THE INVERSE RESONANCE PROBLEM FOR CMV OPERATORS

RUDI WEIKARD AND MAXIM ZINCHENKO

Abstract. We consider the class of CMV operators with super-exponentially decaying Verblunsky coefficients. For these we define the concept of a resonance. Then we prove the existence of Jost solutions and a uniqueness theorem for the inverse resonance problem: Given the location of all resonances, taking multiplicities into account, the Verblunsky coefficients are uniquely determined.

1. Introduction

This paper is a contribution to the inverse spectral theory of CMV operators. These are represented by five-diagonal infinite unitary matrices giving rise to unitary operators on $\ell^2(\mathbb{N}_0)$. They are closely connected with trigonometric moment problems and with orthogonal polynomials and finite measures on the unit circle. The entries in a CMV matrix are determined by the Verblunsky coefficients, a sequence of complex numbers in the unit disk. For more details we refer the reader to Simon's two-volume monograph [17] and accompanying review papers [16], [18], and [19].

The history of CMV operators is rather interesting: The corresponding five-diagonal unitary matrices were first introduced in 1991 by Bunse–Gerstner and Elsner [7] in the context of numerical linear algebra, and subsequently treated in detail by Watkins [22]. Ten years later they were rediscovered by Cantero, Moral, and Velázquez [8] in connection with orthogonal polynomials on the unit circle. Unaware of the earlier works, Simon coined the term "CMV matrix" in his two-volume monograph [17] (cf. the discussion in Simon [19]). Around the same time CMV matrices also appeared in a work of Bourget, Howland, and Joye [2], though in a context different from orthogonal polynomials. In addition, it was noted by Berezansky and Dudkin [1], that CMV matrices also arise as certain generalizations of block Jacobi matrices.

It is well known (see [15], [17], [10], and [9]) that the spectral properties of CMV operators are fully captured by a Weyl-Titchmarsh-type m-function, making the situation rather analogous to that of Schrödinger operators. In the current situation the m-function is defined away from the unit circle as an analytic function. The values of $m(z)$ are in the right half-plane if $|z| < 1$ and in the left half-plane if $|z| > 1$. Generally these functions can not be analytically extended through the unit circle. Nevertheless, as we will show, in the case of super-exponentially decaying Verblunsky coefficients either branch of the m-function does have a meromorphic extension through the unit circle, but the extension does not agree with

Date: 8. February 2010.

Key words and phrases. CMV operators, finite difference operators, inverse problems.

the originally defined m-function so that one ends up with two meromorphic functions defined on $\mathbb{C}$. Since $m(z) = -\overline{m(1/\overline{z})}$ dealing with one of these functions is sufficient. We shall choose the function which agrees with the m-function in the unit disk and call it M. In analogy to Schrödinger operators the poles of M, which are all outside the unit disk, will be called resonances.

Closely associated with a CMV operator U is a recursion relation

$$(u,v)^\top(z,k) = T(z,k)(u,v)^\top(z,k-1), \quad k \in \mathbb{N}, \tag{1.1}$$

with 2×2-transfer matrices $T(z,k)$ given by (2.6). This recursion relation plays the same role as differential equations do for Sturm-Liouville problems. At least under our assumptions on the Verblunsky coefficients there is a square summable solution to this recursion with a prescribed asymptotic behavior at infinity. This solution will be called the Jost solution.

Just as in the case of Schrödinger operators there is an inverse spectral theory asserting that a CMV operator may be recovered from its m-function or the associated spectral measure, see [10] or [9]. We show in this paper, that the m-function and hence the Verblunsky coefficients are uniquely determined from the location of all resonances, taking multiplicities properly into account, that is, we establish the following theorem.

Theorem 1.1. *Assume that there are constants $C, \gamma > 1$ such that the Verblunsky coefficients α_k of a CMV operator U satisfy*

$$|\alpha_k| < C\exp(-k^\gamma), \quad k \in \mathbb{N}. \tag{1.2}$$

Then the resonances of U and their multiplicities uniquely determine the Verblunsky coefficients and hence U itself.

We also prove the following result.

Theorem 1.2. *Under the assumptions of Theorem* 1.1 *the zeros of the Jost function* (*the first element of the Jost solution*)*, their multiplicities, and the value of the first nonzero Verblunsky coefficient uniquely determine all the Verblunsky coefficients.*

Theorem 1.1 extends earlier results on the inverse resonance problems for Schrödinger and Jacobi operators, see [3]–[6] and [11]. Also of interest is the problem of stability of the recovered operators obtained from incomplete or noisy data, a problem which was considered in [13] and [12] for discrete and continuous Schrödinger operators, respectively. An analogue of such a stability result for the case of CMV matrices is in preparation by the present authors.

The paper is organized as follows. In Section 2 we introduce some basic notations, give rigorous definitions of the objects under consideration, and state some central facts which are already known. In Section 3 we establish the existence of Jost solutions. Finally, in Section 4 we prove theorems 1.1 and 1.2.

2. Preliminaries

We start by introducing some of the basic notations used throughout this paper. Detailed proofs of all facts in this preparatory section (and a lot of additional material) can be found in [10] and [17, Ch. 4.2].

In the following, we denote the set of nonnegative integers by $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and the set of all complex-valued sequences defined on $\mathbb{N}_0$ by $\mathbb{C}^{\mathbb{N}_0}$. By $\ell^2(\mathbb{N}_0)$ we denote the Hilbert space of all square summable complex-valued sequences with scalar product $(\cdot,\cdot)_{\ell^2(\mathbb{N}_0)}$, linear in the second argument. Recall that the vectors $\delta_k \in \ell^2(\mathbb{N}_0)$, $k \in \mathbb{N}_0$, defined by the requirement that $\delta_k(n)$ equals Kronecker's $\delta_{k,n}$, form the standard basis in $\ell^2(\mathbb{N}_0)$. Moreover, we denote by $\ell^2(\mathbb{N}_0)^2$ the space of two-component sequences with square summable entries. The open unit disk in the complex plane is denote by $\mathbb{D}$.

The central object of interest in this paper is a sequence $\alpha : \mathbb{N} \to \mathbb{D}$. Following Simon [17], we call the α_k Verblunsky coefficients in honor of Verblunsky's pioneering work in the theory of orthogonal polynomials on the unit circle [20], [21]. We emphasize that

$$|\alpha_k| < 1, \quad k \in \mathbb{N}. \tag{2.1}$$

Given a sequence α satisfying (2.1), we define a sequence of positive real numbers ρ_k by

$$\rho_k = \sqrt{1 - |\alpha_k|^2}, \quad k \in \mathbb{N}. \tag{2.2}$$

Next, we introduce a half-lattice CMV operator U on $\ell^2(\mathbb{N}_0)$ via its matrix representation in the standard basis of $\ell^2(\mathbb{N}_0)$. Let Θ_k be the 2×2 unitary matrices

$$\Theta_k = \begin{pmatrix} -\alpha_k & \rho_k \\ \rho_k & \overline{\alpha}_k \end{pmatrix}, \quad k \in \mathbb{N}, \tag{2.3}$$

and V and W be block diagonal matrices defined by

$$V = \begin{pmatrix} 1 & & & \\ & \Theta_2 & & \\ & & \Theta_4 & \\ & & & \ddots \end{pmatrix}, \quad W = \begin{pmatrix} \Theta_1 & & & \\ & \Theta_3 & & \\ & & \Theta_5 & \\ & & & \ddots \end{pmatrix}. \tag{2.4}$$

Here V has a single 1×1 block followed by 2×2 blocks on the diagonal. Then the five-diagonal CMV matrix U associated with the Verblunsky coefficients α is given by

$$U = VW = \begin{pmatrix} -\alpha_1 & \rho_1 & 0 & & & & \\ -\rho_1\alpha_2 & -\overline{\alpha}_1\alpha_2 & -\rho_2\alpha_3 & \rho_2\rho_3 & & \text{\Large 0} & \\ \rho_1\rho_2 & \overline{\alpha}_1\rho_2 & -\overline{\alpha}_2\alpha_3 & \overline{\alpha}_2\rho_3 & 0 & & \\ & 0 & -\rho_3\alpha_4 & -\overline{\alpha}_3\alpha_4 & -\rho_4\alpha_5 & \rho_4\rho_5 & \\ & & \rho_3\rho_4 & \overline{\alpha}_3\rho_4 & -\overline{\alpha}_4\alpha_5 & \overline{\alpha}_4\rho_5 & 0 \\ \text{\Large 0} & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.5}$$

Note that U is unitary as an operator from $\ell^2(\mathbb{N}_0)$ to itself.

The CMV matrix associated with the identically zero Verblunsky coefficients is called the free CMV matrix.

Next we recall some of the principal results of [10] needed in this paper. While we have defined U, V, and W as operators from $\ell^2(\mathbb{N}_0)$ to itself they can also be considered as acting on $\mathbb{C}^{\mathbb{N}_0}$. We will use the same letters U, V, and W for these extensions without hesitation whenever their meaning is clear from the context.

We define transfer matrices $T(z,k)$, $z\in\mathbb{C}\backslash\{0\}$, $k\in\mathbb{N}$, by

$$T(z,k)=\begin{cases}\frac{1}{\rho_k}\begin{pmatrix}\alpha_k & z\\ 1/z & \overline{\alpha}_k\end{pmatrix}, & k \text{ odd},\\ \frac{1}{\rho_k}\begin{pmatrix}\overline{\alpha}_k & 1\\ 1 & \alpha_k\end{pmatrix}, & k \text{ even}.\end{cases} \tag{2.6}$$

Lemma 2.1. *Let $z\in\mathbb{C}\backslash\{0\}$ and $u(z,\cdot),v(z,\cdot):\mathbb{N}_0\to\mathbb{C}$ be two sequences of complex functions. Then the following items (i)–(iii) are equivalent:*
(i) For all $k\in\mathbb{N}_0$

$$(Uu(z,\cdot))(k)=zu(z,k)+z(v(z,0)-u(z,0))\delta_0(k), \tag{2.7}$$
$$(Vv(z,\cdot))(k)=u(z,k)+(v(z,0)-u(z,0))\delta_0(k). \tag{2.8}$$

(ii) For all $k\in\mathbb{N}_0$

$$(Wu(z,\cdot))(k)=zv(z,k), \tag{2.9}$$
$$(Vv(z,\cdot))(k)=u(z,k)+(v(z,0)-u(z,0))\delta_0(k). \tag{2.10}$$

(iii) For all $k\in\mathbb{N}$

$$\begin{pmatrix}u(z,k)\\ v(z,k)\end{pmatrix}=T(z,k)\begin{pmatrix}u(z,k-1)\\ v(z,k-1)\end{pmatrix}. \tag{2.11}$$

For $z\in\mathbb{C}\backslash\{0\}$ we introduce now the sequences $k\mapsto(p(z,k),r(z,k))^\top$ and $k\mapsto(q(z,k),s(z,k))^\top$ as the two linearly independent solutions of the recursion (2.11) satisfying the following initial conditions,

$$\begin{pmatrix}p(z,0)\\ r(z,0)\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}q(z,0)\\ s(z,0)\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}. \tag{2.12}$$

If $|z|\neq 1$, that is, if z is in the resolvent set of U, then $u(z,\cdot)=2z(U-z)^{-1}\delta_0$ is an element of $\ell^2(\mathbb{N}_0)$. Define $v(z,0)=2+u(z,0)$ and, for $k\geq 1$, $v(z,k)=(V^*u(z,\cdot))(k)$. If also $z\neq 0$, Lemma 2.1 shows that the sequence $(u(z,\cdot),v(z,\cdot))^\top$ satisfies the recursion (2.11) and is therefore a linear combination of $(p,r)^\top$ and $(q,s)^\top$, that is,

$$\begin{pmatrix}u(z,\cdot)\\ v(z,\cdot)\end{pmatrix}=a(z)\begin{pmatrix}q(z,\cdot)\\ s(z,\cdot)\end{pmatrix}+m(z)\begin{pmatrix}p(z,\cdot)\\ r(z,\cdot)\end{pmatrix}\in\ell^2(\mathbb{N}_0)^2 \tag{2.13}$$

for suitable constants $a(z)$ and $m(z)$. From the initial conditions for p, q, r, and s, and the condition $v(z,0)-u(z,0)=2$ we get $a(z)=1$ and

$$m(z)=1+u(z,0)=(\delta_0,(U+z)(U-z)^{-1}\delta_0)_{\ell^2(\mathbb{N}_0)}. \tag{2.14}$$

The function m is called the Weyl–Titchmarsh m-function for the CMV operator U and the sequence $(u,v)^\top$ is called the Weyl–Titchmarsh solution for the recursion (2.11). A priori, the function m is defined for any z in the resolvent set of U except for $z=0$ and hence certainly for $0<|z|\neq 1$. It is a Caratheodory function with the representation

$$m(z)=\oint_{\partial\mathbb{D}}d\mu(\zeta)\,\frac{\zeta+z}{\zeta-z}, \tag{2.15}$$

where $d\mu$ denotes the spectral measure of U. Note that $z = 0$ is a removable singularity for m and that, in fact, $m(0) = (\delta_0, \delta_0)_{\ell^2(\mathbb{N}_0)} = 1$ so that $d\mu$ is a probability measure,

$$\oint_{\partial\mathbb{D}} d\mu(\zeta) = 1. \tag{2.16}$$

It is also clear from the above definition that $u(0,k) = 0$, $k \in \mathbb{N}_0$, and $v(0,0) = 2$, $v(0,k) = 0$, $k \in \mathbb{N}$, hence the Weyl–Titchmarsh solution has a removable singularity at $z = 0$ as well.

Finally, if $(u,v)^\top$ is any ℓ^2-solution of the recursion (2.11) for z not on the unit circle, then it must be a multiple of the Weyl-Titchmarsh solution, since otherwise z would be an eigenvalue of U, which is impossible. In this case the normalization in (2.12) shows that

$$m(z) = \frac{v(z,0) + u(z,0)}{v(z,0) - u(z,0)}, \quad z \in \mathbb{C}\backslash\partial\mathbb{D}. \tag{2.17}$$

One major reason for the importance of the m-function is that it determines uniquely the Verblunsky coefficients and hence the underlying CMV operator. This is shown by the Borg–Marchenko-type uniqueness result [9] or the inverse spectral theorem [10].

3. Construction of the Jost solution

Hypothesis 3.1. *Let $\alpha : \mathbb{N} \to \mathbb{D}$ be a sequence of super-exponentially decaying Verblunsky coefficients, that is, there are constants $C, \gamma > 1$ such that*

$$|\alpha_k| < C\exp(-k^\gamma), \quad k \in \mathbb{N}. \tag{3.1}$$

Theorem 3.2. *Assume Hypothesis 3.1 to hold and that $z \in \mathbb{C}\backslash\{0\}$. Then the recursion (2.11) has a solution $(u(z,\cdot), v(z,\cdot))^\top$ such that*

$$\begin{aligned} \lim_{k\to\infty} \frac{u(z,2k-1)}{2z^k} &= \lim_{k\to\infty} \frac{v(z,2k)}{2z^k} = 1, \\ \lim_{k\to\infty} \frac{v(z,2k-1)}{2z^k} &= \lim_{k\to\infty} \frac{u(z,2k)}{2z^k} = 0. \end{aligned} \tag{3.2}$$

Moreover, for each $k \in \mathbb{N}_0$, the functions $u(\cdot,k)$ and $v(\cdot,k)$ extend to entire functions of growth order zero.

Proof. We start by defining the numbers

$$\zeta_k = \begin{cases} z, & k \text{ odd}, \\ 1, & k \text{ even}, \end{cases} \quad k \in \mathbb{N}, \tag{3.3}$$

and the matrices

$$A_0(z,k) = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_k^2 \end{pmatrix} \quad \text{and} \quad A_1(z,k) = \begin{pmatrix} 0 & -\alpha_k\zeta_k \\ -\overline{\alpha}_k\zeta_k & 0 \end{pmatrix}, \quad k \in \mathbb{N}, \tag{3.4}$$

where z may be any complex number. Given these we solve the Volterra-type equation

$$F(z,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{n=k+1}^{\infty} A_0(z,k+1)\cdots A_0(z,n-1)A_1(z,n)F(z,n), \quad k \in \mathbb{N}_0, \tag{3.5}$$

by iteration. Let $F_0(z,k)=(1,0)^\top$, $k\in\mathbb{N}_0$, and define recursively

$$F_{s+1}(z,k)=\sum_{n=k+1}^{\infty}A_0(z,k+1)\cdots A_0(z,n-1)A_1(z,n)F_s(z,n),\quad k,s\in\mathbb{N}_0. \tag{3.6}$$

Setting

$$\beta(z,k)=\sum_{n=k+1}^{\infty}|\alpha_n|\max\{1,|z|^{2n-1}\},\quad k\in\mathbb{N}_0, \tag{3.7}$$

one shows by induction that

$$\|F_s(z,k)\|\leqslant\frac{\beta(z,k)^s}{s!},\quad k,s\in\mathbb{N}_0, \tag{3.8}$$

since $\|A_0(z,k)\|\le\max\{1,|z|^2\}$ and $\|A_1(z,k)\|\le|\alpha_k|\max\{1,|z|\}$. Here $\|\cdot\|$ denotes the 2-norm in $\mathbb{C}^2$ or the associated matrix norm. Hence, for every $k\in\mathbb{N}_0$, the series $\sum_{s=0}^{\infty}F_s(\cdot,k)$ converges absolutely and uniformly in compact subsets of $\mathbb{C}$ to a function $F(\cdot,k)$ and the sequence $F(z,\cdot)$ satisfies equation (3.5). Moreover, for each $k\in\mathbb{N}_0$ the function $F(\cdot,k)$ is entire.

Equation (3.5) implies immediately that

$$(A_0(z,k)+A_1(z,k))F(z,k)=F(z,k-1),\quad k\in\mathbb{N}, \tag{3.9}$$

for any complex number z. If z is different from zero we also have

$$F(z,k)=\frac{1}{\zeta_k^2\rho_k^2}\begin{pmatrix}\zeta_k^2 & \alpha_k\zeta_k\\ \overline{\alpha_k}\zeta_k & 1\end{pmatrix}F(z,k-1),\quad k\in\mathbb{N}. \tag{3.10}$$

Now define $C_k=\prod_{j=k+1}^{\infty}\rho_j^{-1}$,

$$\begin{pmatrix}u\\ v\end{pmatrix}(z,2k-1)=2z^kC_{2k-1}F(z,2k-1),\quad k\in\mathbb{N}, \tag{3.11}$$

and

$$\begin{pmatrix}u\\ v\end{pmatrix}(z,2k)=2z^kC_{2k}\begin{pmatrix}0&1\\1&0\end{pmatrix}F(z,2k),\quad k\in\mathbb{N}_0. \tag{3.12}$$

Then the sequence $k\mapsto(u(z,k),v(z,k)^\top$ satisfies the recursion (2.11) and, since $F(z,k)$ tends to $(1,0)^\top$ and C_k tends to 1 as k tends to infinity, the asymptotic behavior (3.2).

It remains to prove the statement on the growth of $(u,v)^\top$. In the following we assume $|z|>1$ and let $N(z)=\lfloor\log(2|z|^2)^{\frac{1}{\gamma-1}}\rfloor$. Then (3.7) together with Hypothesis 3.1 and the estimate $\|F(z,k)\|\le\exp(\beta(z,k))$ implies

$$\|F(z,k)\|\leqslant e^C,\quad k\geqslant N(z). \tag{3.13}$$

For $k<N(z)$ we use (3.9) and the estimate $\|A_0(z,n)+A_1(z,n)\|\leqslant|z|^2(1+|\alpha_n|)$ to obtain,

$$\|F(z,k)\|\leqslant\prod_{n=k+1}^{N(z)}\|A_0(z,n)+A_1(z,n)\|\,\|F(z,N(z))\|\leqslant e^C|z|^{2N(z)}\prod_{n=1}^{\infty}(1+|\alpha_n|). \tag{3.14}$$

This completes the proof. □

The solution $(u(z,\cdot), v(z,\cdot))^\top$ constructed in Theorem 3.2 is in $\ell^2(\mathbb{N}_0)^2$ if $|z|<1$. It is called the *Jost solution* of the recursion (2.11).

Recall that there are square summable solutions also for $|z|>1$. In fact, the sequence $k \mapsto (\tilde{u}(z,k), \tilde{v}(z,k))^\top = (\overline{v(1/\overline{z},k)}, \overline{u(1/\overline{z},k)})^\top$ is such a solution because the transfer matrices exhibit the symmetry,

$$\overline{T(1/\overline{z},k)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T(z,k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k \in \mathbb{N},\ z \in \mathbb{C}\backslash\{0\}. \tag{3.15}$$

If $|z|>1$ the solution $(\tilde{u}(z,k), \tilde{v}(z,k))^\top$ is, by definition, the Jost solution. We emphasize that the analytic extension of the Jost solution inside the unit disk does not coincide with the Jost solution outside the unit disk. Instead, for a given z, the Jost solution and the analytic extension of the Jost solution for $1/\overline{z}$ are linearly independent solutions of (2.11).

4. The Inverse Resonance Problem

In this section we prove two inverse results for the half-lattice CMV operators associated with super-exponentially decaying Verblunsky coefficients. We start with an inverse resonance problem. First, we introduce two functions ψ_0 and ψ_1 associated with a CMV operator U,

$$\psi_0(z) = v(z,0) - u(z,0) \quad \text{and} \quad \psi_1 = v(z,0) + u(z,0), \tag{4.1}$$

where $(u(z,k), v(z,k))^\top$ is the Jost solution associated with the recursion (2.11) in the unit disk. It follows from Theorem 3.2 that in our case of super-exponentially decaying coefficients ψ_0 and ψ_1 are entire functions of growth order zero. These functions have no common zeros since otherwise $u(\cdot,k)$ and $v(\cdot,k)$ have a common zero for $k=0$ and hence for all $k \in \mathbb{N}$ as well which contradicts the asymptotic behavior (3.2). It follows from (2.17) that

$$m(z) = \psi_1(z)/\psi_0(z), \quad z \in \mathbb{D}, \tag{4.2}$$

hence, the zeros of ψ_0 correspond to the poles of the m-function. In particular, this implies that ψ_0 has no zeros in $\mathbb{D}$ and the zeros of ψ_0 on the unit circle are the eigenvalues of U. The zeros of ψ_0 in $\mathbb{C}\backslash\overline{\mathbb{D}}$ are called resonances of the CMV operator U.

Our first result shows that the locations of the eigenvalues and the resonances together with their multiplicities uniquely determine U.

Theorem 4.1. *Assume Hypothesis* 3.1. *Then the zeros of ψ_0 and their multiplicities uniquely determine the Verblunsky coefficients and hence the underlying CMV operator.*

Proof. Since the m-function uniquely determines the Verblunsky coefficients (cf. [9], [10]) it suffices to show that the zeros of ψ_0 uniquely determine the m-function.

First, we note that since ψ_0 is entire of growth order zero, it is, up to a constant multiple, uniquely determined by its zeros via Hadamard's factorization formula, that is, $\psi_0(z) = c\Pi(z)$, where

$$\Pi(z) = \prod_{k=1}^{\infty} (1 - z/z_k)^{n_k}, \quad z \in \mathbb{C}, \tag{4.3}$$

when the zeros of ψ_0 and their multiplicities are denoted by z_k and n_k, respectively. The constant c remains unknown at this point.

Next, we derive a Wronskian-type relation for ψ_0 and ψ_1 which will enable us to recover partial information about ψ_1 which in turn will be sufficient to determine the m-function. Because of equation (3.15) we have

$$\begin{pmatrix} u(z,k) & \overline{v(1/\overline{z},k)} \\ v(z,k) & \overline{u(1/\overline{z},k)} \end{pmatrix} = T(z,k) \begin{pmatrix} u(z,k-1) & \overline{v(1/\overline{z},k-1)} \\ v(z,k-1) & \overline{u(1/\overline{z},k-1)} \end{pmatrix}. \tag{4.4}$$

Taking determinants, the fact that $\det T(z,k) = -1$ and the asymptotic behavior (3.2) yield

$$u(z,k)\overline{u(1/\overline{z},k)} - v(z,k)\overline{v(1/\overline{z},k)} = (-1)^{k+1}4, \quad k \in \mathbb{N}_0, \ z \in \mathbb{C}. \tag{4.5}$$

Using (4.1) and (4.5) at $k=0$ we also obtain

$$\psi_0(z)\overline{\psi_1(1/\overline{z})} + \overline{\psi_0(1/\overline{z})}\psi_1(z) = 8, \quad z \in \mathbb{C}, \tag{4.6}$$

or equivalently,

$$M(z) + \overline{M(1/\overline{z})} = \frac{8}{\psi_0(z)\overline{\psi_0(1/\overline{z})}}, \quad z \in \mathbb{C}, \tag{4.7}$$

where M coincides with the Weyl-Titchmarsh m-function in $\mathbb{D}$ but equals the analytic continuation of m through $\partial\mathbb{D}$ in $\mathbb{C}\backslash\overline{\mathbb{D}}$. We note that this extension is different from what one gets from (2.15) which is the m-function in $\mathbb{C}\backslash\overline{\mathbb{D}}$ but is not an analytic extension through $\partial\mathbb{D}$ of the m-function in $\mathbb{D}$.

By a theorem of Wiman [23], the minimum modulus of an entire function of growth order less than $1/2$ is unbounded. Hence there exists an unbounded sequence r_n such that $\min\{|\psi_0(z)| : |z| = r_n\} \geq 1$. This together with (4.7) implies that $M(z)$ is bounded on the circles $|z| = r_n$. Hence we can use Cauchy's residue theorem to recover the m-function from its residues,

$$\frac{1}{2\pi i}\int_{|\zeta|=r_n} \frac{zM(\zeta)}{\zeta(\zeta-z)}d\zeta = M(z) - M(0) + \sum_{|z_k|<r_n} \operatorname{res}_{\zeta=z_k}\left(\frac{zM(\zeta)}{\zeta(\zeta-z)}\right). \tag{4.8}$$

Since $M(0) = 1$ by (2.15) and since the left hand side of (4.8) goes to zero as $n \to \infty$ we obtain

$$M(z) = 1 - z\sum_k \operatorname{res}_{\zeta=z_k}\left(\frac{M(\zeta)}{\zeta(\zeta-z)}\right), \quad z \in \mathbb{C}\backslash\{z_k\}_{k\in\mathbb{N}}. \tag{4.9}$$

After defining $h_k(\zeta) = (\zeta - z_k)^{n_k}/(\zeta(\zeta-z)\psi_0(\zeta))$ we get

$$\begin{aligned} \operatorname{res}_{\zeta=z_k}\left(\frac{M(\zeta)}{\zeta(\zeta-z)}\right) &= \frac{1}{(n_k-1)!}(\psi_1 h_k)^{(n_k-1)}(z_k) \\ &= \frac{1}{(n_k-1)!}\sum_{n=0}^{n_k-1}\binom{n_k-1}{n}\psi_1^{(n)}(z_k)h_k^{(n_k-1-n)}(z_k). \end{aligned} \tag{4.10}$$

It is clear from (4.10) that to obtain the m-function we need to know the constant c that enters ψ_0 in (4.3) and hence the h_k and also the values $\psi_1^{(n)}(z_k)$ for all $n = 0, \dots, n_k - 1$ and $k \in \mathbb{N}$. The latter we find from equation (4.7),

$$\psi_1^{(n)}(z_k) = \left(\frac{8 - \psi_0(z)\overline{\psi_1(1/\overline{z})}}{\overline{\psi_0(1/\overline{z})}}\right)^{(n)}\Bigg|_{z=z_k} = \left(8/\overline{\psi_0(1/\overline{z})}\right)^{(n)}\Bigg|_{z=z_k}, \tag{4.11}$$

where we used the facts that ψ_0 has a zero of order n_k at z_k for each $k \in \mathbb{N}$ and that $\psi_0(1/\overline{z_k}) \neq 0$. Thus, it remains to find the constant c. In fact, combining (4.3) and (4.9)–(4.11) we see that only the absolute value of c enters the m-function,

$$M(z) = 1 - z|c|^{-2}g(z), \tag{4.12}$$

where $g(z)$ is determined by $\{(z_k, n_k) : k \in \mathbb{N}\}$ and does not depend on c. Using this in (4.7) we find

$$|c|^2 = \frac{4}{\Pi(z)\overline{\Pi(1/\overline{z})}} + \frac{1}{2}\left(zg(z) + \frac{1}{z}\overline{g(1/\overline{z})}\right). \tag{4.13}$$

Since the right hand side is known, so is the value of $|c|^2$ and hence the function M. □

Our next inverse result deals with the zeros of the function $u(\cdot, 0)$ which is the first component of the $k = 0$ entry of the Jost solution for the recursion (2.11).

Theorem 4.2. *Assume Hypothesis* 3.1 *and let* $(u(z,\cdot), v(z,\cdot))^\top$ *be the Jost solution of the recursion* (2.11) *as defined by Theorem* 3.2. *Then the zeros of* $u(\cdot, 0)$ *together with their multiplicities and the value of the first nonzero Verblunsky coefficient uniquely determine all of the Verblunsky coefficients and hence the underlying CMV operator* U.

Proof. As in the previous result it suffices to show that the given data uniquely determine the m-function. In the present case though it will be more convenient to work with the associated Schur function ϕ (cf. [10]), which is an analytic function from $\mathbb{D}$ to $\mathbb{D}$ and is given by

$$\phi(z) = \frac{m(z) - 1}{m(z) + 1} = \frac{u(z,0)}{v(z,0)}, \quad z \in \mathbb{D}, \tag{4.14}$$

where $(u(z,\cdot), v(z,\cdot))^\top$ is the Jost solution for the recursion (2.11). As discussed after (4.1), the functions $u(z,0)$ and $v(z,0)$ have no common zeros. This yields that all the zeros of $v(z,0)$ are in $\mathbb{C}\backslash\overline{\mathbb{D}}$. Moreover, we note that since $m(0) = 1$ the Schur function ϕ and hence also $u(\cdot, 0)$ always vanish at the origin. A detailed asymptotic analysis (cf. [10]) shows that $\phi(z) = -z^{n_0}\overline{\alpha}_{n_0} + O(z^{n_0+1})$ as z tends to 0, where $\alpha_{n_0} \neq 0$ is the first nonzero Verblunsky coefficients, that is, $\alpha_k = 0$ for all $k = 1, \dots, n_0 - 1$.

Since $u(z,0)$ is entire of growth order zero it is uniquely determined up to a constant by its zeros via Hadamard's factorization formula,

$$u(z,0) = cz^{n_0}g(z), \quad g(z) = \prod_{k=1}^{\infty}(1 - z/z_k)^{n_k}, \quad z \in \mathbb{C}, \tag{4.15}$$

where $z_k \neq 0$ are the non-trivial zeros of $u(\cdot, 0)$, n_k their respective multiplicities, and $c \in \mathbb{C}$ is an unknown constant. Using the Wronskian relation (4.5) on $\partial\mathbb{D}$ we obtain

$$|v(z,0)|^2 = 4 + |u(z,0)|^2, \quad |z| = 1. \tag{4.16}$$

$v(\cdot, 0)$ is uniquely determined by $u(\cdot, 0)$ via the inner-outer factorization (see, e.g., Theorem 7.17 in Rudin [14]). Specifically, $v(z,0) = P(z)Q(z)$ with outer factor

$$Q(z) = \exp\left(\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(4 + |u(e^{i\theta}, 0)|^2)\frac{d\theta}{4\pi}\right), \quad z \in \mathbb{D}, \tag{4.17}$$

and inner factor P, a constant of modulus one on account of Jensen's formula using the fact that $v(\cdot, 0)$ has no zeros in $\mathbb{D}$. Thus, we have from (4.14) and (4.17),

$$\phi(z) = cz^{n_0} g(z) \frac{1}{P} \exp\left(- \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log(4+|cg(e^{i\theta})|^2) \frac{d\theta}{4\pi} \right), \quad z \in \mathbb{D}, \tag{4.18}$$

and it remains to find the value of c/P. The asymptotic behavior of ϕ at the origin yields,

$$-\overline{\alpha}_{n_0} = \big(z^{-n_0}\phi(z)\big)\big|_{z=0} = \frac{c}{P} \exp\left(- \int_0^{2\pi} \log(4+|cg(e^{i\theta})|^2) \frac{d\theta}{4\pi} \right). \tag{4.19}$$

Hence $\arg(c/P) = \arg(-\overline{\alpha}_{n_0})$ and $|c/P| = F^{-1}(\log|\alpha_{n_0}|)$ where

$$F(x) = \int_0^{2\pi} \log \frac{x^2}{4+x^2|g(e^{i\theta})|^2} \frac{d\theta}{4\pi} \tag{4.20}$$

is strictly monotone increasing for $x > 0$ and hence invertible. □

Remark 4.3. Looking at the case of a single nonzero coefficient α_{n_0} for some $n_0 \in \mathbb{N}$, we conclude that the previous result is sharp in the sense that the zeros of $u(\cdot, 0)$ alone do not determine all of the Verblunsky coefficients. Indeed, in this case $\phi(z) = -z^{n_0}\overline{\alpha}_{n_0}$, hence $u(\cdot, 0)$ has a single zero at the origin and its location is independent of the value of α_{n_0}.

References

[1] Yu. M. Berezansky and M. E. Dudkin, *The direct and inverse spectral problems for the block Jacobi type unitary matrices*, Meth. Funct. Anal. Top. **11**, 327–345 (2005).

[2] O. Bourget, J. S. Howland, and A. Joye, *Spectral analysis of unitary band matrices*, Commun. Math. Phys. **234**, 191–227 (2003).

[3] B. M. Brown, I. Knowles, R. Weikard, *On the inverse resonance problem*, J. London Math. Soc. (2) **68**(2), 383-401 (2003).

[4] B. M. Brown, S. Naboko, R. Weikard, *The inverse resonance problem for Jacobi operators*, Bull. London Math. Soc. **37**, 727-737 (2005).

[5] M. Brown, S. Naboko, and R. Weikard, *The inverse resonance problem for Hermite operators*, Constr. Approx. **30**, 155–174 (2009).

[6] B. M. Brown, R. Weikard, *The inverse resonance problem for perturbations of algebro-geometric potentials*, Inverse Problems **20**(2), 481-494 (2004).

[7] A. Bunse-Gerstner and L. Elsner, *Schur parameter pencils for the solution of unitary eigenproblem*, Lin. Algebra Appl. **154/156**, 741–778 (1991).

[8] M. J. Cantero, L. Moral, and L. Velázquez, *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle*, Lin. Algebra Appl. **362**, 29–56 (2003).

[9] S. Clark, F. Gesztesy, and M. Zinchenko, *Borg–Marchenko-type uniqueness results for CMV operators*, Trans. Roy. Norw. Soc. Sci. Lett. **4**, 1–20 (2008).

[10] F. Gesztesy and M. Zinchenko, *Weyl–Titchmarsh theory for CMV operators associated with orthogonal polynomials on the unit circle*, J. Approx. Th. **139**, 172–213 (2006).

[11] E. Korotyaev, *Inverse resonance scattering on the half line*, Asymptot. Anal. **37**(34), 215-226 (2004).

[12] M. Marlettta, R. Shterenberg, R. Weikard, *On the inverse resonance problem for Schrödinger operators*, to appear in Commun. Math. Phys.

[13] M. Marlettta and R. Weikard, *Stability for the inverse resonance problem for a Jacobi operator with complex potential*, Inverse Problems **23**(4), 1677–1688 (2007).

[14] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York, third edition, 1987.

[15] B. Simon, *Analogs of the m-function in the theory of orthogonal polynomials on the unit circle*, J. Comp. Appl. Math. **171**, 411-424 (2004).

[16] B. Simon, *Orthogonal polynomials on the unit circle: New results*, Intl. Math. Res. Notices, **53**, 2837–2880 (2004).

[17] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, Part 2: Spectral Theory*, AMS Colloquium Publication Series, Vol. 54, Providence, R.I., 2005.
[18] B. Simon, *OPUC on one foot*, Bull. Amer. Math. Soc. **42**, 431–460 (2005).
[19] B. Simon, *CMV matrices: Five years later*, J. Comp. Appl. Math. **208**, 120–154 (2007).
[20] S. Verblunsky, *On positive harmonic functions: A contribution to the algebra of Fourier series*, Proc. London Math. Soc. (2) **38**, 125–157 (1935).
[21] S. Verblunsky, *On positive harmonic functions (second paper)*, Proc. London Math. Soc. (2) **40**, 290–320 (1936).
[22] D. S. Watkins, *Some perspectives on the eigenvalue problem*, SIAM Rev. **35**, 430–471 (1993).
[23] A. Wiman, *Über eine Eigenschaft der ganzen Funktionen von Höhe Null*, Math. Ann. **76**, 197–211 (1915).

Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294, USA
E-mail address: rudi@math.uab.edu
URL: http://www.math.math.uab.edu/~rudi/

Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA
E-mail address: maxim.zinchenko@wmich.edu