{j,k}$ and $\dot\omega_j(\ell,\lambda,0)=\dot\chi(\lambda)\delta_{j,\ell}$ also hold. Hence we get

$$\int_T \omega(k,\lambda,\cdot)\omega(\ell,\lambda,\cdot)=\chi(\lambda)\dot{\omega_k}'(\ell,\lambda,0)-\dot\chi(\lambda)\omega_\ell'(k,\lambda,0)$$

and, in particular,

$$\int_T \omega(k,E,\cdot)\omega(\ell,E,\cdot)=-\dot\chi(E)\omega_\ell'(k,E,0). \tag{7.1}$$

We introduce the symmetric matrix $N(E)$ by setting

$$N_{k,\ell}(E)=\int_T \omega(k,E,\cdot)\omega(\ell,E,\cdot).$$

Proof of Theorem 1.2. We will prove this theorem by computing the Neumann data of an orthonormal basis of Dirichlet eigenfunctions from the given data. Let E be a Dirichlet eigenvalue of multiplicity $\mu(E)$. Since the naming of the boundary vertices is unimportant we assume now that $K(E)=\{1,\dots,\mu(E)\}$. We may introduce an orthonormal basis $\varphi(k,E,\cdot)$, $k=1,\dots,\mu(E)$ of the eigenspace of E by writing

$$\begin{pmatrix}\omega(1,E,\cdot)\\ \vdots\\ \omega(\mu(E),E,\cdot)\end{pmatrix}=C\begin{pmatrix}\varphi(1,E,\cdot)\\ \vdots\\ \varphi(\mu(E),E,\cdot)\end{pmatrix}$$

with a lower triangular matrix whose diagonal elements are non-zero. In fact, $CC^\top$ is the LU-factorization of the upper left $\mu(E)\times\mu(E)$ block of $N(E)$. Thus, this block determines the matrices C and C^{-1} uniquely.

Now, the Neumann data of the orthonormal Dirichlet eigenfunctions $\varphi(k,E,\cdot)$ are given as linear combinations (in terms of the matrix C) of the Neumann data of the $\omega(k,E,\cdot)$, $k=1,\dots,\mu(E)$ which in turn are uniquely determined, according to equation (7.1) by the first $\mu(E)$ columns of $N(E)$ and the known function χ. $\square$

References

[1] S. Avdonin and P. Kurasov. Inverse problems for quantum trees. Preprint, 2006.

[2] M.I. Belishev. Boundary spectral inverse problem on a class of graphs (trees) by the BC method. *Inverse Problems*, 20(3):647–672, 2004.

[3] M.I. Belishev and A.F. Vakulenko. Inverse problems on graphs: recovering the tree of strings by the BC-method. *J. Inverse Ill-Posed Probl.*, 14(1):29–46, 2006.

[4] C. Bennewitz. A note on the Titchmarsh-Weyl m-function. Preprint ANL-87-26, Vol. 2, 105–111. Argonne National Laboratory, Argonne, Illinois, USA, 1988.

[5] G. Borg. Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte. *Acta Math.*, 78:1–96, 1946.

[6] B.M. Brown and R. Weikard. A Borg-Levinson theorem for trees. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 461(2062):3231–3243, 2005.

[7] R. Carlson. Inverse eigenvalue problems on directed graphs. *Trans. Amer. Math. Soc.*, 351(10):4069–4088, 1999.

[8] E.A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. Robert E. Krieger, Malabar, Florida, 1984.

[9] E.B. Curtis and J.A. Morrow. Determining the resistors in a network. *SIAM J. Appl. Math.*, 50(3):918–930, 1990.

[10] E.B. Curtis and J.A. Morrow. The Dirichlet to Neumann map for a resistor network. *SIAM J. Appl. Math.*, 51(4):1011–1029, 1991.

[11] N.I. Gerasimenko. The inverse scattering problem on a noncompact graph. *Teoret. Mat. Fiz.*, 75(2):187–200, 1988.

[12] M. Harmer. Inverse scattering on matrices with boundary conditions. *J. Phys. A*, 38(22):4875–4885, 2005.

[13] P. Kuchment. Graph models for waves in thin structures. *Waves Random Media*, 12(4):R1–R24, 2002.

[14] P. Kuchment. Quantum graphs. I. Some basic structures. *Waves Random Media*, 14(1):S107–S128, 2004. Special section on quantum graphs.

[15] P. Kurasov and F. Stenberg. On the inverse scattering problem on branching graphs. *J. Phys. A*, 35(1):101–121, 2002.

[16] P. Kurasov and M. Nowaczyk. Inverse spectral problem for quantum graphs. *J. Phys. A*, 38(22):4901–4915, 2005.

[17] N. Levinson. The inverse Sturm-Liouville problem. *Mat. Tidsskr. B.*, 1949:25–30, 1949.

[18] V.A. Marčenko. Concerning the theory of a differential operator of the second order. *Doklady Akad. Nauk SSSR.* (*N.S.*), 72:457–460, 1950.

[19] A. Nachman, J. Sylvester, and G. Uhlmann. An n-dimensional Borg-Levinson theorem. *Comm. Math. Phys.*, 115(4):595–605, 1988.

[20] V.N. Pivovarchik. Ambartsumyan's theorem for the Sturm-Liouville boundary value problem on a star-shaped graph. *Funktsional. Anal. i Prilozhen.*, 39(2):78–81, 2005.

[21] V.N. Pivovarchik. Inverse problem for the Sturm-Liouville equation on a simple graph. *SIAM J. Math. Anal.*, 32(4):801–819 (electronic), 2000.

[22] V. Yurko. Inverse spectral problems for Sturm-Liouville operators on graphs. *Inverse Problems*, 21(3):1075–1086, 2005.

B.M. Brown
School of Computer Science, University of Wales
Cardiff, PO Box 916
Cardiff CF2 3XF, UK
e-mail: Malcolm.Brown@cs.cf.ac.uk

R. Weikard
Department of Mathematics
University of Alabama at Birmingham
Birmingham
Alabama 35294-1170, USA
e-mail: rudi@math.uab.edu